Boundary layers for non-linear flows in pipes and channels

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Vorticity concentration
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- Michael Taylor, UNC-Chapel Hill
Fluid equations

Viscous fluid flow modeled by the *Navier-Stokes* equations

\[
\begin{align*}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu &= \nu \Delta u^\nu - \nabla p^\nu + f, & q \in \Omega, \ t \in (0, T), \\
\text{div} \ u^\nu &= 0, & q \in \Omega, \ t \in (0, T),
\end{align*}
\]

where \( p^\nu \) pressure, \( u^\nu \) velocity, \( \nu \) viscosity coeff., \( f \) forcing.

\( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \) with unit normal \( n \).

Impose no-slip boundary conditions
\( u^\nu(t, q) = \nu^B(t, q), \ q \in \partial \Omega, \ \nu^B(t, q) \) a vector field tangent to \( \partial \Omega \)
\( \Rightarrow \) force flow at boundary, even in \( f \equiv 0 \).

Impose initial conditions \( u(0, q) = u_0(q), \ q \in \Omega \). Only normal component of \( u_0 \) required to vanish at \( \partial \Omega \) (rigid boundary).
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\(\text{(NS)}\)

where \(p^\nu\) pressure, \(u^\nu\) velocity, \(\nu\) viscosity coeff., \(f\) forcing. 

\(\Omega\) is a smooth bounded domain in \(\mathbb{R}^3\) with unit normal \(n\).

Impose no-slip boundary conditions 

\(u^\nu(t, q) = v^B(t, q), \ q \in \partial \Omega, \ v^B(t, q)\) a vector field tangent to \(\partial \Omega\) 

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Impose initial conditions \( u(0, q) = u_0(q), \ q \in \Omega \). Only normal component of \( u_0 \) required to vanish at \( \partial \Omega \) (rigid boundary).
Assume $u^0 := \lim_{\nu \to 0+} u^\nu$

Then $u^0$ should be a weak solution of the Euler equations, modeling inviscid fluid flow:

\[
\begin{align*}
\partial_t u^0 + u^0 \cdot \nabla u^0 &= -\nabla p^0 + f, & q \in \Omega, \ t \in (0, T), \\
\text{div} \ u^0 &= 0, & q \in \Omega, \ t \in (0, T),
\end{align*}
\]

(E)

$u$ must satisfy only slip boundary conditions $u \cdot n = 0$, $q \in \partial \Omega$, where $n$ is the unit outer normal to $\partial \Omega$.

Impose same initial condition $u(0, q) = u_0(q)$, $q \in \Omega$ (this can be easily relaxed, e.g. $u^\nu(0) \to u(0)$).
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Zero-viscosity limit

Say that the vanishing viscosity limit holds, if $u^\nu$ converges to $u^0$ in the energy norm, i.e.,:

$$u^\nu \rightarrow u^0 \text{ strongly in } L^\infty([0, T], L^2(\Omega)).$$

No general convergence results even for 2D flows in a disk.

Converge of $u^\nu$ to $u$ cannot be uniform in $\Omega$ due to different boundary conditions $\Rightarrow$ a boundary layer appears where flow is potentially violent.

Vorticity $\omega = \text{curl } u$ is formed in the layer that may propagate in the bulk (boundary layer separation)
$\Rightarrow$ Von Kármán vortices shed in the wake.

Flow reversal may occur due to an adverse pressure gradient.
Zero-viscosity limit

Say that the **vanishing viscosity limit** holds, if \( u^{\nu} \) converges to \( u^0 \) in the energy norm, i.e.,:

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\lim_{\nu \to 0^+} u^{\nu} \to u^0 \quad \text{strongly in } L^\infty([0, T], L^2(\Omega)).
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Von Kármán vortices shed in the atmosphere by Guadalupe Island (©NASA).

Experiment, laminar boundary layer separation (An Album of Fluid Motion, Van Dyke, Parabolic Press)
Assume flow occupy half-plane \( \{ x \in \mathbb{R}, y > 0 \} \).

If \( \mathbf{u}^\nu = (u^\nu, v^\nu) \) is the NS solution, assume:

\[
\begin{align*}
  v^\nu(x, y, t) & \approx v^0(x, y, t) + \sqrt{\nu} v(x, y/\sqrt{\nu}, t), \\
  u^\nu(x, y, t) & \approx u^0(x, y, t) + u(x, y/\sqrt{\nu}, t).
\end{align*}
\]

in a layer of thickness \( \sqrt{\nu} \Rightarrow \)

\( u^0, v^0 \) satisfy Euler with slip b.c. and pressure \( p_0 \), \( u \) and \( v \) solve Prandtl equations:

\[
\begin{align*}
  u_t + uu_x + vv_y &= u_{yy} + p_x^0, \\
  u_x + v_y &= 0,
\end{align*}
\]

with boundary conditions:

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\begin{align*}
  \lim_{y \to \infty} u(x, y, t) &= u^0(x, t), \\
  u(x, 0, t) &= v(x, 0, t) = 0,
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Prandtl equations are not well-posed in Sobolev spaces without extra assumptions (Gérard-Varet & Dormy ’10, Gérard-Varet & Nguyen‘10, Guo & Nguyen ’11).

Ill-posedness given by instability for the linearized equation around certain shear flows, and lack of Lipschitz continuity of non-linear flow.

Prandtl blow up in finite time (Van Dommelen & Shen ‘77, E & Enquist ‘97) for certain smooth, compactly supported initial conditions.

Solutions with complex singularities (Gargano-Sammartino-Sciacca ‘10).

Form of $\nu$-expansion not generally valid in $L^\infty H^1$ (Grenier ‘00).
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Convergence for generic flows

The zero viscosity limit holds for $\mathbb{R}^n$ in Sobolev and Besov norms (Swann 71, Kato 74, Constantin-Wu 95/96, Masmoudi 07, Kelliher-Cozzi 07, Cozzi 13).

Convergence holds for less singular free b. c (Lions ’69).

$$u^\nu \cdot n = 0, \quad \omega^\nu = 0 \quad \text{on } \partial \Omega \quad (2D),$$

$n$ normal unit vector to $\partial \Omega$, and for Navier friction b. c.

$$u \cdot n = 0, \quad [\text{Def}(u^\nu)n] \cdot \tau = -\alpha u^\nu \cdot \tau \quad \text{on } \partial \Omega,$$

with $\alpha$ friction coefficient, $\tau$ unit tangent vector to $\partial \Omega$ (Clopeau-Mikelic-Robert 98, Lopes-Planas 05, Iftimie-Planas 06, Rusin 06, Kelliher 06, Xiao-Xin 07, Beirão da Veiga-Crispo 09, Berselli-Spirito 10, Masmoudi-Rousset 11, Kelliher-Gie 12).
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Criteria for convergence

Kato’s criterion (84): Convergence holds \( \Leftrightarrow \) rate of energy dissipation in layer \( \sim \nu \) vanishes as \( \nu \to 0 \):

\[
\nu \to u \text{ in } C([0, T], L^2(\Omega)) \Leftrightarrow \nu \int_0^T \| \nabla u_\nu \|^2_{\{1-c\nu <|x|<1\}} \, dt \to 0
\]

Enough that tangential dissipation vanishes (Temam-Wang 97, Wang 01), but in a thicker layer \( \delta(\nu)/\nu = o(1) \):

\[
\nu \int_0^{T'} \| \nabla_T u_\nu \|^2_{\{1-\delta(\nu)<|x|<1\}} \, dt \to 0, \quad \nu \to 0,
\]

Here \( \nu^B \neq 0 \), regular in time, \( \nu^B \in H^{1/2}([0, T]) \).

(Kato-type criterion for the Ekman layer in rotating, stratified fluids, Grenier-Masmoudi 97).
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Prandtl theory and boundary layer analysis

- **analytic or monotonic** data, special geometry (half plane) (Oleinik 60’s, Asano ’88, Caflisch-Sammartino ’96-’01, Lombardo-Cannone-Sammartino, Kukavica-Vicol ’13 analytic in tangential direction, Gerard-Varé - Masmoudi ’13 Gevrey class, Kukavica-Masmoudi-Vicol-Wong ‘14 multiple monotonic regions, Constantin-Kukavica-Vicol ‘14 no back flow + lower vorticity bound);
- **linearized** equations, special geometry (channel, exterior disk) (Temam-Wang ’96, Lombardo-Sammartino ’01, Lombardo-Caflish-Sammartino ’01);
- **non-characteristic data**, e.g. injection-suction, or favorable pressure gradient (Temam-Wang ’96,Xin-Zhang ’04);
- half plane, **vorticity away** from boundary (Maekawa ’12);
Taylor-Couette-type flows

Consider special classes of flows in 2D or 3D, where symmetry depletes nonlinearity and control gradient of pressure. Flows are laminar, no boundary layer separation.

Boundary layer can be studied for:
- Plane-parallel flows in periodic channels;
- certain parallel flows in circular, straight, periodic pipes.

When boundary data regular enough, these flows satisfy Kato’s criterion (in Temam-Wang’s formulation).

Consider rigid boundary motions that may be irregular in time.
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Consider rigid boundary motions that may be irregular in time.
Consider plane parallel channel flow with horizontal periodic boundary conditions (DiPerna-Majda 84, Wang 01). Periodicity ensures uniqueness of the flow.

Velocity field has form

\[ u^\nu(t, x, y, z) = (v^\nu(t, z), w^\nu(t, x, z), 0) \]

in \( \mathcal{O} = \{(x, y, z) \mid x, y \in \mathbb{R}/\mathbb{Z}, z \in [0, 1]\} \).

Flow is automatically divergence free, but not planar.

Take regular initial data: \( u_0(x, y, z) = (V(z), W(x, z), 0) \).
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3D Navier-Stokes reduces to 2D weakly non-linear system:

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\begin{align*}
    \partial_t v^\nu &= \nu \partial_z^2 v^\nu, \\
    \partial_t w^\nu + v^\nu \partial_x w^\nu &= \nu (\partial_x^2 w^\nu + \partial_z^2 w^\nu),
\end{align*}
\]

No pressure gradient, advection term already divergence free.

\( u_0 \) does not satisfy b. c. \( \Rightarrow \) initial layer for NS.

Can allow for rigid motions of walls \( (\nu^B \neq 0) \).

For simplicity, set \( u^\nu|_{\partial \Omega} = 0, t > 0 \).
Fluid equations revisited

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\partial_t v &= 0, \\
\partial_t w + v \partial_x w &= 0,
\end{aligned}
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\(x \in \mathbb{R}/\mathbb{Z}, \ z \in [0, 1], \ t > 0\)

with \(v(0) = V, \ w(0) = W\), and no boundary conditions.

Explicit solution given by:

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\begin{aligned}
v(t, x, z) &= V(z), \\
w(t, x, z) &= W(x - tV(z), z) = e^{-tX} W(x, z).
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where \(X = V(z) \partial_x\) vector field with domain \(H^1(\Omega)\).

\(\Rightarrow\) vanishing viscosity limit is \(v^\nu \to V, \ w^\nu \to e^{-tX} W\) as \(\nu \to 0\).
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\( \Rightarrow \) vanishing viscosity limit is \( v^\nu \to V, \ w^\nu \to e^{-tX}W \) as \( \nu \to 0. \)
Write Navier-Stokes in mild form, using Duhamel’s principle:

\[ v^\nu = e^{\nu t} \partial_z^2 V \]

\[ w^\nu = e^{t(\nu \Delta - X)} W + \int_0^t e^{(t-s)(\nu \Delta - X)} [(V - v^\nu) \partial_x w^\nu] \, ds. \]

Write \((w^\nu - w)(t) = [e^{t(\nu \Delta - X)} - e^{-t X}] W + R^\nu(t)\), with

\[ R^\nu(t, x, z) := \int_0^t e^{(t-s)(\nu \Delta - X)} [(V(z) - v^\nu(s, z)) \partial_x w^\nu(s, x, z)] \, ds. \]

Remainder \(R^\nu\) less singular as \(\nu \rightarrow 0\) due to good convergence \(v^\nu \rightarrow V\) (heat semigroup estimates).
**Equivalent NS formulations**

Write Navier-Stokes in mild form, using Duhamel’s principle:

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Conormal Sobolev spaces

For $k \in \mathbb{Z}_+$, define

$$\mathcal{V}^k(O) = \{ f \in L^2(O) : Y_1 \cdots Y_j f \in L^2(O), \ \forall j \leq k \},$$

where $Y_j$ tangent to $\partial O$. $\mathcal{V}^k$ conormal or b-Sobolev space (Melrose). Cf. work of Masmoudi-Rousset.

Energy estimates $\Rightarrow$ $e^{t(\nu \Delta - X)} f \to e^{-tX} f$ strongly in $\mathcal{V}^k(O)$, $\forall k \in \mathbb{Z}_+$ uniformly in $t > 0$.

Since $\partial_x^k w^\nu$ solves same equation as $w^\nu$,

$$u^\nu \to u \text{ in } L^\infty([0, T], \mathcal{V}^k), \forall k \in \mathbb{Z}_+.$$
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No boundary layer separation.
Boundary layer analysis

Behavior of $\nu^\nu$ in boundary layer by the method of images.

Let $\delta(z) = \max(|z - 1|, |z|)$. Then, for $0 \leq t \leq T$,

$$|\nu^\nu(z, t) - V(z) \leq C(T) \sup_{0 \leq s \leq t} \psi((s\nu)^{-1/2}\delta(z)),$$

where $\psi(s)$ denote a function that vanishes at 0 and satisfies as $s \to \infty$: $\psi(s) \lesssim (1 + s)^{-1/2}$. \Rightarrow

$$\nu^\nu(z, t) \longrightarrow V(z) \longrightarrow 0,$$

uniformly in $t \in [0, T]$, for $\delta(z) \geq \eta(\nu)$, where $\sqrt{\nu}/\eta(\nu) = o(1)$.

Boundary layer has thickness $O(\sqrt{\nu})$. 
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Boundary layer for $w^\nu$

Convergence of $\nu^\nu \rightarrow V$ implies that $R^\nu(x, z, t) \rightarrow 0$ outside the same layer of thickness $\sim \sqrt{\nu}$.

Next study limit $e^{t(\nu \Delta - X)} W \rightarrow e^{-tX} W$. Recall $X = V(z) \partial_x$.

Limit is more singular than semiclassical, as $X$ is first order.

Observe that $g^\nu = e^{tX} e^{t(\nu \Delta - X)} W$ solves:

$$\partial_t g^\nu = L_\nu g^\nu, \quad g^\nu(0) = W, \quad g_\nu = 0 \text{ on } \partial \Omega \times \mathbb{R}_+,$$

where $L_\nu = e^{tX} \nu \Delta e^{-tX}$ smooth family of strongly elliptic operators $\Rightarrow$ study boundary layer using a double layer potential.
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Layer Potentials

Extend \( X \) and \( W \) smoothly to the whole space (periodic in \( x \)).

Compare \( g_\nu \) to \( G_\nu - V_\nu \),

where \( G_\nu \) is the “whole-space” solution with periodic b.c. in \( x \) and initial condition \( W \).

\( V_\nu \) is the “boundary correction”, solving:

\[
\begin{cases}
    \partial_t V_\nu = L_\nu V_\nu, & \text{on } \mathbb{R} \times \partial \mathcal{O}, \\
    V_\nu = \chi_{\mathbb{R}^+}(t)G_\nu, & \text{on } \mathbb{R} \times \partial \mathcal{O} \\
    V_\nu = 0, & \text{on } (-\infty, 0) \times \mathcal{O}.
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Both \( g_\nu \) and \( G_\nu - V_\nu \) satisfy homogeneous boundary cond.

Construct \( V_\nu \) approximately via layer potentials.
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Both \( g_\nu \) and \( G_\nu - V_\nu \) satisfy homogeneous boundary cond.

Construct \( V_\nu \) approximately via layer potentials.
Let $g_s$ be pull-back of Euclidean metric on $\mathcal{O}$ by the flow of $X = V(z)\partial_x$ to time $s$. Set $dS_s(y), \partial/\partial n_{s,y}$ the area element and outer normal to $\partial \mathcal{O}$ in this metric.

Define, for $t \in \mathbb{R}, x \in \mathcal{O}$,

$$D_\nu h(t, x) = \nu \int_0^\infty \int_{\partial \mathcal{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(\nu, t, s, x, y) dS_s(y) \, ds,$$

where "whole-space" parametrix $H$ constructed recursively using symbol calculus. Note $H(1, t, s, x, y) = \nu H(\nu, t, s, x, y)$.

Then $V^\nu = D_\nu \Phi^\nu$, if $\Phi$ satisfies on the boundary:

$$\left(\frac{1}{2} I + \nu N_\nu\right) \Phi^\nu = \chi_{\mathbb{R}^+}(t) G^\nu,$$

with $N_\nu$ the formal limit of $D_\nu$ to the boundary.
Layer potential cont.

Let $g_s$ be pull-back of Euclidean metric on $O$ by the flow of $X = V(z)\partial_x$ to time $s$. Set $dS_s(y), \partial/\partial n_{s,y}$ the area element and outer normal to $\partial O$ in this metric.

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Layer potential cont.

Enough to consider leading term $H_0$ in the parametrix (in local coordinates):

$$H_0(1, t, s, x, y) \sim \frac{1}{4\pi t} \text{Det}(g_s(y))^{-1/2} \text{Det}(G(t, s, x, y))^{1/2} \exp(-(x - y)^T G(t, s, x, y)(x - y))/4(t - s), \quad t > s > 0$$

where $G^{-1}(t, s, x, y) = \frac{1}{t - s} \int_s^t g_\tau^{-1}(x, y) d\tau$.

Let $D^0_\nu$ be the associated double layer potential.

Pseudodifferential estimates ($N_\nu$ has symbol of order $-1/2$) give that all other terms are smaller and smoother.

Also can solve the boundary equation recursively to infinite order. In fact, here $\phi_\nu \approx 2\chi_{\mathbb{R}^+}(t)G_\nu$, remainder already $O(\sqrt{\nu})$. 

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Boundary layers
Layer potential cont.

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Layer potential analysis gives:

\[ \| g_\nu - (G_\nu - 2D_\nu^0(\chi_{\mathbb{R}^+} + t) G_\nu) \|_{L^\infty([0,T] \times \partial \mathcal{O})} \leq C \nu^{1/2} \| G_\nu \|_{L^\infty([0,T] \times \partial \mathcal{O})} \]

Recall \( e^{t(\nu \Delta - X)} W = e^{-tX} g_\nu \). Use the maximum principle to relate \( G_\nu \) to \( W = G_\nu^{t=0} \):

\[ \|(e^{t(\nu \Delta - X)} W - e^{-tX} W) - 2D_\nu^0(\chi_{\mathbb{R}^+} + t) W\|_{L^\infty([0,T] \times \mathcal{O})} \leq C(T) \nu^{1/2} \| W \|_{L^\infty(\overline{\mathcal{O}})} \]

Interpret \( 2D_\nu^0 \chi_{\mathbb{R}^+} (t) W \) as the zero-order *corrector* to the inviscid flow (cf. with Prandtl theory).
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Interpret \( 2D^0_\nu \chi_{\mathbb{R}}+ (t) W \) as the zero-order \textit{corrector} to the inviscid flow (cf. with Prandtl theory).
Consider parallel pipe flow with periodic boundary cond. along pipe axis (Wang 01).

Again, periodicity ensures uniqueness of the flow.

Velocity field has form (note flow is not planar):

\[
u^\nu = u^\nu_\phi(t, r)e_\phi + u^\nu_x(t, r, \phi)e_x\]

in \(Q = \Omega \times \mathbb{R} \setminus L\mathbb{Z}\), \(\Omega\) unit disk. \((r, \phi)\) polar coordinates in \(\Omega\).

Similar ansatz imposed on the Euler velocity \(u\).

This form preserved by Navier-Stokes and Euler evolution (if \(f\) suitable).
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3D Navier-Stokes reduces to 2D weakly non-linear system:

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\partial_t u^\nu_\phi = \frac{\nu}{r} \partial_r (r \partial_r u^\nu_\phi) - \frac{\nu}{r^2} u^\nu_\phi + f_1(t, r),
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\[
\partial_t u^\nu_x + \frac{u^\nu_\phi}{r} \partial_\phi u^\nu_x = \frac{\nu}{r} \partial_r (r \partial_r u^\nu_x) + \frac{\nu}{r^2} \partial_{\phi\phi} u^\nu_x + f_2(t, r, \phi),
\]

Horizontal component \( v^\nu = u_\phi e_\phi \) is a 2D circularly symmetric flow evolving independently.

Force flow by moving walls:

\[
 u^\nu|_{r=1} = v^B = \beta := \beta_\phi(t)e_\phi + \beta_x(t, \phi)e_x.
\]

Take regular initial data: \( u^\nu|_{t=0} = u_0 := a(r)e_\phi + b(r, \phi)e_x. \)
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3D Euler reduces to 2D weakly non-linear system:

\[
\begin{align*}
\partial_t u_0^0 &= f_1, \\
\partial_t u_x^0 + \frac{u_0^0}{r} \partial_\phi u_x^0 &= f_2,
\end{align*}
\]

with same initial condition \( u_0 \) and no boundary conditions.

Pressure \( p^\nu = p^\nu(t, r) \) determined (up to a constant) from:

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-(u_\phi^\nu)^2 + r \partial_r p^\nu = 0,
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for both NS and Euler.
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Flow in the boundary layer can be rigorously modeled in some cases using a corrector $\theta^{\nu}$ to the Euler flow near walls.

$\theta^{\nu}$ approximates $u^{\nu} - u^{0}$ (Vishik-Lyusternik 57, Lions73).

$\Rightarrow u^{0} + \theta^{\nu}$ approximate NS solution.

For pipe flow postulate:

$$u^{\text{app}}(t, r, \phi) = u^{\text{ou}}(t, r, \phi) + \theta^{0}(t, \frac{1 - r}{\sqrt{\nu}}, \phi),$$

$$p^{\nu}(t, r, \phi) \approx p^{\text{ou}}(t, r) + q^{0}(t, \frac{1 - r}{\sqrt{\nu}}).$$

$u^{\text{ou}}$ is the outer solution, valid away from walls. Outer solution is Euler solution $u^{0}$ at zero order. Similarly for the pressure.

$\theta^{0}, q^{0}$ are the zero-order correctors. Can take $q^{0} \equiv 0$. 
Flow correctors

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\( \theta^0, q^0 \) are the zero-order correctors. Can take \( q^0 \equiv 0. \)
Derive effective equation for the corrector from NS. Show convergence of approx solution to NS solution.

Must impose **compatibility conditions** on the data. At zero order:

\[(u_0)_\phi(1) = \beta_\phi(0, 1), \quad (u_0)_x(1, \phi) = \beta_x(0, \phi),\]

where \(\beta\) is velocity at wall.

These conditions prevent formation of initial layer at \(t = 0\).

Next, introduce stretch variable:

\[Z = \frac{1 - r}{\sqrt{\nu}}.\]

In \(Z\) domain **unbounded** \([0, +\infty) \Rightarrow \) use radial cut-off \(\rho(r)\).

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NS and E are weakly non-linear equations, Euler velocity will affect profile of corrector near boundary.

Taylor expand all functions that are not boundary layer functions (e.g., outer solution) in $Z$ around $Z = 0$ \Rightarrow

Corrector satisfies:

$$
\partial_t \theta^0_\phi - \partial ZZ \theta^0_\phi = 0,
$$

$$
\partial_t \theta^0_x + \theta^0_\phi \partial_\phi u^0_x(t, 1, \phi) + \theta^0_\phi \partial_\phi \theta^0_x + u^0_\phi(t, 1) \partial_\phi \theta^0_x = \partial ZZ \theta^0_x,
$$

$$(\theta^0_\phi, \theta^0_x)|_{Z=0} = (\beta_\phi(t) - u^0_\phi(t, 1), \beta_\phi(t) - u^0_x(t, 1, \phi)),
$$

$$(\theta^0_\phi, \theta^0_x)|_{Z=\infty} = 0, (\theta^0_\phi, \theta^0_x)|_{t=0} = (0, 0).
$$

Approximate Navier-Stokes solution

$$
u^{\text{app}}(t, r, \phi) = u^0(t, r, \phi) + \rho(r) \theta^0(t, \frac{1 - r}{\sqrt{\nu}}, \phi)
$$

then satisfies NS with extra small forcing.
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$$\partial_t \theta^0_\phi - \partial_{ZZ} \theta^0_\phi = 0,$$

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$$(\theta^0_\phi, \theta^0_x)|_{Z=0} = (\beta_\phi(t) - u^0_\phi(t, 1), \beta_x(t, \phi) - u^0_x(t, 1, \phi)),$$

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$$\partial_t \theta_\phi^0 - \partial ZZ \theta_\phi^0 = 0,$$

$$\partial_t \theta_x^0 + \theta_\phi^0 \partial_\phi u_x^0(t, 1, \phi) + \theta_\phi^0 \partial_\phi \theta_x^0 + u_\phi^0(t, 1) \partial_\phi \theta_x^0 = \partial ZZ \theta_x^0,$$

$$(\theta_\phi^0, \theta_x^0)|_{Z=0} = (\beta_\phi(t) - u_\phi^0(t, 1), \beta_x(t, \phi) - u_x^0(t, 1, \phi)),$$

$$(\theta_\phi^0, \theta_x^0)|_{Z=\infty} = 0, (\theta_\phi^0, \theta_x^0)|_{t=0} = (0, 0).$$

Approximate Navier-Stokes solution

$$u^{\text{app}}(t, r, \phi) = u^0(t, r, \phi) + \rho(r) \theta^0(t, \frac{1 - r}{\sqrt{\nu}}, \phi)$$

then satisfies NS with extra small forcing.
Higher order compatibility

Given inductively $p$-Cauchy data for NS, the $m$-th order compatibility conditions:

$$\partial_t^p u^\nu |_{r=1, t=0} = \partial_t^p \beta^\nu |_{r=1, t=0}, \quad p = 1, \ldots, m.$$  

At first-order, $m = 1$:

$$\partial_t \beta_\phi(0) = \nu (a'(1) + a''(1) - a(1)) + f_1(0, 1),$$

$$\partial_t \beta_x(0, \phi) = \nu (\partial_r b(1, \phi) + \partial_{rr} b(1, \phi)$$

$$+ \partial_{\phi \phi} b(1, \phi)) - a(1) \partial_\phi b(1, \phi) + f_2(0, 1, \phi).$$

$\Rightarrow \nu$-independent initial data must have given profile at the boundary:

$$\partial_r b(1, \phi) + \partial_{rr} b(1, \phi) + \partial_{\phi \phi} b(1, \phi) = 0,$$

$$\partial_t \beta_x(0, \phi) = -a(1) \partial_\phi b(1, \phi) + f_2(0, 1, \phi),$$

Obtain higher regularity and decay for the correctors.
Main results

Theorem

If $u_0 \in H^m(\Omega)$, $\beta \in H^2(0, T; H^m(\Omega))$, $f \in L^2(0, T; H^m)$, $m > 4$, satisfying the $[m/2] - 1$-order compatibility:

\[
\left\| u^\nu - \tilde{u}^{app} \right\|_{L^\infty(0, T; L^2(\Omega))} \leq c \nu^{\frac{3}{4}},
\]
\[
\left\| u^\nu - \tilde{u}^{app} \right\|_{L^2(0, T; H^1(\Omega))} \leq c \nu^{\frac{1}{4}},
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\[
\left\| u^\nu - \tilde{u}^{app} \right\|_{L^\infty(0, T; H^1(\Omega))} \leq c \nu^{\frac{1}{4}},
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\[
\left\| p^\nu - p^0 \right\|_{L^\infty(\Omega \times [0, T])} \leq c \nu^{\frac{1}{2}},
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\left\| p^\nu - p^0 \right\|_{L^\infty(0, T; H^1(\Omega))} \leq c \nu^{\frac{1}{4}}.
\]
Follow the approach of Z. Xin and T. Yanagisawa for linearized, compressible Navier-Stokes ⇒ tangential velocity there satisfies a boundary-layer-type equation.

Since \( \| \theta \|_{L^\infty(0,T;L^2(Q))} \approx \nu^{1/4} \), get optimal rate of convergence.

**Corollary**

*Under the same assumptions in main theorem:*

\[
C_1 \nu^{1/4} \leq \| u'(t) - u_0 \|_{L^\infty(0,T;L^2(\Omega))} \leq C_2 \nu^{1/4},
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where \( C_1, C_2 \) independent of \( \nu \).
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*Under the same assumptions in main theorem:*

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*where $C_1, C_2$ independent of $\nu$.***
Sketch of proof. $L^2$, $L^\infty$ estimates.

1. Introduce **cut-off function** $\rho$ for the corrector so that outer solution plus correctors satisfy b.c. *exactly*.

2. Show that extra forcing in the NS equation for $u^{app}$ are small $\Rightarrow$ use regularity and decay estimates for corrector:

   $\theta_0^\phi \in \cap_{j=0}^{\lfloor m/2 \rfloor} C^j([0, T]; H^{m-2j}(0, +\infty))$ and

   $$\langle Z \rangle^I \frac{\partial^\alpha}{\partial_\alpha} \theta_0^\phi \in C(0, T; L^2(0, +\infty)), \quad \alpha \leq 2,$$

   $$\langle Z \rangle^I \frac{\partial^\alpha}{\partial_\alpha} \theta_0^\phi \in L^\infty([0, T] \times [0, +\infty)), \quad \alpha = 0, 1,$$

   and similar estimates holds for $\theta_0^x$.

3. $L^\infty_t L^2, L^2_t H^1$ bounds by energy estimates, $L^\infty((0, T) \times \Omega)$ by maximum principle.
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   \]

   \[
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Sketch of proof cont., $L^\infty(H^1)$ estimate.

5 $L^\infty(H^1)$ estimates require two bounds, one near boundary, the other in the interior $\Rightarrow$ additional cut-off function $\psi(r)$.

6 Estimate near the boundary consequence of better control on tangential derivatives:

$$\left\| \frac{1}{r} \partial_\phi u_{x_{app}} \right\|_{L^\infty(\Omega \times [0,T])} \leq c,$$

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with $w_b = \psi u_{x_{app}}$, $c$ depending on initial data, not on $\nu$.

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Vorticity production at the boundary $\Gamma$ of the cylinder $Q$ should not vanish in the limit $\nu \to 0_+$. 

Take $f \equiv 0$, but rotate the cylinder with angular velocity $\alpha(t)$. Assume $\alpha \in BV(\mathbb{R})$, $\alpha(t-) = 0$, $t \leq 0$.

**Theorem**

In the limit as $\nu \to 0_+$, for all $\psi \in C(\overline{Q})^3$, point-wise in $t$:

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(\text{curl } u^\nu(t), \psi)_{L^2(Q)} \to (\text{curl } u^0, \psi)_{L^2(Q)} + (u^0 \times n, \psi)_{L^2(\Gamma)} + \left(\frac{\alpha(t-)}{2\pi}, \psi\right)_{L^2(\Gamma)}
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where $n$ normal to boundary.
Vorticity concentration

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If Euler flow is **steady**, accumulation of vorticity at the boundary arises in two cases:

1. when the initial data is **ill prepared** ⇒ \( u_0 \mid_{\Gamma} \neq 0 \).
2. when **acceleration** is applied to \( \Gamma \) ⇒ \( \alpha(t-) \neq 0 \) (\( \alpha(0-) = 0 \)).

Explicit examples of **inviscid**, almost **linear** flows producing a non-zero torque at the boundary.

Convergence in **distribution sense** follow directly from vanishing viscosity limit.

If \( \nu, v \in H^1(Q) \), \( \nu \rightarrow v \) in \( L^2(Q) \), then weakly in \( H^{-1}(\mathbb{R}^3) \):

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\text{curl } \nu \rightharpoonup \text{curl } v + v' \times \nabla \chi_Q = \text{curl } v + (v \times n) \sigma,
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where \( \sigma \) surface area on \( \Gamma \), and \( v' \) extension of \( v \) to the whole space (De Giorgi).
Vorticity concentration cont.

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Sketch of proof

Cylinder rotation affects directly only horizontal components $\Rightarrow$ assume flow is planar ($u_x = 0$) $\Rightarrow$ vorticity $\omega$ scalar.

Then $u^\nu(x, t) = (v^\nu(x, t), 0)$, with $v^\nu(x, t) = u^\nu(r, t) e_\phi$ $\Rightarrow v^\nu$ solves heat equation on $\Omega$, unit disk, with Dirichlet boundary conditions.

Initial data $v^\nu(0) = v_0$ exact stationary solution of Euler.

Do not assume $v_0 = 0$ at $\partial \Omega$ (initial layer for NS affects vorticity production).

The vanishing viscosity limit: $v^\nu \rightarrow v_0$ as $\nu \rightarrow 0$.

$\omega^\nu$ radial function $\Rightarrow v^\nu \cdot \nabla \omega^\nu = 0$ (as $v^\nu \parallel e_\phi$).

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Vorticity production at boundary

Derive $L^1$-bounds for $\omega^\nu$ uniform in viscosity.

Need a boundary condition for vorticity $\Rightarrow$ follows from symmetry.

Need $\alpha \in BV([0, \infty))$ to obtain $L^1$ estimates on vorticity.

Enough that initial data $v_0 \in L^2(\Omega)$ circularly symmetric, $\omega_0 \in L^1(\Omega)$.

Convergence of derivatives of $v^\nu$ in the interior of $\Omega$

Vorticity $\omega^\nu \rightarrow \omega_0$ in $L^1(\mathcal{O})$ for any subset $\mathcal{O} \subset \Omega$. 
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If $\alpha$ regular, integration by parts gives:

$$\begin{cases} \partial_t \omega^\nu = \nu \Delta \omega^\nu, & \text{on } (0, \infty) \times \Omega, \\ \frac{x}{|x|} \cdot \nabla \omega^\nu(t, x) = \frac{\alpha'(t)}{2\pi \nu}, & \text{on } (0, \infty) \times \partial \Omega, \end{cases}$$

(1)

Regularize $\alpha \in \text{BV}$ and approximate $L^1$-norm with strictly convex functionals $\Rightarrow$

$$\|\omega^\nu(t)\|_{L^1(\Omega)} \leq \|\omega_0\|_{L^1(\Omega)} + \|\alpha\|_{\text{BV}([0,t])}. \quad (2)$$

Hence $\omega^\nu$ converges weakly to a Radon measure $\mu$ as $\nu \to 0$ for $\alpha \in \text{BV}$, necessarily supported on $\partial \Omega$.

If $v_0 \in H^1_0(\Omega)$ circularly symmetric,

$$\text{curl } v^\nu = \text{curl } e^{\nu t} \Delta_D v_0 = e^{t \Delta_N} \text{curl } v_0 = \omega^\nu.$$
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By Stokes’ Theorem,

\[ \nu \times n \mid_{\partial \Omega} = \int_{\omega} \nu(t, x) \, dx = \int_{[0,t)} d\alpha(t) = \alpha(t^-), \quad t > 0. \]

Boundary corrections then give (using (3) for \( t > 0 \) + rotation invariance):

\[ \lim_{\nu \searrow 0} \text{curl} \ u^\nu(t) = \omega_0 + (\alpha(t^-) - b) \frac{d\phi}{2\pi}, \]

weakly in the space of Radon measures, where \( b = \int_{\Omega} \omega_0 \, dx \).
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Non-planar pipe flows

In the general case of pipe flows \( u_x \neq 0 \), the main difficulty is to obtain a boundary condition for the vorticity.

Follow the idea of Lightill, use the equation restricted at the boundary (flow is regular).

Obtain boundary conditions for the vorticity

\[ \omega_\nu = \omega_r^\nu e_r + \omega_\phi^\nu e_\phi + \omega_x^\nu e_x \text{ (for } \alpha = 0) : \]

\[
\begin{aligned}
\frac{\partial \omega_\phi^\nu}{\partial r} + \omega_\phi^\nu &= 0, \quad \text{on } \Gamma, \\
\frac{\partial \omega_\phi^\nu}{\partial r} &= -\nu^{-1} (*) \quad \text{on } \Gamma, \\
\omega_r^\nu &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

where

\[
(*) = -\frac{\partial u_\phi^\nu}{\partial t} + \nu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_\phi^\nu}{\partial r} \right) - \frac{1}{r^2} u_\phi^\nu \right\}.
\]
Future work/Open problems

- Boundary layer analysis for Oseen equation.
- Stability of boundary layer (circularly symmetric, planar flows).
THANK YOU