A few questions in random matrix theory

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Random matrix theory

Wigner (50’s) Wishart (30’s)

Standard random matrices:

- Wigner random matrix: $H = H^*$ of size $N \times N$ with complex or real entries s.t.

\[
H_{ij}, 1 \leq i < j \leq N \text{ i.i.d. } \mathbb{E}H_{ij} = 0, \text{Var}H_{ij} = \sigma^2 < \infty,
\]

and independent centered entries on the diagonal.

Archetypal ensemble: GUE/GOE (Hermitian/real symmetric) $H_{ij} \sim \mathcal{N}(0, 1)$.

- Sample covariance matrices $M = XX^*$ with $X$ of size $N \times p$ and the entries of $X$ are i.i.d. centered r.v. with finite variance $\sigma^2$. $p$ and $N$ are comparable.

Random matrix theory: what is the asymptotic spectral properties of these matrices as the dimension goes to infinity? (eigenvalues and associated eigenvectors)
Motivations

• mathematical physics: statistics of energy levels of heavy nuclei (Wigner).

• mathematical statistics in high dimension: huge amount of data (genetics, microarrays, etc) estimation of the "true" covariance with the sample covariance matrix?

• finance (risk of a large portfolio depends e.g. on extreme eigenvalues), communication theory (capacity of some communication channel can be expressed in terms of RMT objects)

• connections with many other fields of mathematics have now emerged (longest increasing subsequence, etc)
Global behavior of the spectrum and Wigner’s theorem

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \] ordered eigenvalues of \( H_N := \frac{1}{\sqrt{N}} H. \)

**Theorem** Wigner [57]

Let \( f \) be a bounded continuous function and \( \sigma^2 := \text{Var}(H_{ij}) \). Then, almost surely,

\[ \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \to \int f d\rho_{sc}, \quad \frac{d\rho_{sc}(x)}{dx} := \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} 1_{|x| \leq 2\sigma}. \]

Proof: moment method or resolvent approach
Local behavior: the complex Gaussian case

Density w.r.t Lebesgue measure on $\mathcal{H}_N$: $\sigma^2 = 1$,

$$dP_N(H_N) := \exp \left\{-N \text{Tr} H_N^2 / 2 \right\} dH.$$

Consequences :

- explicit joint eigenvalue density

$$f(x_1, \ldots, x_N) = \prod_{i<j} (x_i - x_j)^2 \exp \left\{-N \sum_{i=1}^{N} x_i^2 / 2 \right\}. $$

- eigenvector distribution is explicit: $U$ the matrix of eigenvectors is distributed according to the Haar measure on the unitary group.
  Thus uniformly distributed along the $N$ coordinates.
Spa\v{c}ing and edge statistics

Eigenvalue statistics: let $f : \mathbb{R}^m \to \mathbb{R}$ be a symmetric function with compact support. Let also $\rho_N$ be a scaling and $u \in [-2\sigma, 2\sigma]$.

$$S_N(f) := \sum_{1 \leq i_1 \neq i_2 \ldots \neq i_m \leq N} f(\rho_N(\lambda_{i_1} - u), \ldots, \rho_N(\lambda_{i_m} - u)).$$

- in the bulk: fix $u \in (-2\sigma, 2\sigma)$, $\rho_N = N \rho_{sc}(u)$.

$$\mathbb{E}S_N(f) \to \int_{\mathbb{R}^m} f(x_1, \ldots, x_m) \text{det} K_{\sin}(x_i, x_j) \prod_{i=1}^m dx_i. K_{\sin}(t, s) := \frac{\sin \pi(t - s)}{\pi(t - s)}.$$

- at the edges $u = \pm 2\sigma$, $\rho_N = 2\sigma N^{2/3}$.

$$\mathbb{E}S_N(f) \to \int_{\mathbb{R}^m} f(x_1, \ldots, x_m) \text{det} K_A(x_i, x_j) \prod_{i=1}^m dx_i. K_A(s, t) = \int_0^\infty Ai(t + u)Ai(s + u)du.$$
The universality conjecture

The same local statistics shall be true for a general Wigner matrix

- In the bulk provided $\sigma^2 < \infty$,

- at the edge provided $\mathbb{E}|H_{ij}|^4 < \infty$ AND $\mathbb{E}H_{ij} = 0$.

- eigenvectors shall be "asymptotically Haar distributed" provided $\mathbb{E}|H_{ij}|^4 < \infty$ and...
**Universality results**


Assume that

$$\exists c > 0, \mathbb{P}(\lvert H_{ij} \rvert \geq tc) \leq e^{-t}$$

with support on at least 3 points and assume that the first four moments of $H_{ij}$ match those of a Gaussian r.v. $\mathbb{E}\lvert \Re H_{ij} \rvert^k \lvert \Im H_{ij} \rvert^l$ identical for all $k + l \leq 4$.

Then, universality in the bulk and at the edge (correlation functions are asymptotically the same).

Let $F$ be a smooth bounded function $F : \mathbb{C}^k \rightarrow \mathbb{R}$ with $\lvert \nabla^j F(x) \rvert = O(N^\delta)$, $j = 1 \ldots, 5$.

Then

$$\lvert \mathbb{E}F((\sqrt{N}u_{ia,pb}), 1 \leq a, b \leq k) - \mathbb{E}F((\xi_{ia,pb}), 1 \leq a, b \leq k) \rvert = o(1)$$

as $N \rightarrow \infty$. Here eigenvectors are determined by $u_{i,1} > 0$, $k \leq N^\delta$ and the $\xi_j$'s are i.i.d. Gaussian $\mathcal{N}(0, 1)$ random variables.
Some ideas of the proof

Start with a GUE;

replace one entry (modulo the symmetry assumption) by another r.v. with same moments.

Impact on both eigenvalues and eigenvectors of this perturbation: Hadamard’s variation formulae for fixed rank perturbation of matrices (algebraic)+ the fact that eigenvectors are delocalized.

Another approach by Erdös-Schlein-Yau and Knowles-Yin: strong connection with local ”semi-circle law”.

Comparison argument
Extensions to other ensembles

One can relax:

• the independence assumption

• the moments assumptions: heavy tailed distribution

\[ \mathbb{P}(|H_{ij}| \geq t) = L(t)t^{-\alpha}, \alpha < 4. \]

• Force some of the entries to be zero (introduce some geometry which breaks the invariance of the matrix by permutation)

Focus on the third and second point.
Other ensembles: easy case

Block decomposition:

\[ H = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \]

where \( X \) is \( N \times p \) with i.i.d. entries.

Equivalent to sample covariance matrices \( XX^* \). The global behavior is different (Marchenko-Pastur law) but the asymptotic local eigenvalue statistics are the SAME. (Peche, Erdos-Schlein-Yau, Tao-Vu).

Note that local eigenvalue statistics are more robust than global ones.
Random band matrices

Assume that

\[ H_{ij} = 0 \text{ unless } |i - j| \leq W/2 \text{ (or } |i - j|_N \leq W/2) . \]

\( W \) bandwidth \( W = cN^\mu \) for some \( 0 < c < 1 \) and \( 0 < \mu \leq 1 \).

Other entries are i.i.d. with finite variance.

\( \mu = 0 \) diagonal matrix with i.i.d. entries \( \rightarrow \) Wigner random matrix when \( c = \mu = 1 \).


\[ H_N = \frac{1}{\sqrt{W}} H. \]

Periodic matrices : Wigner semi-circle.
If \( 0 < \mu < 1 \), non periodic : Wigner semi-circle; if \( \mu = 1 \) and \( c < 1 \) then another distribution (not explicit).
An interesting ensemble

Believed to be more complicated than Wigner ensembles: no reference ensemble. There does not exist a “simple” band random matrix ensemble for which eigenvalue/eigenvector statistics can be explicitly computed as for the GUE/GOE.

A flavor of Anderson transition: Band model believed to exhibit a phase transition, depending on $\mu$.

A vector is said to have localization length $L$ if most of its $l^2$ norm is carried on $L$ coordinates.

Fyodorov-Mirlin (1991) (superanalysis) explain that for Gaussian entries, the localization length of a typical eigenvector in the bulk is of order $L = O(N^{2\mu})$ so that eigenvectors should be localized (resp. delocalized or extended) if $\mu < 1/2$ (resp. $> 1/2$). There exists an intuitive explanation for that scale...
Some eigenvalue statistics

Edge:

**Theorem** (Sodin 2010)

Consider band matrices with entries having sub-Gaussian tails. Assume that $\mu > 5/6$ then $\lambda_{max}(HW^{-1/2})$ fluctuates in the scale $N^{-2/3}$ and the same limiting distribution as for the GUE (Tracy-Widom law).

If $\mu > 5/6$ no explicit limiting distribution (existence). The joint distribution of largest eigenvalues is unknown.

Eigenvectors (localization/delocalization):

**Theorem** (Erdos-Yau-Knowles 2011-2012)

Assume that $\mu > 4/5$ and sub-exponential decay of the entries. Then eigenvectors in the bulk are completely delocalized in the large $N$ limit.

**Theorem** Schenker [2009]. Therein it is proved that $L \ll N^{8\mu}$ for all bulk eigenvectors of some random band matrices with i.i.d. Gaussian entries on the band.
More questions: sparsity, the role of moments for band matrices

• No band structure: there are at most $W$ non zero entries per row. Still Wigner if $\mu < 1$. Eigenvalue statistics? Eigenvector : delocalized or not? Is it sparsity that plays a role? or need some structure (not aware of any conjecture)?

• What is the role of moments in the localization of eigenvectors (conjecture of Bouchaud and Cizeau (1994))? Consider heavy tailed entries

$$\mathbb{P}(|H_{ij}| \geq x) \sim x^{-\alpha}L(x), \quad \alpha > 0 \quad L \text{ slowly varying.}$$

If $\alpha < 2(1 + \mu^{-1})$, then eigenvectors associated to extreme eigenvalues are almost carried by 2 coordinates. This is true for all $\mu \leq 1$. (Soshnikov (2006), Auffinger-Ben Arous-P (2007)).

If $\alpha > 2(1 + \mu^{-1})$, then no eigenvector localized on less than $L := \lfloor N^c \rfloor$ coordinates, for any $c$ s.t. $c < \frac{2}{5} \mu \frac{\alpha - 2}{\alpha - 1}$.
Conclusion

• New techniques are needed: no comparison if the band or sparse matrix is not close to Wigner.

• The geometry of null entries seems to play a role: compare Sample covariance matrices/band/ Wigner: on global scale but on local ones this is not so obvious (apart from band matrices).

• The understanding of eigenvectors is probably more complicated ( Wigner: asymptotically Haar; here no idea). Band matrices: some reasons to believe that eigenvectors are localized on successive coordinates, what about sparse matrices?

• The edge of the spectrum seems to be different: indeed expect more localization. No conjectured localization length but there are some reasons to believe that $W^{9/5}$ shall be the one.

• The role of moments in universality for both eigenvalues and eigenvectors.