Portfolio management under risk constraints

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12 septembre 2013
Journée des doctorants DIM
Plan

- Key-tool: BSDEs
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  - Reflected backward stochastic differential equations with jumps and partial integro variational inequalities

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  - Double barrier reflected BSDEs with jumps and generalized Dynkin games
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  - Portfolio management under amount constraints
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  - BSDEs with jumps and weak nonlinear terminal condition
  - BSDEs with weak reflection
Key-tool: BSDEs

Definition BSDEs
A process \((X, Z)\) is said to be a solution of the BSDE associated with driver \(f\) and terminal condition \(\xi\) if

\[
-dX_t = f(t, X_t, Z_t)\, dt - Z_t\, dW_t
\]

\[
X_T = \xi_T
\]

The interest for this kind of stochastic equations has increased steadily. This is due to the strong connections of these equations with mathematical finance and the fact that they gave a generalization of the well known Feynman-Kac formula to semi-linear partial differential equations and fully nonlinear equations (probabilistic representations).
Reflected BSDEs with jumps

A process $(Y, Z, k(\cdot), A)$ is said to be a solution of the reflected BSDE with jumps associated with driver $g$ and obstacle $\xi$, if

$$-dY_t = g(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - Z_t dW_t - \int_{R^*} k_t(u) \tilde{N}(dt, du)$$

$$Y_T = \xi_T$$

$$\xi_t \leq Y_t \ 0 \leq t \leq T \ a.s.,$$

$$\int_0^T (Y_t - \xi_t) dA^c_t = 0 \ a.s. \ and \ \Delta A^d_t = \Delta A^d_t 1_{\{Y_t = \xi_t\}}$$
Reflected BSDEs with jumps in the Markovian case

For each \((t, x) \in [0, T] \times \mathbb{R}\), let \(\{X_s^{t,x}, t \leq s \leq T\}\) be the unique \(\mathbb{R}\)-valued solution of the SDE with jumps:

\[
X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) \, dr + \int_t^s \sigma(X_r^{t,x}) \, dW_r + \int_t^s \int_{\mathbb{R}^*} \beta(X_r^{t,x}, e) \tilde{N}(dr, de)
\]

The obstacle \(\xi_s^{t,x}\) and driver \(f\) are of the following form:

\[
\begin{align*}
\xi_s^{t,x} &:= h(s, X_s^{t,x}), \quad s < T \\
\xi_T^{t,x} &:= g(X_T^{t,x}) \\
f(s, X_s^{t,x}(\omega), y, z, k) &:= \varphi(s, X_s^{t,x}(\omega), y, z, \int_{\mathbb{R}^*} k(e) \gamma(x, e) \nu(de)) \mathbf{1}_{s \geq t}
\end{align*}
\]
Related obstacle problem for a PIDE

\[
\begin{aligned}
\min(u(t, x) - h(t, x), \\
\frac{\partial u}{\partial t}(t, x) - Lu(t, x) - f(t, x, u(t, x), (\sigma \frac{\partial u}{\partial x})(t, x), Bu(t, x)) = 0, \\
(t, x) \in [0, T) \times \mathbb{R} \\
u(T, x) = g(x), x \in \mathbb{R}
\end{aligned}
\]

(2)

where

- \( L := A + K \)

- \( A\phi(x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2}(x) + b(x)\frac{\partial \phi}{\partial x}(x), \phi \in C^2(\mathbb{R}) \)

- \( K\phi(x) := \int_{\mathbb{R}^*} \left( \phi(x + \beta(x, e)) - \phi(x) - \frac{\partial \phi}{\partial x}(x)\beta(x, e) \right) v(de) \)

- \( B\phi(x) := \int_{\mathbb{R}^*} (\phi(x + \beta(x, e)) - \phi(x))\gamma(x, e) v(de). \)
RBSDEs with jumps and PIDVIs.
Literature/Contribution

**Literature** Links between RBSDEs and obstacle problems for PDEs in the continuous case.

**Contribution** We extend the previous results in the case when the dynamic of $Y$ and the obstacle admit jumps.

- **In the general case:**
  
  We establish new a priori estimates for RBSDEs with jumps.

- **In the Markovian case:**
  
  We define:

  $$u(t, x) := Y_t^{t,x}. \quad (3)$$

  - We introduce the related obstacle problem for a parabolic PIDE (2)
  - We establish an existence result. More precisely, we show that $u$ defined by (3) is a solution of the PIDVI (2). The proof we propose is different and simpler than the one used in the previous literature in the continuous case.
We prove properties of the function $u$ defined by (3).

We give an uniqueness result in the class of continuous functions with polynomial growth. In order to obtain it, we establish a comparison theorem. The proof of this theorem: technical difficulties!
Financial application: optimal stopping problem for dynamic risk measures induced by BSDEs with jumps.

In the framework of risk measures: the state process \( X = \) an index, an interest rate process, an economic factor, an indicator of the market, the value of a portfolio, which has an influence on the risk measure and the position.

- Dynamic risk measure of the financial position \( \zeta \) at time \( t \):
  \[
  \rho_t(\zeta, S) := -\mathcal{X}_t(\zeta, S)
  \]
  where \( \mathcal{X}_t(\zeta, S) = \mathcal{X}_t \) denotes the solution of the following BSDE:
  \[
  -d\mathcal{X}_t = f(t, \mathcal{X}_t, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbb{R}^*} l_t(u) \tilde{N}(dt, du); \quad \mathcal{X}_S = \zeta
  \]

- The minimal risk measure:
  \[
  \nu(S) := \text{ess inf}_{\tau \in \mathcal{T}_S} \rho_S(\xi_\tau, \tau) \quad (4)
  \]
Proposition 1. The minimal risk measure at time $S$ satisfies

$$v(S) = -Y_S = -u(S, X_S) \quad \text{a.s.}$$

(5)

where $u$ is the unique viscosity solution of the PIDIV (2). Moreover, the stopping time $\tau^*_S$ defined by

$$\tau^*_S := \inf\{ t \geq S, \ Y_t = \xi_t \} = \inf\{ t \geq S, \ u(t, X_t) = \bar{h}(t, X_t) \}$$

is optimal for (4), that is $v(S) = \rho_S(\xi_{\tau^*_S}, \tau^*_S) \text{ a.s.}$
DBSDEs with jumps and generalized Dynkin games. Mathematical tools

- Double barrier reflected BSDEs with jumps

\[-dY_t = g(t, Y_t, Z_t, k_t(\cdot))dt + dA_t - dA'_t - Z_t dW_t\]
\[-\int_{\mathbb{R}^*} k_t(u) \tilde{N}(dt, du) \quad Y_T = \xi_T\]
\[\xi_t \leq Y_t \leq \zeta_t, \quad 0 \leq t \leq T \text{ a.s.,}\]
\[
\begin{cases}
\int_0^T (Y_t - \xi_t) dA_t^c = 0 \text{ a.s. and } \int_0^T (\zeta_t - Y_t) dA'_t^c = 0 \text{ a.s.} \\
\Delta A_t^d = \Delta A_t^d \mathbf{1}_{\{Y_t = \xi_t\}} \text{ and } \\
\Delta A'_t^d = \Delta A'_t^d \mathbf{1}_{\{Y_t = \zeta_t\}} \text{ a.s.}
\end{cases}
\] (7)

- Double barrier reflected BSDEs with jumps (in the Markovian case)

- Related obstacle problem for a parabolic PIDE
DBSDEs with jumps and generalized Dynkin games. Mathematical tools

- **Classical Dynkin games**
  
  Consider the gain (or payoff):

  \[ I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u)du + \xi_\tau 1_{\{\tau \leq \sigma\}} + \zeta_\sigma 1_{\{\sigma < \tau\}} \]

  The upper and lower value functions at time \( S \) are defined respectively by

  \[ \overline{V}(S) := \text{ess inf}_\sigma \text{ess sup}_\tau \mathbb{E}[I_S(\tau, \sigma) | \mathcal{F}_S] \]
  \[ \underline{V}(S) := \text{ess sup}_\tau \text{ess inf}_\sigma \mathbb{E}[I_S(\tau, \sigma) | \mathcal{F}_S] \]

  The game is said to be fair if it admits a value function, i.e. \( \overline{V}(S) = \underline{V}(S) \).
DBSDEs with jumps and generalized Dynkin games. 
Mathematical tools

- **General Dynkin games** Let \( I(\tau, \sigma) \) be the \( \mathcal{F}_{\tau \wedge \sigma} \)-measurable random variable defined by

\[
I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.
\]

For each stopping time \( S \), the upper and lower value functions at time \( S \) are defined respectively by

\[
\overline{V}(S) := \text{ess inf} \bigg( \text{ess sup} \mathcal{E}_{S, \tau \wedge \sigma}(I(\tau, \sigma)) \bigg),
\]

\[
\underline{V}(S) := \text{ess sup} \bigg( \text{ess inf} \mathcal{E}_{S, \tau \wedge \sigma}(I(\tau, \sigma)) \bigg).
\]
DBSDEs with jumps and generalized Dynkin games.

Literature/Contribution

**Literature** Papers on DBBSDEs in the continuous case and a few papers on DBBSDEs with jumps and continuous obstacles.

**Contribution** We extend some of the previous results in the case when the obstacles also admit jumps; we introduce a more general class of game problems.

More precisely:

- **Existence and uniqueness of the solution of the DBBSDE with jumps and RCLL obstacles.**

- **Links between the DBBSDDE and classical Dynkin games** We show that the value function of the classical game coincides with the solution of the DBBSDE.
DBSDEs with jumps and generalized Dynkin games.

Literature/Contribution

- **An alternative characterization of the solution of the DBBSDE.** We introduce the general Dynkin games. We show that this game is fair and its value function also coincides with the solution of the DBBSDE.

- **Comparison theorems.** We establish a comparison theorem and a strict comparison theorem for DBBSDEs with jumps (which has not been the case even in the continuous case in the previous literature).

- **A priori estimates and links with obstacle problems for PIDE.** We establish new a priori estimates for DBBSDEs and prove existence and uniqueness of the solution of the associated obstacle problem (in the Markovian framework).
DBSDEs with jumps and generalized Dynkin games. Financial application

Result applied in mathematical finance to deal with American game (or recallable) options whose underlying derivates contain a Poisson part. The nonarbitrage price process of the recallable option is equal to the value process of the zero-sum Dynkin game associated with $\xi$ and $\zeta$.

- The value process of an European option with maturity $T$ and payoff $\xi$ is equal to the $g$-conditional expectation $(\mathcal{E}_t, T(\xi))_{0 \leq t < T}$.
- The recallable option: the seller is allowed to cancel the recallable option and the buyer is allowed to exercise it at any stopping time up to the maturity $T$. If the buyer decides to exercise at $\sigma$ or the seller to cancel at $\tau$, then the seller pays the amount:

$$l(\tau, \sigma) = \xi_\sigma 1_{\{\sigma \leq \tau\}} + \zeta_\tau 1_{\{\tau < \sigma\}}.$$ 

The quantity $\xi_\sigma$ (resp. $\zeta_\tau$) is the amount that the buyer obtains (resp. the seller pays) for her decision to exercise (resp., cancel) first at $\sigma$ (resp., $\tau$). The difference $\zeta - \xi$ represents the compensation process.
"Exact replication under Delta constraints" (R. Elie, JF Chassagneux, I. Kharroubi)

- Mathematical tools: BSDEs
- Super-replication price of a contingent claim under K (convex) constraint

\[ u(t, x) = \inf \{ y \in \mathbb{R} : \exists \Delta \in \mathcal{A}_{t,x}^K, y + \int_t^T \Delta_s dX_{s,x}^T \geq h(X_{T,x}^t) \mathbb{P} \text{ a.s.} \} \]

where \( \Delta \) = number of assets \( X \).

- \( \Delta \) is solution of a BSDE. It is obtained a necessary and sufficient condition on the driver of this BSDE under which super-hedging any claim under Delta constraints is equivalent to simply hedge the facelift transform of this claim (viability approach for BSDEs).
Goal: extend the previous results in the case when we impose convex constraints on the amount invested in the assets.

Partial results: The amount invested is also solution of a multidimensional BSDE. We have obtained a necessary and sufficient condition on the driver in the case of half-spaces (by using a viability approach).
BSDEs with jumps and weak nonlinear terminal condition. Literature

"BSDEs with weak terminal condition" (B. Bouchard, R. Elie, A. Reveillac)

- **Problem formulation** Given \( \psi \) and \( m \), find the minimal solution \((Y, Z)\) to

\[
Y_t \geq Y_T + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s
\]

satisfying

\[
E[\psi(Y_T)] \geq m.
\]

- "Weak terminal condition": no fixed terminal condition, but a constraint in expectation.

- **Financial application**: the minimal initial value of \( Y \) corresponds to the super-hedging price with controlled loss:

\[
\inf\{y \in \mathbb{R} : \exists Z : E[\psi(Y_T^{y,Z})] \geq m\}.
\]
• **Goal:** Extend the previous results in the case of jumps and when the terminal condition is of the form $\mathcal{E}^f[\psi(Y_T)] \geq m$. In the Markovian case, write the associated PDE.

• **Partial results:** In the Non-markovian setting, we have obtained the dynamic programming principle.

• **Financial application:** Risk measures
BSDEs with weak reflection. Formulation of the problem

- **Super-replication price of American options:**

  \[ \{ \inf y \in \mathbb{R} : \exists \theta \in \mathcal{A}, Y_{\tau}^{y,\theta} \geq \Phi(S_{\tau}), \forall \tau \} \]

  - \( Y \): portfolio dynamic
  - \( S \): asset price
  - \( \theta \): strategy
  - \( y \): initial capital

- **Goal** Compute the super-replication price in the case of controlled loss (the price is smaller, but the seller takes risk!):

  \[ \{ \inf y \in \mathbb{R} : \exists \theta \in \mathcal{A}, P(Y_{\tau}^{y,\theta} \geq \Phi(S_{\tau})) \geq \mu, \forall \tau \} \]
BSDEs with weak reflection. Ideas

In the case of european options (B. Bouchard, R. Elie, N. Touzi):

- Super-replication price partial hedge
- Super-replication price total hedge

Reduction method:

- \( \overline{S} = (S, P) \), \( P \) : dynamic probability of super-replication
- Th. martingale representation

\[
P_t^{p, \alpha} = p + \int_0^t \alpha_u dW_u
\]

- The "new" control: \((\theta, \alpha)\)
- Mathematical tool: BSDEs with weak terminal condition

In our case: We generalize the above method in the case of American options. For instance, we have obtained a control independent of \( \tau \). We’ll need to introduce a new class of BSDEs with weak reflection.