Mach stems, kink modes and so on...

Strongly nonlinear geometric optics for hyperbolic boundary value problems

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Paris, Mathematical hydrodynamics, June 2014
Plan of the talk

1. A little bit of history (and motivation)
2. A classification of hyperbolic boundary value problems
3. The Mach stem equation for nonresonant wavetrains
4. The Mach stem equation for pulses
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Where I got interested in Mach stems...

**Majda-Rosales** 1983 proposed a theory for **Mach stems** formation on shock waves. A Mach stem is a wave pattern in **compressible inviscid** fluid flows that involves three **shock waves** and one **contact discontinuity** emanating from a single point (in 2D).

*Figure: Source: Majda-Rosales, SIAM Journal on Applied Mathematics, 1983.*
Mach stems also arise in the *reflection* of planar shocks past wedges, see, e.g., Courant-Friedrichs 1948, or the review article Serre 2007. (Classical benchmark for numerical codes.)

**Figure**: Mach reflection past a wedge. Credit: Stuttgart Numerics Research Group [http://nrg.iag.uni-stuttgart.de/](http://nrg.iag.uni-stuttgart.de/).
Mach stems occur in "real life"...

**Figure:** Mach Stem formation after nuclear explosion. Credit: Atomic Archive http://www.atomicarchive.com.
Similar expansions were derived by Artola-Majda 1987 for the formation of kinks on two-dimensional contact discontinuities (vortex sheets), giving rise to a **kink mode** (CD-CD-S-R pattern).

**Figure:** Source: Artola-Majda, Physica D, 1987.
In both articles by Majda-Rosales and Artola-Majda, the authors "construct" what they call weakly nonlinear asymptotic expansions of the form
\[
u_\varepsilon(t, x) \sim u + \varepsilon U_1 \left( t, x, \frac{\Phi(t, x)}{\varepsilon} \right) + \varepsilon^2 U_2 ...\]
that are formal solutions to free boundary value problems for the compressible Euler equations (either shock waves or contact discontinuities).
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that are formal solutions to free boundary value problems for the compressible Euler equations (either shock waves or contact discontinuities).

More precisely, the authors derive a leading order amplitude equation from which the above (finite/infinite ?) asymptotic expansion is (presumably) constructed.
The main feature of these expansions is an amplification phenomenon:

- Oscillating waves of amplitude $\varepsilon^2$ and frequency $\sim 1/\varepsilon$ that hit the free boundary may give rise, when reflected, to waves of amplitude $O(\varepsilon)$. 
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After some algebra (...), the construction of the leading order term reduces to solving a scalar equation:
- Burgers equation with an extra integro-differential term for shock waves,
- Burgers equation for contact discontinuities.
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- Burgers equation with an extra integro-differential term for shock waves,
- Burgers equation for contact discontinuities.

Some causality arguments are used in order to discard some terms that might arise in the expansion...

The new Burgers type equation seems to display similar "wave breaking" behavior as the Burgers equation. Majda-Rosales 1984.
Main goals

Five-six years ago, the objective was to construct highly oscillating shock waves/contact discontinuities, whose leading order expansion was precisely the one given by Majda-Rosales or Artola-Majda...
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What we have achieved so far:

- Justify geometric optics expansions with amplification for **linear** boundary value problems (joint work with O. Guès).

- Justify weakly nonlinear geometric optics with amplification for **semilinear** problems. Part of the problem is to understand the appropriate scaling (joint work with O. Guès and M. Williams).

- Give a systematic construction of Artola-Majda-Rosales type expansions to produce high order approximate solutions for **quasilinear** problems. This regime corresponds to strongly nonlinear geometric optics, in analogy with Cheverry-Guès-Métivier 2003 (joint work with M. Williams).
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4. The Mach stem equation for pulses
We consider a linear \textit{hyperbolic} first order PDE system in the \textit{half-space} \( \{x_d > 0\} \):

\[
\begin{aligned}
\partial_t u + \sum_{j=1}^d A_j \partial_{x_j} u &= F(t, x), & t &\in [0, T], & x &\in \mathbb{R}^d, \\
B u|_{x_d=0} &= g(t, y), & t &\in [0, T], & y &\in \mathbb{R}^{d-1}, \\
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The number of boundary conditions is well-understood:

**Assumption**

The matrix \( A_d \) is invertible (non-characteristic boundary). Moreover the matrix \( B \) has maximal rank \( p \), and \( p \) equals the number of **positive** eigenvalues of \( A_d \) (incoming characteristics).
The well-posedness theory is based on a normal modes decomposition.

We consider the global in time problem:

\[
\begin{cases}
\partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^d, \\
B \ u|_{x_d=0} = 0, & t \in \mathbb{R}, y \in \mathbb{R}^{d-1},
\end{cases}
\]

for which we look for solutions under the form

\[\exp(i \ z \ t + i \ \eta \cdot y) \ U(x_d),\]

with \( \text{Im} \ z < 0, \ \eta \in \mathbb{R}^{d-1} \) and \( U \in L^2(\mathbb{R}^+). \)
We thus need to study an ODE system:

\[
\begin{align*}
U' &= -i A_d^{-1} \left( z I + \sum_{j=1}^{d-1} \eta_j A_j \right) U, \quad x_d > 0, \\
B U(0) &= 0,
\end{align*}
\]

for which the following result holds:

**Lemma (Hersh 1963)**

The above ODE system is hyperbolic for \( \text{Im} z < 0 \), and its stable subspace \( E^s(z, \eta) \) has dimension \( p \).
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If \( E^s(z, \eta) \cap \text{Ker} \, B \neq \{0\} \) for some pair \((z, \eta)\), then the boundary value problem is violently ill-posed (the instability is of Kelvin-Helmholtz type).
Technical but crucial result (Kreiss 1970, Métivier 2000)

If the system is hyperbolic with constant multiplicity, the stable subspace $E^s(z, \eta)$ extends continuously to nonzero pairs $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1}$, that is to say to $\text{Im} \, z = 0$. 

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**Warning**: for $\tau \in \mathbb{R}$, the vectors in the extended stable subspace $E^s(\tau, \eta)$ may correspond to oscillating solutions of the ODE. $E^s(\tau, \eta)$ may contain (part of) the central subspace.
There are more or less three main classes of problems for which one can hope to prove something:

**Class 1 (the best, Kreiss-Sakamoto 1970)**

\[ E^s(z, \eta) \cap \text{Ker } B = \{0\} \text{ for all } \text{Im } z \leq 0 \text{ and } \eta \in \mathbb{R}^{d-1}. \] The boundary value problem is said to satisfy the uniform Kreiss-Lopatinskii condition.

Majda’s shock wave stability theory (1982-1983) is based on a similar ”spectral stability” condition.
Classification of hyperbolic boundary value problems

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**Weakly nonlinear** geometric optics expansions for **quasilinear** problems have been rigorously justified by Williams 1996-2002, Lescarret 2007, C.-Guès-Williams 2011.

**Scaling**: amplitude \( \varepsilon \), frequency \( 1/\varepsilon \), as for the Cauchy problem (e.g., Hunter-Majda-Rosales 1986, Joly-Métivier-Rauch 1993-1995...).
Classification of hyperbolic boundary value problems

Class 2 (a little less nice, Sablé-Tougeron 1986)

\[ E^s(z, \eta) \cap \text{Ker } B = \{0\} \text{ when } \text{Im } z < 0, \text{ but for some frequency there holds } E^s(\tau, \eta) \cap \text{Ker } B \neq \{0\} \text{ and corresponding nonzero solutions } U \text{ to the ODE have exponential decay (surface waves). The best known example are Rayleigh waves in elastodynamics.} \]

Weakly nonlinear geometric optics expansions for quasilinear problems have been rigorously justified by Marcou 2010 (for wavetrains). Same scaling as for the Cauchy problem.
Classification of hyperbolic boundary value problems

Class 3 (the toughest one, Benzoni-Gavage, Rousset, Serre, Zumbrun 2002...)

\[ E^s(z, \eta) \cap \text{Ker } B = \{0\} \text{ when } \text{Im } z < 0, \text{ but for some frequency there holds } E^s(\tau, \eta) \cap \text{Ker } B \neq \{0\} \text{ and corresponding nonzero solutions } U \text{ to the ODE oscillate.} \]

The Artola-Majda-Rosales expansions are based on such a spectral configuration.
In the linear case (no interaction between oscillations), our main result (C.-Guès 2010) justifies the amplification of outgoing oscillations and of the boundary source term:

- Outgoing wave packets: amplitude $\varepsilon$, frequency $\sim 1/\varepsilon$,
- Incoming wave packets: amplitude 1, frequency $\sim 1/\varepsilon$. 

Geometric optics expansions
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- Incoming wave packets: amplitude 1, frequency $\sim 1/\varepsilon$.

Outgoing wave packets only hit the boundary once, so one single amplification is allowed in the geometry of the half-space.

In this situation, what is the weakly nonlinear scaling?
Consider the semilinear oscillating problem:

\[
\begin{aligned}
&\partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = F(u), & t \in [0, T], \ x \in \mathbb{R}^d_+, \\
&Bu|_{x_d=0} = \varepsilon^{\alpha} g \left( t, y, \frac{\varphi_0(t,y)}{\varepsilon} \right), & t \in [0, T], \ y \in \mathbb{R}^{d-1}, \\
&u|_{t=0} = 0, & x \in \mathbb{R}^d_+.
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Expected behavior of the solution:

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B u|_{x_d = 0} = \varepsilon^\alpha g \left( t, y, \frac{\varphi_0(t, y)}{\varepsilon} \right), & t \in [0, T], \ y \in \mathbb{R}^{d-1} , \\
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\]

Expected behavior of the solution:

- The boundary source term produces incoming wave packets of amplitude \( \varepsilon^{\alpha-1} \).

- **Quadratic interaction** between two resonant incoming oscillations may produce an outgoing wave packet of amplitude \( \varepsilon^{2 \alpha-2} \).

- Amplified reflection of the outgoing wave packet produces incoming wave packets of amplitude \( \varepsilon^{2 \alpha-3} \).
If such an incoming + incoming → outgoing resonance occurs, \textbf{weakly nonlinear} geometric optics corresponds to $\alpha = 2$ for \textbf{semilinear} problems ! Rigorous justification in C.-Guès-Williams 2014.
If such an

\[\text{incoming } + \text{ incoming} \rightarrow \text{ outgoing}\]

resonance occurs, *weakly nonlinear* geometric optics corresponds to \(\alpha = 2\) for *semilinear* problems! Rigorous justification in C.-Guès-Williams 2014.

Same formal argument for *quasilinear* problems: if resonance occurs, weakly nonlinear geometric optics corresponds to \(\alpha = 3\) and \(O(\varepsilon^2)\) solutions. This is *much weaker* than the Artola-Majda-Rosales expansions.

**Partial conclusion**

Deal with Artola-Majda-Rosales type expansions in *nonresonant* cases (nonresonant wavetrains or pulses).
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We wish to construct highly oscillating solutions to the quasilinear problem:

\[
\begin{align*}
\partial_t u + \sum_{j=1}^{d} A_j(u) \partial_{x_j} u &= 0, & t \in [0, T], x \in \mathbb{R}^d, \\
B(u) u|_{x_d=0} &= \varepsilon^2 g \left( t, y, \frac{\varphi_0(t, y)}{\varepsilon} \right), & t \in [0, T], y \in \mathbb{R}^{d-1}, \\
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We wish to construct highly oscillating solutions to the **quasilinear** problem:

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\( u|_{t=0} = 0, \quad x \in \mathbb{R}^d. \)

**Assumptions**

We assume that the linearized problem at 0 belongs to the third class (WR), and

\[ \varphi_0(t, y) = \tau t + \eta \cdot y, \]

is a **critical** planar phase.

Furthermore, \( g \) is \( \Theta \)-periodic and has zero mean with respect to its last argument.
We formally expand the solution as:

\[ u_\varepsilon(t, x) \sim \sum_{n \geq 1} \varepsilon^n U_n \left( t, x, \frac{\Phi(t, x)}{\varepsilon} \right) , \]

with

\[ \Phi = (\varphi_1, \ldots, \varphi_M), \]

denotes the collection of phases with trace \( \varphi_0 \) on the boundary:

\[ \varphi_m(t, x) = \varphi_0(t, y) + \omega_m x_d . \]

Then, we compute...
Nonresonance assumption

If \( n \in \mathbb{N}^M \) has two nonzero coordinates, then \( n \cdot \Phi \) is not a characteristic phase.
Nonresonance assumption

If \( n \in \mathbb{N}^M \) has two nonzero coordinates, then \( n \cdot \Phi \) is not a characteristic phase.

For a strictly hyperbolic system, the leading profile reads:

\[
U_1(t, x, \theta_1, \ldots, \theta_M) = \sum_{m \text{ incoming}} \sigma_m(t, x, \theta_m) r_m,
\]

where the \( \sigma_m \)'s satisfy:

- \( \Theta \)-periodicity and zero mean with respect to \( \theta_m \),
- the traces of the \( \sigma_m \)'s are all proportional to a single function \( a(t, y, \theta) \),
- decoupled Burgers equations in \( \{x_d > 0\} \).
On the boundary \( \{ x_d = 0 \} \), the function \( a \) satisfies an equation of the form:

\[
\partial_t a + \mathbf{v} \cdot \nabla_y a + c_1 \partial_\theta (a^2) + c_2 \mathbb{F}_{\text{per}}(\partial_\theta a, \partial_\theta a) = G(t, y, \theta),
\]

where \( \mathbb{F}_{\text{per}} \) is a bilinear Fourier multiplier with respect to \( \theta \):

\[
\mathbb{F}_{\text{per}}(a, b)(\theta) := \sum_{k \in \mathbb{Z}^*} \left( \sum_{k_1 + k_2 = k \atop k_1, k_2 \neq 0} \frac{a_{k_1} b_{k_2}}{k_1 \Omega_1 + k_2 \Omega_2} \right) \, e^{2 i \pi k \theta/\Theta}.
\]
Well-posedness of the leading amplitude equation

It seems hard at this stage to avoid making a small divisor assumption...

**Assumption (control of small divisors)**

The numbers \((\Omega_1, \Omega_2)\) satisfy the estimate

\[
|k_1 \Omega_1 + k_2 \Omega_2| \geq \frac{c}{|k|^{\gamma}},
\]

for given \(c > 0\) and \(\gamma > 0\).
Then the term $F_{\text{per}}(\partial_\theta a, \partial_\theta a)$ acts as a **semilinearity**:

**Proposition**

For $k \in \mathbb{N}$ large enough, we have

$$
\|F_{\text{per}}(\partial_\theta a, \partial_\theta a)\|_{H^k} \lesssim \|a\|_{H^k}^2,
$$

and the leading amplitude equation is **locally in time solvable** in $H^k$.

This result completes the construction of the leading order term, and the correctors are constructed by solving analogous equations.
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We consider the same high frequency problem as before but now change the functional framework.

**Assumption**

The source term $g$ has **polynomial decay** at infinity with respect to its last argument.
The leading amplitude equation is derived by similar methods, and now reads as in the Majda-Rosales article:

\[
\partial_t a + \mathbf{v} \cdot \nabla_y a + c_1 \partial_\theta (a^2) + c_2 \mathbb{F}_{\text{pul}}(\partial_\theta a, \partial_\theta a) = G(t, y, \theta),
\]

with:

\[
\mathbb{F}_{\text{pul}}(a, b)(\theta) := \int_0^{+\infty} a(\theta + s \Omega_1) b(\theta + s \Omega_2) \, ds.
\]
New functional framework.

**Definition (weighted Sobolev spaces)**

\[ \Gamma^k(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) := \left\{ u \in L^2(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) \mid \theta^\alpha \partial^\beta u \in L^2(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) \right\} \]

\[ \alpha + |\beta| \leq k \]
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**Proposition**

For \( k \in \mathbb{N} \) large enough, we have

\[ \| F_{\text{pul}}(\partial_\theta a, \partial_\theta a) \|_{\Gamma^k} \lesssim \| a \|_{\Gamma^k}^2, \]

and the leading amplitude equation is **locally in time solvable** in \( \Gamma^k \).
Remarks

In the case of wavetrains, the construction of the leading profile requires an algebraic control of small divisors. This is in contrast with ”classical” weakly nonlinear geometric optics (Joly-Métivier-Rauch 1993, Williams 1999), where small divisors occur only in the construction of correctors to the leading profile.
Remarks

- In the case of wavetrains, the construction of the leading profile requires an algebraic control of *small divisors*. This is in contrast with "classical" weakly nonlinear geometric optics (Joly-Métivier-Rauch 1993, Williams 1999), where small divisors occur only in the construction of *correctors* to the leading profile.

- In the case of pulses, the small divisor assumption does not seem to appear any longer... However, the functional framework plays a key role...
In weighted Sobolev spaces, we have derived the bound:

\[ \| \mathcal{F}_{\text{pul}}(\partial_\theta a, \partial_\theta a) \|_{\Gamma^k} \lesssim \| a \|_{\Gamma^k}^2, \]

and this bound does not hold in standard Sobolev spaces.
Remarks (continued)

In weighted Sobolev spaces, we have derived the bound:

$$\| F_{\text{pul}}(\partial_\theta a, \partial_\theta a) \|_{\Gamma^k} \lesssim \| a \|^2_{\Gamma^k},$$

and this bound does not hold in standard Sobolev spaces.

As a matter of fact, for all integer $k \in \mathbb{N}$, there exists a bounded sequence $(a_n)$ in $H^k$ such that

$$\sup_n \left| \langle a_n, F(\partial_\theta a_n, \partial_\theta a_n) \rangle_{H^k} \right| = +\infty.$$  

This prevents from applying the energy method in $H^k$ (but does not prove ill-posedness in $H^k$...).

Same type of construction as in Benzoni-C.-Tzvetkov 2011.
We have a (formal) result that makes the link between the two quadratic nonlinearities $\mathbb{F}_{\text{per}}$ and $\mathbb{F}_{\text{pul}}$, as the period $\Theta$ tends to infinity. The argument is based on the Poisson summation formula ($f \in \mathcal{S}$):

$$\sum_{n \in \mathbb{Z}} f(x + n \Theta) = \frac{1}{\Theta} \sum_{k \in \mathbb{Z}} \hat{f} \left( \frac{2k \pi}{\Theta} \right) e^{2i \pi k x / \Theta}.$$
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Final main objective

Understand the stability of the constructed approximate solutions.

Wide open !!!