Mathematical models for microstructured optical fibre (MOF) fabrication

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The inverse problem in fibre drawing

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Microstructured optical fibres (MOFs)

the fibre draw process

preform drawn into fibre

outer boundary

cross-sectional area $S_0$

cross-plane

channels

feed speed $U_0$

neck-down length $L$

cross-sectional area $S_1$

draw speed $U_1$

the fibre draw process
**Fibre drawing**

a method for fabricating optical fibres from preforms

Diagram of fibre drawing:

- Preform fed into furnace, heated, drawn into a fibre, wound around spool.

Draw tower (IPAS)
Problem: fibre geometry differs from preform geometry
Geometrical cross section deforms under both draw tension and surface tension.

For a preform with circular channels, channels in the fibre are not circular.
Deformation is undesirable, difficult to quantify.

Aim: to understand, and predict, this deformation

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(Issa et al., Optics Letters, 2004)
The experiments

Controlled by experimenters

- preform radius \( R_0 \approx 10 \text{ mm} \)
- feed speed \( U_0 \approx 1 \text{ mm/min} \)
- draw speed \( U_1 \approx 1 \text{ m/min} \)
- temperature \( T \approx 1000^\circ \text{C} \)
- viscosity \( \mu \approx 10^5 \text{ Pa s} \)
- surface tension \( \gamma \approx 0.25 \text{ N/m} \)

Measured

- fibre radius \( R_1 \approx 0.1 \text{ mm} \)
- draw tension \( \sigma \approx 10 \text{ g} \)

- Often given \textit{draw ratio} \( D = U_1/U_0. \)
- \( \mu, \) but not \( \gamma, \) strongly depends on \( T. \)

Experimenters want to know:

- How to pick feed speed, draw speed, temperature?
- How to design preform?

Currently done by trial and error. Expensive. Wastes time and resources.
Forward and inverse problem

- **Forward problem**: For given preform geometry & draw parameters, what will the final fibre look like?
- **Inverse problem**: For desired fibre geometry, what preform geometry & draw parameters will produce it?

Outline of talk

- Fluid dynamics of fibre drawing
- Decoupling into “axial” and “cross-plane” problems (slender fibres)
- Complex variable formulation and conformal geometry
- Model for prediction of draw tension needed for desired fibre geometry
- "EPM" (elliptical pore model) for the inverse problem

The presentation here is based on work in [Stokes, Buchak, Crowdy & Ebendorff-Heidepriem, *J. Fluid Mech.*, (2014)]
Assuming *Newtonian fluid, low Reynolds number*,

Flow governed by Stokes equations,

$$
-\nabla p + \mu \nabla^2 \vec{u} = 0 \\
\nabla \cdot \vec{u} = 0
$$

with surface tension and imposed pressures on free boundaries,

$$
\sigma \cdot \hat{n} = -\gamma \kappa \hat{n} - p_k \hat{n} \\
\vec{x}_t \cdot \hat{n} = \vec{u} \cdot \hat{n}
$$

**Difficult 3D free boundary problem**
Numerical approach

Given its importance there have been several numerical attempts to study the problem. Among others:

- Xue, Tanner, Barton, Lwin, Large & Poladian, *J. Lightwave Tech.*, (2005a)
- Xue, Tanner, Barton, Lwin, Large & Poladian, *J. Lightwave Tech.*, (2005b)
- Xue, Tanner, Barton, Lwin, Large & Poladian, *J. Lightwave Tech.*, (2005c)

These are based on finite element methods, boundary integral techniques.

Inverse problem not easily studied this way
Slender fibres: asymptotic separation

Our approach is based on a “slender fibre” formulation from Cummings & Howell, JFM, 1999

▶ They studied drawing of a solid fibre (no “holes” or “channels”)
▶ Key insight: for slender fibres, fully 3D problem can be split into two simpler problems, a 1D stretching and a 2D sintering problem
“Axial” and “cross-plane” problems

For slender fibres where

\[ \epsilon = \frac{\sqrt{S_0}}{L} \ll 1 \]

3D problem splits into two coupled problems

2D cross-plane problem

Axial stretching problem

We have devised a way to exploit this decoupling for MOFs with any number of channels
Axial flow equations

The equations governing the steady axial flow problem are

\[ S_t + (uS)_x = 0 \]
\[ 3(Su_x)_x + \frac{1}{2}\gamma^* \sum_{n=0}^{M} (\Gamma_n)_x = \sum_{n=0}^{M} \pm (p_B^{(n)})_x A_n. \]

\( S(x, t) \) is the cross-sectional area of the fluid region

\( u(x, t) \) is the axial fluid velocity,

\( A_n(x, t), \Gamma_n(x, t), \) and \( p_B^{(n)}(x, t) \) are the area, perimeter, and pressure associated with the \( n \)-th channel.

(These equations encode steady mass and momentum balance)
Cross-plane equations

In each cross-plane the flow is governed by Stokes equations,

\[-\nabla p + \mu \nabla^2 \vec{u} = 0\]
\[\nabla \cdot \vec{u} = 0\]

with surface tension and imposed pressures on free boundaries,

\[\sigma \cdot \hat{n} = -\kappa \hat{n} - p_k \hat{n}\]
\[\vec{x}_\tau \cdot \hat{n} = \vec{u} \cdot \hat{n}\]

Crucial difference: notice the appearance of the reduced time

\[\tau = \gamma_* \int_t^0 \frac{dt}{\mu \sqrt{S}}\]

This is the classical 2D surface-tension driven Stokes flow problem, but now in this reduced time variable \(\tau\).
Important “base-line” case

An important “base-line” case to study has the following assumptions:

- zero Reynolds number
- absence of channel pressurizations
  \[ p_n(t) = 0, \quad \forall n \]
- constant viscosity (independent of temperature)
- no temperature, heat transfer effects

Once this problem is understood, add other effects later
Zero channel pressures: integration of axial problem

When all channel pressures vanish the axial flow problem admits an explicit integral [Stokes, Buchak, Crowdy & Ebendorff-Heidepriem, *J. Fluid Mech.*, (2014)]

Introduce

\[ H(\tau) = \int_0^\tau \tilde{\Gamma}(\tau') d\tau', \]

where \( \tilde{\Gamma}(\tau) \) is the total boundary perimeter in the cross-plane.

\( \tilde{\Gamma}(\tau) \) can be readily computed once the (unit area) cross-plane problem has been solved.

With \( H(\tau) \) determined from the cross-plane problem

\[ S(\tau) = \left( 1 - \frac{\sigma}{\gamma^*} \int_0^\tau H(\tau') d\tau' \right)^2 / H(\tau)^2, \]

\[ x(\tau) = -\frac{1}{\sigma} \log(H(\tau) \sqrt{S(\tau)}). \]

The parameter \( \sigma \) is the scaled fibre tension.
For zero channel pressurizations, with a given initial cross-plane geometry, the set of possible cross-plane geometry shapes is fully determined \textit{a priori} (from the 2D cross-plane solution).

What is not known yet is at what axial position $x$ and with what total cross-sectional area $S$ that cross-section will appear.

But these are given explicitly by the relations

\[
S(\tau) = \left(1 - \frac{\sigma}{\gamma^*} \int_0^\tau H(\tau') d\tau' \right)^2 / H(\tau)^2, \\
x(\tau) = -\frac{1}{\sigma} \log(H(\tau)\sqrt{S(\tau)}).
\]

where only $H(\tau)$ is “fed in” from the cross-plane solution.

The particular draw conditions are reflected in the $\gamma^*$ and $\sigma$ parameters.

The reduced time $\tau$ provides a natural parameter for all cross-plane geometries.
2D cross-plane free surface problem

Incompressible two-dimensional flow described by streamfunction $\psi$

$$\nabla^4 \psi = 0$$

Complex variable formulation. Let $z = x + iy$.

General biharmonic field given by

$$\psi = \text{Im}[zf(z, t) + g(z, t)]$$

where $f(z, t), g(z, t)$ are analytic in fluid region ("Goursat functions")

Physical variables given by

\[
\begin{align*}
\frac{p}{\mu} - i\omega &= 4f'(z, t) \\
u + iv &= -f(z, t) + zf'(z, t) + g'(z, t) \\
e_{11} + ie_{12} &= zf''(z, t) + g''(z, t)
\end{align*}
\]

$p$ is pressure, $\omega$ is vorticity, $e_{ij}$ is fluid rate-of-strain tensor
Velocity field is $(u, v)$
Stokes flow fundamental singularities

The well-known fundamental singularities of Stokes flow (e.g. stokeslets, stresslets, rotlets) now manifest themselves as isolated singularities of these two Goursat functions.

For example:

**Stresslet**: a stresslet singularity of strength \( \lambda \) at \( z_0 \) is equivalent to

\[
f(z) = \frac{\lambda}{z - z_0}, \quad g'(z) = \frac{\lambda z_0}{(z - z_0)^2}
\]

**Source/sink**: an irrotational source/sink of strength \( m \) at \( z_0 \) is given by

\[
f(z) = \text{locally analytic}, \quad g'(z) = \frac{m}{2\pi(z - z_0)}
\]
An important advantage for introducing a complex variable formulation is that the free surface stress condition can be integrated once (with respect to arclength $s$) yielding

$$f(z, t) + z f'(z, t) + g'(z, t) = -\frac{i\gamma}{2} \frac{dz}{ds}$$

$\gamma$ is surface tension

(This renders any numerical method less “stiff”.)
Advantage 2: can use conformal mapping technology

Time-dependent conformal maps can be used to track the evolving interfaces

\[ \rho < |\zeta| < 1 \]

Numerical methods for the cross-plane problem based on conformal mapping descriptions have been devised

[Buchak & Crowdy, *J. Comp. Phys*, (2014)]
Advantage 3: exact solutions derivable using complex analysis

Remarkably, a variety of exact solutions to the time-evolving 2D free surface evolution under surface tension are known.

Among others:

Case study: sintering annular array of $N$ discs

$\rho < |\zeta| < 1$

$N = 4$

Solutions for the evolution of the initial fluid domain on the right can be found in exact form
Exact solution: sintering annular array of $N$ discs

Conformal map, from the annulus $\rho < |\zeta| < 1$, describing the fluid domain evolution exactly is found to be

$$z(\zeta, t) = A(t)\zeta \frac{P_N(\zeta \rho^{2/N}/a(t), \rho(t))}{P_N(\zeta/a(t), \rho(t))}$$

where

$$P_N(\zeta, \rho) = \prod_{k=1}^{N} P(\zeta \omega_k, \rho), \quad \omega_k = e^{2\pi i (k-1)/N}$$

and

$$P(\zeta, \rho) \equiv (1 - \zeta) \prod_{n=1}^{\infty} (1 - \rho^{2n} \zeta)(1 - \rho^{2n}/\zeta)$$

Solution depends on just three time-evolving parameters $A(t), a(t)$ and $\rho(t)$
They satisfy a system of 3 nonlinear first-order ODEs


\textbf{Drawing} \( N = 4 \) \textit{circular fibres} \( \rightarrow \) \textit{single fibre}

This exact cross-plane solution can be combined with the axial equations to give the solution for the pulling of these four touching discs into a single fibre:

\[
\begin{align*}
\tau &= 0.00 \quad x = 0.00 \\
\tau &= 0.10 \quad x = 0.13 \\
\tau &= 0.20 \quad x = 0.34 \\
\tau &= 0.30 \quad x = 1.00
\end{align*}
\]

Just need evolution of total cross-plane perimeter (i.e. to compute \( H(\tau) \))

(Useful for separate application of “fibre optic couplers”)
The general model: back to axial flow equations

With $H(\tau)$ determined from the cross-plane problem we have

$$S(\tau) = \left(1 - \frac{\sigma}{\gamma^*} \int_0^\tau H(\tau') d\tau'\right)^2 / H(\tau)^2,$$

$$x(\tau) = -\frac{1}{\sigma} \log(H(\tau) \sqrt{S(\tau)}).$$

The condition that $x(\tau_L) = L$ (i.e. that required final geometry is attained) is equivalent to the following “$P - Q$ relation”:

$$\frac{1}{Q} + \frac{\log Q}{P} = 1$$

where

$$P = \frac{\gamma^*}{\int_0^{\tau_L} H(\tau') d\tau'}, \quad Q = \frac{\sqrt{D}}{H(\tau_L)}.$$
Balance of draw tension and surface tension

\[ \frac{1}{Q} + \log \frac{Q}{P} = 1 \]

where

\[ P = \frac{\gamma^*}{\int_0^{\tau_L} H(\tau') d\tau'}, \quad Q = \frac{\sqrt{D}}{H(\tau_L)}. \]

\( \tau_L \) decides the final geometry. This gives denominators in \( P \) and \( Q \). Cross-plane solution “fed in” via these denominators only.
Our model predicts required draw tension

Given initial geometry, desired final geometry (prescribed by $\tau_L$), and surface tension (generally constant over the range of experimental draw parameters) the key relation

$$\frac{1}{Q} + \log \frac{Q}{\mathcal{P}} = 1$$

where

$$\mathcal{P} = \frac{\gamma^*}{\int_0^{\tau_L} H(\tau') d\tau'}, \quad Q = \frac{\sqrt{D}}{H(\tau_L)}.$$ 

tells us the draw tension required to attain the required final geometry.

Temperature is key control parameter that must be used to control fluid viscosity, and hence draw tension.

Important to be able to accurately monitor tension during the draw

This has been a crucial new insight for the experimentalists.
What about the inverse problem?

Earlier exact solutions, with \( N \to \infty \), highlight a difficulty for “inverse problem”:

\[
\begin{array}{c}
N = 20 \text{ solutions} \\
[\text{Crowdy, (2003)}]\end{array}
\]

Different initial conditions can give indistinguishable profiles in forward time
(physically, surface tension “irons out” ripples)
Simply “running backwards in time” will be an unstable process
Exact solution for compressible **elliptical** bubble in linear ambient

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**Key fact:** *an initially elliptical bubble in linear Stokes flow remains elliptical as it evolves*

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The ellipse describable by the time-evolving conformal mapping

\[
z(\zeta, t) = z_0(t) + \alpha(t) \frac{\zeta}{\zeta} + \beta(t) \zeta
\]

with ODEs for \(\alpha(t)\) and \(\beta(t)\) known from [DGC, J. Fluid Mech., 476, 345-356, (2003)]
The “elliptical pore model” (EPM)

- For inverse problem we need a constrained way to run the simulation backwards in time (i.e. to “filter out” ripples, or follow “slow manifold”)

Crowdy, *J. Fluid Mech.*, [2004] has proposed an “elliptical pore model” for multi-pore interactions (based on exact elliptical solutions just given)

- “Outer”: each elliptical channel is a point stresslet + point source/sink;
  “Inner”: evolution of ellipse given by previous exact solution in linear flow generated by superposition of singularities due to the other channels
“Outer solution”: fundamental singularities

The well-known fundamental singularities of Stokes flow (e.g. stokeslets, stresslets, rotlets) now manifest themselves as isolated singularities of these two Goursat functions.

The two used to describe each elliptical channel “in its far-field” are:

**Stresslet**: a stresslet singularity of strength $\lambda$ at $z_0$ is equivalent to

$$f(z) = \frac{\lambda}{z - z_0}, \quad g'(z) = \frac{\lambda z_0}{(z - z_0)^2}$$

**Source/sink**: an irrotational source/sink of strength $m$ at $z_0$ is given by

$$f(z) = \text{locally analytic}, \quad g'(z) = \frac{m}{2\pi(z - z_0)}$$
The ODEs for the EPM

- Model gives system of ordinary differential equations dictating how channels move & evolve.

\[
\frac{dZ_n}{d\tau} = - \sum_{j \neq n} \frac{\mu_j(Z_n - Z_j)}{2\pi(Z_n - Z_j)^2} + \sum_{j \neq n} \frac{m_j}{2\pi(Z_n - Z_j)} - \sum_{j \neq n} \frac{\mu_j}{2\pi(Z_n - Z_j)}
\]

\[
\frac{d\alpha_n}{d\tau} = -\alpha I_n(0) - \frac{1}{2}\alpha(p_n - p_B^{(n)})
\]

\[
\frac{d\beta_n}{d\tau} = -\beta I_n(0) + \frac{1}{2}\beta(p_n - p_B^{(n)}) + 2k_n\alpha_n + i\omega_n\beta_n
\]

\[
\frac{dR}{d\tau} = \frac{M}{2\pi R} \quad \leftarrow \text{assume outer boundary circular}
\]

- Computationally inexpensive – short Matlab script is available on web.
- Much more computational efficient than full numerical simulation
- It can be run forwards or backwards in a stable manner
- It is a “rational” exclusion of extraneous growth of unwanted modes
Testing the accuracy of EPM

How does EPM predictions compare with full numerical solution?

(solid – model; dashed – numerical solution)

Excellent agreement with numerical solution.
Testing the accuracy of EPM

(solid – model; dashed – numerical solution)

Model is computationally inexpensive, even for high connectivity geometries.
EPM can help solve the inverse problem (stably)

- Easy to evolve fibre configuration backwards in time to get preform.

How to design preforms?

In the following examples, we pick a fibre geometry and show possible preforms, each with required parameters (inverse problem).

We take:

- \( L = 5 \text{ cm} \)
- \( \mu = 5 \times 10^5 \text{ Pa s} \)
- \( \gamma = 0.25 \text{ N/m} \)
- \( R_0 = 10 \text{ mm} \)
- \( R_1 = 0.5 \text{ mm} \)
EPM predictions I

Fiber configuration

\[ \tau_1 = 0.001 \quad D = 399 \quad U_0 = 45.8 \text{ mm/min} \]
\[ \tau_1 = 0.012 \quad D = 386 \quad U_0 = 2.3 \text{ mm/min} \]
\[ \tau_1 = 0.024 \quad D = 368 \quad U_0 = 1.2 \text{ mm/min} \]

preform configurations, with draw parameters
draw

Allowing longer deformation time \( \tau_1 \) lets feed speed \( U_0 \) be lower.
EPM predictions II

Fiber configuration

\[ \tau_1 = 0.003 \text{ D}=393 \text{ U}_0 = 8.4 \text{ mm/min} \]
\[ \tau_1 = 0.007 \text{ D}=385 \text{ U}_0 = 4.2 \text{ mm/min} \]
\[ \tau_1 = 0.014 \text{ D}=369 \text{ U}_0 = 2.2 \text{ mm/min} \]
\[ \tau_1 = 0.021 \text{ D}=352 \text{ U}_0 = 1.5 \text{ mm/min} \]
\[ \tau_1 = 0.028 \text{ D}=333 \text{ U}_0 = 1.2 \text{ mm/min} \]
\[ \tau_1 = 0.035 \text{ D}=313 \text{ U}_0 = 1.0 \text{ mm/min} \]

preform configurations, with draw parameters

Tradeoff between circular preform (easy to make) and low feed speed (easy to draw).
Summary

- New model, for slender fibres, showing the importance of draw tension
- New numerical schemes, based on conformal geometry, to solve forward problem for any geometry
- Approximate EPM to solve forward or inverse problem for fibres with elliptical channels

Current work

- Experimental validation (in progress)
- Generalize to allow temperature-dependent viscosity
- Generalize the EPM to other channel shape classes

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