On water waves with angled crests

Sijue Wu

University of Michigan, Ann Arbor

June 2014
We consider the motion of the interface separating air from water.

We assume:
- air density = 0
- water density = 1
- water region is below the air region. At time $t$, water region is $\Omega(t)$, the interface is $\Sigma(t)$.

We assume that the water is:
- inviscid, incompressible, irrotational.
- The surface tension is zero.
- The water is subject to the influence of gravity $\mathbf{g} = (0, -g)$.
- $g > 0$
\[ \text{density} = 0 \]

\[ \text{density} = 1 \]

\[ \mathbf{g} = (0, -g) \]
The motion of the fluid is described by

\[
\begin{align*}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} &= (0, -g) - \nabla P \quad \text{in } \Omega(t) \\
\text{div } \mathbf{v} &= 0, \quad \text{curl } \mathbf{v} = 0, \quad \text{in } \Omega(t) \\
\mathbf{v} &= 0, \quad \text{on } \Sigma(t) \\
P &= 0, \quad \text{on } \Sigma(t)
\end{align*}
\]

\( (1) \)

\( \mathbf{v} \) is the fluid velocity, \( P \) is the fluid pressure.

When surface tension is zero, the motion can be subject to the Taylor instability.

- Taylor condition:

\[
- \frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0
\]

on the interface \( \Sigma(t) \). \( \mathbf{n} \) is the unit normal to \( \Sigma(t) \) pointing out of the water region \( \Omega(t) \).
G. I. Taylor: linearize about the flat interface,

- **air above water**
  
  \[-\frac{\partial P}{\partial n} > 0 : \text{stable.}\]

- **water above air**
  
  \[-\frac{\partial P}{\partial n} < 0 : \text{unstable.}\]
Known results:

**Local wellposedness for arbitrary data**


- The Taylor condition always holds for the water wave motion, i.e.
  \[-\frac{\partial P}{\partial n} \geq c_0 > 0\]
  as long as the interface is non-selfintersecting.

- **Local existence and uniqueness in Sobolev spaces:**
  There exist a unique solution for a finite time period, for any initially non-selfintersecting interface, and any given initial velocity (incompressible & irrotational).
The proof of the fact \(-\frac{\partial P}{\partial n} > 0\):

Applying div to both sides of the Euler equation gives

\[ \Delta P = -|\nabla \mathbf{v}|^2 \leq 0 \]

Maximum principle implies \(-\frac{\partial P}{\partial n} \geq 0\).

For \(C^1\) interface, \(-\frac{\partial P}{\partial n} \geq c_0 > 0\), by Green's identity.
Earlier results:

1. T. Beale, T. Hou & Lowengrub (1992). Linear wellposedness assuming the presumed solution satisfies the Taylor sign condition:

\[- \frac{\partial P}{\partial n} \geq c_0 > 0.\]

2. Nalimov (1974): infinite depth, 2-D, small data, local wellposedness

3. Yosihara (1982): finite depth, 2-D, small data. local wellposedness
Recent works:

Local wellposedness with additional effects: nonzero surface tension, finite depth, nonzero vorticity, assuming the Taylor sign condition holds.

Global behavior for small and smooth data

More recently, there are results on almost global or global well-posedness for small and smooth initial data, for the infinite depth zero surface tension water wave problem, in two and three dimensions.
Global behavior for small and smooth data

- S. Wu (2009): almost global well-posedness for 2-D,
- S. Wu (2011): global well-posedness for 3-D
- Germain, Masmoudi & Shatah (2012): global well-posedness for 3-D
- Alazard & Delort (2013): 2-D water waves, global existence and modified scattering
- Ionescu & Pusateri (2013): similar result
- Ifrim & Tataru (2014): 2-D water wave, global existence.
Singularities:

- Alazard, Burq, Zuily (2012): Local wellposedness in low regularity Sobolev space—the interface is $C^{3/2+\epsilon}$.

- Alazard, Burq, Zuily (2014): Local wellposedness in low regularity Sobolev space—the interface is $C^{3/2-\epsilon}$. 
Singularities:

What are some typical singular behaviors? How does it form? What are some basic structures of the singularities?
S. Wu (2012): construction of self-similar solution for 2-D water waves in the regime where convection is in dominance:

- $z \sim t$, or in hyperbolic scaling: $s = 1$.
- neglecting gravity and surface tension.
- satisfies the Taylor sign condition $-\frac{\partial P}{\partial n} \geq 0$. 
\( \nu < \frac{1}{2}, \quad \mu > \frac{1}{2} \)
Question:
Q: How relevant are the self-similar solutions? Are they stable?

In all earlier work, either it is assumed there is no bottom, or there is a bottom $\gamma$, of a positive distance away from the interface $\Sigma(t)$

$$dist(\Sigma(t), \gamma) \geq h_0 > 0$$

Question
We consider the following problem:
Q: the interaction of the free surface with a fixed rigid boundary?
In the presence of a fixed rigid boundary $\Gamma$, the motion of the fluid is described by

\[
\begin{aligned}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} &= (0, -g) - \nabla P & \text{in } \Omega(t) \\
\text{div } \mathbf{v} &= 0, & \text{curl } \mathbf{v} &= 0, & \text{in } \Omega(t) \\
P &= 0, & \text{on } \Sigma(t) \\
\mathbf{v} \cdot \mathbf{n} &= 0, & \text{on } \Gamma
\end{aligned}
\]

(2)

$v$ is the fluid velocity, $P$ is the fluid pressure. $\mathbf{n}$ is a normal vector to $\Gamma$.

\[
\partial \Omega(t) = \Sigma(t) \cup \Gamma.
\]

While this problem maybe hard....
We look at a specific problem:

If the fixed rigid boundary $\gamma$ is a vertical wall $\{x = 0\}$, and the fluid domain $\Omega(t)$ is the domain to the right of $\{x = 0\}$. Then the velocity field $\mathbf{v} = (v_1, v_2)$ satisfies $v_1(0, y; t) = 0$. By Schwarz reflection:

$\mathbf{v}(-x, y; t) = (-v_1(x, y; t), v_2(x, y; t)); \; P(-x, y; t) = P(x, y; t)$

we can reduce the problem to the one on the symmetric domain without a fixed wall.

This is new only when the angle of the wave with the wall is other than $90^\circ$. 
Q: Is it possible for the angle between the interface and the wall to be other than $\frac{\pi}{2}$?
We study the following problem:

Assume the rigid boundary $\gamma$ is consisting of two vertical walls:

$$\gamma = \{x = 0\} \cup \{x = 1\}$$

Assume the free interface $\Sigma(t)$ makes a 90° angle with the wall $\{x = 0\}$, but we allow a possible non-trivial angle at $\{x = 1\}$. We make a Schwarz reflection about $\{x = 0\}$:
Q: Can the angle \( \nu \) be other than \( 90^\circ \)?

Q: local existence in this framework?
Yes, there is local existence in this framework. In fact, besides a non-trivial angle $\nu$, the interface can have angled crests.

The angle $\nu$ must be no more than $\frac{\pi}{2}$, and the interior angles of the angled crests cannot be more than $\pi$.

The water wave problem admits such solutions.

A prior estimate: joint work with Rafe Kinsey.
In our framework, we have

1. \( -\frac{\partial P}{\partial n} \geq 0 \), but
2. \( -\frac{\partial P}{\partial n} = -\mathbf{n} \cdot \nabla P = 0 \) at the wall when there is a non-trivial angle, and at the points on the interface where there are angled crests.
3. \( \mathbf{n} \) outward unit normal.
Let the free surface be

$$\Sigma(t) : z = z(\alpha, t), \quad \alpha \text{ Lagrangian coordinate.}$$

- $z = x(\alpha, t) + iy(\alpha, t)$, in complex form;
- $z_t = z_t(\alpha, t)$ is the velocity;
- $z_{tt}$ is the acceleration;
- $-i$ is the gravity;
- $P = 0$ on $\Sigma(t)$ implies: $\nabla P \perp \Sigma(t)$
- $\nabla P = i\alpha z_\alpha$, where $\alpha = -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial n}$
\[ \text{div} \mathbf{v} = \text{curl} \mathbf{v} = 0 \text{ implies } \tilde{\mathbf{v}} \text{ is holomorphic in } \Omega(t). \]
\[ \mathbf{v}(-1 + iy; t) = \mathbf{v}(1 + iy; t) \text{ is purely imaginary.} \]
\[ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } x = \pm 1. \]
\[ \bar{z}_t(\alpha, t) = \bar{\mathbf{v}}(z(\alpha, t), t), \text{ the boundary value of periodic holomorphic function } \mathbf{v}. \]
\[ \bar{z}_t = \mathcal{S}_z \bar{z}_t + c \]

Equation of the free surface:

\[
\begin{aligned}
Z_{tt} + i &= i\alpha Z_{\alpha} \\
\bar{Z}_t &= \mathcal{S}_z \bar{Z}_t + c
\end{aligned}
\]  
\( (3) \)
Quasilinear equation:

\[ \ddot{Z}_{ttt} + i\alpha \dot{Z}_{t\alpha} = -i\alpha_t \dot{Z}_\alpha \]

Let

- \( u = \dot{Z}_t \)
- \( i\dot{Z}_{t\alpha} = \nabla_n u \)

Free surface equation:

\[ (\partial_t^2 + a \nabla_n) u = l.o.t \]

is degenerate hyperbolic, if \( \alpha \) can be zero.
\( \alpha = 0 \) at the singularities:

Free surface equation: \( z_{tt} + i = i \alpha z_{\alpha} := \nabla P, \, \alpha \in \mathbb{R} \) implies:

\[
-x_{\alpha} = \frac{y_{tt} + 1}{y_{\alpha}}.
\] (4)

\[
\tan \nu = -\frac{x_{\alpha}}{y_{\alpha}} = \frac{y_{tt} + 1}{x_{tt}}.
\] (5)

- \( x_t(\pm 1; t) = 0 \) implies \( x_{tt}(\pm 1; t) = 0. \)
- If \( \nu \neq \frac{\pi}{2} \), then \( y_{tt} + 1 = 0 \) at \( x = \pm 1. \)
- Therefore \( \nabla P = 0 \) at the corner \( x = \pm 1. \)
- \( \alpha = 0 \) at the corner \( x = \pm 1. \)
Let $\Phi : \Omega(t) \to \mathcal{P} := [-1, 1] \times (-\infty, 0]$ be the Riemann mapping, taking corners of $\Omega(t)$ to $(\pm1, 0)$, $-\infty$ to $-\infty$:

Let $h(\alpha; t) := \Phi(z(\alpha; t); t)$.

$h(\cdot; t) : [-1, 1] \to [-1, 1]$ is strictly increasing.
Let

- \( h^{-1} \) be:  \( h(h^{-1}(\alpha'; t); t) = \alpha' \)
- \( Z(\alpha'; t) = z(h^{-1}(\alpha'; t), t) := z \circ h^{-1}; Z_{,\alpha'} = \partial_{\alpha'} Z(\alpha', t) \)
- \( Z_t(\alpha'; t) := z_t \circ h^{-1}; Z_{tt}(\alpha'; t) := z_{tt} \circ h^{-1} \)
- \( A \circ h = \alpha h_{\alpha} \)
- \( \mathbb{H} f(\alpha') = \frac{1}{2i} \int_{-1}^{1} \cot \left( \frac{\pi}{2} (\alpha' - \beta') \right) f(\beta') \, d\beta' \)

be the Hilbert transform

Free surface equation in Riemann mapping variable \( \alpha' \):

\[
\begin{aligned}
Z_{tt} + i &= iAZ_{,\alpha'} \\
\overline{Z}_t &= \mathbb{H}\overline{Z}_t + c
\end{aligned}
\]
\[ h(\alpha; t) := \Phi(z(\alpha; t); t) \] implies that

\[ Z(\alpha', t) = \Phi^{-1}(\alpha'; t); \quad Z_{,\alpha} = \partial_{z'}\Phi^{-1}(\alpha'; t) \]

\[ \bar{Z}_{,\alpha'}(Z_{tt} + i) = iA|Z_{,\alpha'}|^2 := iA_1 \]

We can show that \( A_1 \geq 1 \) (this was shown in [S. Wu 1997] for the whole line case. )

\[ \frac{1}{Z_{,\alpha'}} = i\frac{\bar{Z}_{tt} - i}{A_1} \]
Fact 1: $\nu \leq \frac{\pi}{2}$

at the corner:
$\Phi^{-1}(z') \approx (z')^r$, where $\nu = \frac{\pi}{2} r$.

$Z_{,z'} = \partial_z \Phi^{-1}(z') \approx (z')^{r-1}$.

if $\nu > \frac{\pi}{2}$, i.e. if $r > 1$, then $Z_{,z'} = 0$ at the corner, so $Z_{tt} = \infty$, so $y_{tt} = \infty$ at the corner, since $x_{tt} = 0$,
Recall

\[ \tan \nu = -\frac{x_\alpha}{y_\alpha} = \frac{y_{tt} + 1}{x_{tt}}. \]  

(7)

this implies

\[ \tan \nu = \infty \]

therefore

\[ \nu = \frac{\pi}{2} \]

So \( \nu \) cannot be greater than \( \frac{\pi}{2} \).

Similarly,

- Interior angle of the angled crests cannot be more than \( \pi \).
• Fact 2:

\[-\frac{\partial P}{\partial \mathbf{n}} = a|z_\alpha| = \frac{A_1}{|Z_{\alpha'}|} \circ h \geq 0\]

Recall: \(Z_{\alpha'} = \partial_{z'} \Phi^{-1}(\alpha'; t)\)

(S.Wu, 1997) If the interface \(\Sigma(t) \in C^1\), then

\(0 < c_0 \leq |\partial_{z'} \Phi^{-1}(\alpha'; t)| \leq C_0 < \infty\), then

\[-\frac{\partial P}{\partial \mathbf{n}} \geq c_1 > 0\]

• If the angle \(\nu < \frac{\pi}{2}\), or if the free surface has angled crests with interior angle \(< \pi\), then \(r < 1\), then \(\frac{1}{Z_{\alpha'}} \rightarrow 0\) at the corner or at the crests, this implies

\[-\frac{\partial P}{\partial \mathbf{n}} = 0\]

at the corner if \(\nu < \frac{\pi}{2}\) and at the crests where the interior angle is \(< \pi\).
We introduce a special derivative

\[ D_\alpha f = \frac{1}{z_\alpha} \partial_\alpha f, \quad D_{\alpha'} g = \frac{1}{Z_{\alpha',\alpha}} \partial_{\alpha'} g \]

If \( f \) is the boundary value of a periodic holomorphic function \( F \) on \( \Omega(t) \), \( f(\alpha, t) = F(z(\alpha, t), t) \), then

\[ D_\alpha f = \partial_z F(z(\alpha, t); t) = -i \partial_y F(z(\alpha, t); t) \]

\( D_\alpha \) preserves the holomorphicity and periodicity of periodic holomorphic functions.
Recall the quasilinear equation of the free surface:
\[
(\partial_t^2 + i\alpha \partial_{\alpha})\bar{Z}_t = -i\alpha_t\bar{Z}_{\alpha},
\]
higher order equation
\[
(\partial_t^2 + i\alpha \partial_{\alpha})\theta = G_{\theta}.
\]
where \(\theta = D_{\alpha}^k \bar{Z}_t\), \(G_{\theta} = D_{\alpha}^k(-i\alpha_t\bar{Z}_{\alpha}) + [\partial_t^2 + i\alpha \partial_{\alpha}, D_{\alpha}^k]\bar{Z}_t\).
Energy:
\[
e = \int |\theta_t|^2 + \Re \int (i\alpha \partial_{\alpha} \theta) \bar{\theta}
\]
We construct the energy: let \( \alpha_0 \in [-1, 1] \) be fixed

\[
E = E_{a,D_{\alpha}^2 \bar{z}_t} + E_{b,D_{\alpha} \bar{z}_t} + |\bar{z}_{tt}(\alpha_0; t) - i|
\]

where

\[
E_{a,\theta} = \int_{-1}^{1} \frac{h_{\alpha}}{A_1 \circ h} |\theta_t|^2 \, d\alpha + \Re \int_{-1}^{1} \frac{h_{\alpha}}{A_1 \circ h} (i\alpha \partial_{\alpha} \theta) \bar{\theta} \, d\alpha + \text{l.o.t.}
\]

\[
E_{b,\theta} = \int_{-1}^{1} \frac{1}{\alpha} |\theta_t|^2 \, d\alpha + \Re \int_{-1}^{1} (i\partial_{\alpha} \theta) \bar{\theta} \, d\alpha + \text{l.o.t.}
\]

\[
\alpha \approx h_{\alpha} \approx - \frac{\partial P}{\partial n}
\]

\( E_a \) and \( E_b \) have roughly inverse singular weights \( h_{\alpha} \) and \( \frac{1}{\alpha} \).

\[
A_1 \circ h = \frac{\alpha |z_{\alpha}|^2}{h_{\alpha}}
\]
Main Result

**Theorem (A priori estimate, joint work with Rafe Kinsey)**

There exists a polynomial $p = p(x)$ with universal coefficients, such that, for any solution of water wave equations with $E(t) < \infty$ for all $t \in [0, T]$,

$$\frac{d}{dt} E(t) \leq p(E(t))$$

(10)

for all $t \in [0, T]$.

**Theorem (local existence)**

For any initial data satisfying $E(0) < \infty$, there exists $T > 0$, depending only on $E(0)$, such that the water wave equation is solvable for time $t \in [0, T]$, with $E(t) < \infty$ for $t \in [0, T]$. 
A characterization of the energy

\[ E(t) \leq C\left( \| \tilde{Z}_{t,\alpha'} \|_{L^2}, \| D_{\alpha'}^2 \tilde{Z}_t \|_{L^2}, \| \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_{L^2}, \| D_{\alpha'} \tilde{Z}_t \|_{H^{1/2}}, \| D_{\alpha'} \tilde{Z}_t \|_{H^{1/2}}, \| \frac{1}{Z,\alpha'} \|_{L^\infty} \right), \quad (11) \]

\[ \| \tilde{Z}_{t,\alpha'} \|_{L^2}, \| D_{\alpha'}^2 \tilde{Z}_t \|_{L^2}, \| \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_{L^2}, \]

\[ \| D_{\alpha'} \frac{1}{Z,\alpha'} \|_{L^2}, \| \frac{1}{Z,\alpha'} D_{\alpha'}^2 \tilde{Z}_t \|_{H^{1/2}}, \| D_{\alpha'} \tilde{Z}_t \|_{H^{1/2}}, \| \frac{1}{Z,\alpha'} \|_{L^\infty} \leq C(E) \]

\[ (12) \]

\( C \): universal polynomial.
The self-similar solution (S. Wu 2012) has finite energy.

In general, surfaces that have angled crests of interior angle $\angle \leq \frac{\pi}{2}$, and the angle $\nu$ of the wave with the vertical wall $\nu < \frac{\pi}{4}$ have finite energy.

Stokes wave of maximum height do not have finite energy.
Thank you very much for your attention!