Sparse Tensor Approximations of PDEs on high-dimensional parameter spaces

Christoph Schwab
Seminar for Applied Mathematics ETH Zürich

Albert Cohen (Paris, France) , Ron DeVore (Texas A& M)
Roman Andreev, Marcel Bieri, Claude Gittelson

“High-dimensional Problems & Solutions”
WS Paris, France, June 21+22, 2010

Supported in part by the Swiss Nat’l Science Foundation and ERC

Outline

1. Elliptic BVP with stochastic data
2. infinite - Dimensional Deterministic BVP
3. Regularity
4. Convergence Rates of sGFEM
5. Complexity
6. Sparse Tensor sGFEM
7. Conclusions & References

Elliptic BVP with stochastic data

Given:

- probability space (Ω, Σ, P) on data space $X(D) \subseteq L^\infty(D)$, $V \subseteq H^1(D)$,
- random diffusion coefficient $a(x,\omega) \in L^\infty(\Omega, dP; X(D))$,
- deterministic source term $f \in H^{-1}(D) = (H_0^1(D))' =: V'$,

(sBVP) Find $u(x,\omega) \in L^2(\Omega, dP; V) = L^2(\Omega, dP; H_0^1(D))$ such that

$$\mathbb{E}\left[\int_D a(x,\cdot)\nabla_x u(x,\cdot) \cdot \nabla_x v(x,\cdot) dx\right] = \mathbb{E}\left[\int_D f(x)v(x,\cdot)dx\right]$$

for all $v \in L^2(\Omega, dP; V)$

$a \in L^\infty(\Omega, dP; X(D))$ and $a_{max} \geq a(\cdot,\cdot) \geq a_{min} > 0 \Rightarrow \exists! u \in L^2(\Omega, dP; H_0^1(D))$.

Karhunen-Loève expansion

- separation of deterministic and stochastic variables -

Example 1 (Karhunen-Loève)
If $a \in L^2(\Omega, dP; L^\infty(D))$ then in $L^2(\Omega, dP; L^2(D))$,

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \psi_m(x) Y_m(\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(x) Y_m(\omega),$$

- $(\lambda_m, \varphi_m)_{m\geq 1}$ eigensequence of (cpt., s.a.) **covariance operator**

$$\mathcal{C}[a] : L^2(D) \to L^2(D) \quad (\mathcal{C}[a]v)(x) := \int_D C_a(x,x')v(x')dx' \quad \forall v \in L^2(D),$$

-

$$C_a = \mathbb{E}\left[(a - \mathbb{E}[a]) \otimes (a - \mathbb{E}[a])\right]$$

-

$$Y_m(\omega) := \frac{1}{\sqrt{\lambda_m}} \int_D (a(x,\omega) - \mathbb{E}[a](x))\varphi_m(x)dx : \Omega \to \Gamma_m \subseteq \mathbb{R} \quad m = 1, 2, \ldots$$

Parametric Deterministic BVP

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{j\geq 1} \psi_j(x) Y_j(\omega) \simeq a(x,y) := \mathbb{E}[a](x) + \sum_{j\geq 1} \psi_j(x) y_j,$$

$$y = (y_1, y_2, \ldots) \in U = (-1,1)^{\mathbb{N}}, \quad \mathbb{P}(d\omega) = \rho(dy) = \rho(y)dy = \bigotimes_{j\geq 1} \rho_j(y_j) dy_j$$

parametric deterministic form of (sBVP) on U:

$$\text{Find } u \in L^2(U,V,d\rho) \quad \text{s.t.} \quad B(u,v) = F(v) \quad \forall v \in L^2(U,V,d\rho),$$

$$B(u,v) = \int_U \left(\int_D a(x,y) \nabla u(x,y) \cdot \nabla v(x,y) dx \right) \rho(dy), F(v) = \int_U \left(\int_D f(x) v(x,y) dx \right) \rho(dy)$$

sGFEM

$\mathcal{F}$: set of all sequences $\nu = (\nu_j)_{j\geq 1}$ of nonnegative integers such that only finitely many ν_j are non-zero.

Notation:

$$|\nu| := \sum_{j\geq 1} \nu_j = \|\nu\|_{\ell^1}, \quad \nu! = \prod_{j\geq 1} \nu_j!, \quad \nu \in \mathcal{F}.$$

Tensorized ($L^2(-1,1;1/2dx)$-normalized) Legendre polynomials:

$$L_\nu(y) = \prod_{j\geq 1} L_{\nu_j}(y_j).$$

Proposition 2:
Assume $Y_j \sim \mathcal{U}(-1,1)$, i.e. $\rho_j(dy_j) = \mu_j(dy_j) := 1/2dy_j$, $\mu := \bigotimes_{j\geq 1} \mu_j(dy_j)$.

Then

- $(L_\nu)_{\nu\in\mathcal{F}}$ is a complete orthonormal system in $L^2(U, \mu(dy))$.

- each $v \in L^2(U, V, d\mu)$ has a representation

$$v = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu, \quad \text{where} \quad v_\nu = \int_U v(\cdot, y) L_\nu(y) d\mu(y) \in V$$

  and there holds Parseval's equation

$$\|v\|_{L^2(U,V,d\mu)} = \|(\|v_\nu\|_V)\|_{\ell^2(\mathcal{F})}.$$

sGFEM

For any finite subset $\Lambda \subset \mathcal{F}$, define

$$X_\Lambda := \{v_\Lambda(x,y) = \sum_{\nu \in \Lambda} v_\nu(x) L_\nu(y) \ ; \ v_\nu \in V\} \subset L^\infty(U,V) \subset L^2(U,V,d\mu)$$

where $\{L_\nu\}_{\nu \in \mathcal{F}}$ is the basis of Legendre polynomials.

Galerkin approximation: $u_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu \in X_\Lambda$ s.t.

$$u_\Lambda \in X_\Lambda \quad \text{such that} \quad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \quad \forall v_\Lambda \in X_\Lambda.$$

Cea's lemma:

$$\|u - u_\Lambda\|_{L^2(U,V,d\mu)} \le C_1 \inf_{v_\Lambda \in X_\Lambda} \|u - v_\Lambda\|_{L^2(U,V,d\mu)},$$

where $C_1 := \sqrt{\frac{a_{\max}}{a_{\min}}}$.

sGFEM

Legendre expansion of $u(x,y)$:

$$u(x,y) = \sum_{\nu\in\mathcal{F}} v_\nu(x) L_\nu(y) \text{ where } v_\nu := \int_U u(\cdot,y) L_\nu(y) d\mu(y) \in V.$$

$$\|u - u_\Lambda\|_{L^2(U,V,d\mu)} \le C_1 \left(\sum_{\nu\notin\Lambda} \|v_\nu\|_V^2\right)^{\frac{1}{2}}.$$

Choice of Λ? Summability of sequence $\alpha_\nu := \|v_\nu\|_V$, $\nu \in \mathcal{F}$.
Lemma 3 (Stechkin):
Let $0 < p \le q \le \infty$ and assume $\alpha = (\alpha_\nu)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$.
If $\mathcal{F}_N$ is the set of indices corresponding to N largest $|\alpha_\nu|$,

$$(\sum_{\nu\notin\mathcal{F}_N} |\alpha_\nu|^q)^{\frac{1}{q}} \le \|\alpha\|_{\ell^p(\mathcal{F})} N^{-r},$$

where $r := \frac{1}{p} - \frac{1}{q} \ge 0$.

Regularity

$$b = (b_j)_{j=1}^{\infty}, \quad b_j := \frac{\|\psi_j\|_{L^\infty(D)}}{a_{min}}, \quad j = 1, 2, \ldots$$

Theorem 4
Assume

$$\|b\|_{\ell_1(\mathbb{N})} = \sum_{j \geq 1} \frac{\|\psi_j\|_{L^\infty(D)}}{a_{min}} \leq \kappa < 1.$$

Then

$$\|v_\nu\|_V \leq \left(\frac{|\nu|!}{\nu!} b^\nu \right) \frac{\|f\|_{V^*}}{a_{min}}, \quad b^\nu = b_1^{\nu_1} b_2^{\nu_2} \ldots, \quad \nu \in \mathcal{F}.$$

Regularity

Theorem 5 (Sequence Summability)
For $0 < p \leq 1$,

$$(\tfrac{|\nu|!}{\nu!} b^\nu)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F}) \text{ iff } \|b\|_{\ell^1(\mathbb{N})} < 1 \ \wedge b \in \ell^p(\mathbb{N}).$$

Moreover,

$$\|(\frac{|\nu|!}{\nu!} b^\nu)\|_{\ell^p(\mathcal{F})} \leq \frac{2}{\eta} \exp\left(\frac{2(1-p)(J(\eta) + \|b\|^p_{\ell^p(\mathbb{N})})}{p^2 \eta}\right),$$

where $\eta := (1 - \|b\|_{\ell^1(\mathbb{N})} a_{min})/2$ and $J(\eta)$ is the smallest positive integer such that $\sum_{j>J} |b_j|^p \leq \frac{\eta}{2}$.

Convergence rates

- semi-discretization in stochastic variable -

Focus on approximation in
L^2 (mean square, maximum likelihood) and L^∞ (worst case).

Tool: Classical Legendre basis $(P_n)_{n\geq 0}$:

$$\|P_n\|_{L^\infty([-1,1])} = P_n(1) = 1. \tag{1}$$

L^2 normalized Legendre Sequence $L_n(t) = \sqrt{2n+1}P_n(t)$,

$$\int_{-1}^{1} |L_n(t)|^2 \frac{dt}{2} = 1.$$

Note

$$L_0 = P_0 = 1.$$

For $\nu \in \mathcal{F}$, recall tensorized polynomials

$$P_\nu(y) := \prod_{j\geq 1} P_{\nu_j}(y_j) \text{ and } L_\nu(y) := \prod_{j\geq 1} L_{\nu_j}(y_j). \tag{2}$$

Note:

- $(L_\nu)_{\nu\in\mathcal{F}}$ orthonormal basis of $L^2(U,d\mu)$, $\mu = \bigotimes_{j\geq 1} \frac{1}{2} dy_j$
- every $u \in L^\infty(U,V,d\mu) \subset L^2(U,V,d\mu)$ admits expansions

$$u(y) = \sum_{\nu\in\mathcal{F}} u_\nu P_\nu(y) = \sum_{\nu\in\mathcal{F}} v_\nu L_\nu(y)\ . \qquad (3)$$

  These converge in $L^2(U,V,d\mu)$ with $u_\nu, v_\nu \in V$ defined by

$$v_\nu := \int_U u(y)L_\nu(y)d\mu(y) \text{ and } u_\nu := \left(\prod_{j\geq 1}(1+2\nu_j)\right)^{1/2} v_\nu. \qquad (4)$$

Estimates of $\|u_\nu\|_V$, $\|v_\nu\|_V$ through complex analysis. Note $\|v_\nu\|_V \leq \|u_\nu\|_V$.

Uniform Ellipticity Assumption: $\mathrm{UEA}(r, R)$
$\exists 0 < r \leq R < \infty$ *such that* $\forall x \in D$ *and* $\forall y \in U$

$$0 < r \leq a(x, y) \leq R < \infty. \tag{5}$$

Define

$$\mathcal{A}_\delta = \{z \in \mathbb{C}^{\mathbb{N}} : \delta \leq \Re(a(x,z)) \leq |a(x,z)| \leq 2R \quad \text{for every} \quad x \in D\} \, . \tag{6}$$

Polydiscs: For $\rho = (\rho_j)_{j\geq 1}$ sequence of radii $\rho_j > 0$ define:

$$\mathcal{U}_\rho := \bigotimes_{j\geq 1} \{z_j \in \mathbb{C} : |z_j| \leq \rho_j\}. \tag{7}$$

When $\overline{a}$ and ψ_j are real valued, $\mathbf{UEA}(r,R) \Longrightarrow \forall x \in D$ and $\forall z \in \mathcal{U}$,

$$0 < r \leq \Re(a(x,z)) \leq |a(x,z)| \leq 2R, \tag{8}$$

$\rho = (\rho_j)_{j\geq 1}$ is δ-*admissible* iff

$$\forall x \in D : \quad \sum_{j\geq 1} \rho_j |\psi_j(x)| \leq \Re(\overline{a}(x)) - \delta. \tag{9}$$

If $\rho = (\rho_j)_{j\geq 1}$ is δ-admissible, then $\mathcal{U}_\rho \subset \mathcal{A}_\delta$.

Lemma 6:
Assume $\mathbf{UEAC}(r, R)$ for some $0 < r \le R < \infty$. Let $\rho = (\rho_j)_{j \ge 1}$ be δ-admissible for some $0 < \delta < r$ with $\rho_j > 1$ for all j such that $\nu_j \ne 0$.

Then for any $\nu \in \mathcal{F}$

$$\|v_\nu\|_V \le \|u_\nu\|_V \le \frac{\|f\|_{V^*}}{\delta} \prod_{j \ge 1, \nu_j \ne 0} \phi(\rho_j)(2\nu_j + 1)\rho_j^{-\nu_j}, \qquad (10)$$

where $\phi(t) := \frac{\pi t}{2(t-1)}$ for $t > 1$.

Theorem 7:
Assume

- $a(x,z)$ satisfies $\mathbf{UEAC}(r,R)$ for some $0 < r \le R < \infty$
- $(\|\psi_j\|_{L^\infty(D)})_{j\ge 1} \in \ell^p(\mathbb{N})$ for some $p < 1$,

Then

$$(\|u_\nu\|_V)_{\nu\in\mathcal{F}}, \quad (\|v_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$$

for the same value of p.

The Legendre expansions (3) converge in $L^\infty(U,V)$:

i.e. $S_{\Lambda_N}u(y) := \sum_{\nu\in\Lambda_N} u_\nu(x)P_\nu(y) = \sum_{\nu\in\Lambda_N} v_\nu(x)L_\nu(y)$ converge

$$\lim_{N\to+\infty} \sup_{y\in U} \|u(y) - S_{\Lambda_N}u(y)\|_V = 0 \tag{11}$$

for $(\Lambda_N)_{N\ge 1} \subset \mathcal{F}$ *any* sequence of finite sets which exhausts $\mathcal{F}$.

Moreover,...

1. L^∞ ("worst case") convergence:

   If Λ_N is the set of $\nu \in \mathcal{F}$ corresponding to indices of N largest $\|u_\nu\|_V$, then holds the convergence estimate

$$\sup_{y \in U} \|u(y) - S_{\Lambda_N} u(y)\|_V \leq \|(\|u_\nu\|_V)\|_{\ell^p(\mathcal{F})} N^{-r}, \quad r = \frac{1}{p} - 1 \geq 0. \tag{12}$$

2. L^2 ("maximum likelihood") convergence:

   If Λ_N is the set of $\nu \in \mathcal{F}$ corresponding to indices of N largest $\|v_\nu\|_V$, then

$$\|u - S_{\Lambda_N} u\|_{L^2(U,V,d\mu)} \leq \|(\|v_\nu\|_V)\|_{\ell^p(\mathcal{F})} N^{-r}, \quad r = \frac{1}{p} - \frac{1}{2} \geq \frac{1}{2}. \tag{13}$$

Legendre coefficient estimate in Lemma 6 and ν-dependent δ-admissible ρ: fix $1 < \kappa \leq 2$ such that

$$(\kappa - 1) \sum_{j \geq 1} \|\psi_j\|_{L^\infty(D)} \leq \frac{r}{8}. \tag{14}$$

Choose J_0 as smallest integer such that

$$\sum_{j > J_0} \|\psi_j\|_{L^\infty(D)} \leq \frac{r(\kappa - 1)}{18\pi\kappa}, \tag{15}$$

Then define

$$\rho_j := \begin{cases} \kappa & \text{if} \quad 1 \leq j \leq J_0, \\ 2 + \frac{r\nu_j}{4|\nu_F| \|\psi_j\|_{L^\infty(D)}}, & \forall j \geq J_0 \text{ such that } \nu_j \neq 0, \\ 1 & \forall j \geq J_0 \text{ such that } \nu_j = 0\,. \end{cases} \tag{16}$$

Then ρ is $\frac{r}{2}$ admissible.

Number Theory (finding Λ_ℓ) - I

Problem: Theorem 7 does *not* give a constructive way of finding sets

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_N \subset \ldots \subset \mathcal{F}$$

1. Adaptive sGFEM, or 2. A-priori selection of the Λ_N.

Consider **2**:
given $b = (b_1, b_2, \ldots) > 0$ s.t. $1 > b_1 \geq b_2 \geq \ldots \geq b_m \to 0$ and $\varepsilon > 0$, define

$$\Lambda_\varepsilon(b) := \{\nu \in \mathcal{F} \quad | \quad b^\nu \geq \varepsilon\} \quad = \quad \{\nu \in \mathcal{F} \quad | \quad b^{-\nu} \leq 1/\varepsilon\}$$

If our bounds on coefficients $\|v_\nu\|_V$ *are sharp,*
$\Lambda_\varepsilon(b)$ *will contain essentially the* $\#\Lambda_\varepsilon(b)$ *'largest coefficients'* $\|v_\nu\|_V$.

Monotonicity: for any b, $\overline{b} \in \ell^\infty(\mathbb{N})$ as above and any $\bar{\varepsilon}, \varepsilon > 0$

1. $M_\varepsilon(b) := \max_{m\in\mathbb{N}}\{b_m \geq \varepsilon\} < \infty$,
2. $\forall \nu \in \Lambda_\varepsilon(b) : \mathrm{supp}(\nu) \subset \{1, 2, \ldots, M_\varepsilon(b)\}$,
3. $\bar{\varepsilon} \geq \varepsilon$ implies $\Lambda_{\bar{\varepsilon}}(b) \subseteq \Lambda_\varepsilon(b)$,
4. $b \preceq \overline{b}$ implies $\Lambda_\varepsilon(b) \subseteq \Lambda_\varepsilon(\overline{b})$.

Number Theory (finding Λ_ℓ) - II

Consider *comparison sequences* $b_\sigma \in \ell^\infty(\mathbb{N})$ such that

$$b \preceq b_\sigma := \{m^{-\sigma} : m = 1, 2, ...\}, \qquad \sigma > 0.$$

Note:

$$b \preceq b_\sigma \quad \Rightarrow \quad \forall \nu \in \mathcal{F} : \qquad b^\nu \leq b_\sigma^\nu$$

Scaling: given $\sigma > 0$, $\varepsilon > 0$, it holds

$$\Lambda_\varepsilon(b_\sigma) = \Lambda_{\varepsilon^{1/\sigma}}(b_1) = \left\{ \nu \in \mathbb{N}_0^{\mathbb{N}} : \prod_{m \in \mathbb{N}} m^{\nu_m} \leq \varepsilon^{-1/\sigma} \right\} .$$

$\nu \in \Lambda_\varepsilon(b_\sigma)$ **iff** $\nu \in \mathcal{F}$ **is multiplicative partition of an integer** $x < \varepsilon^{-1/\sigma}$**.**

Complexity (finding Λ_ℓ)

Proposition 8 (R. Andreev '08)
1. $\Lambda_\varepsilon(b_\sigma)$ can be localized in work and memory growing log-linearly in $\#\Lambda_\varepsilon(b_\sigma)$.

2. As $\varepsilon \to 0$,

$$\#\Lambda_\varepsilon(b_\sigma) \sim x \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}} \quad \text{with} \quad x = \varepsilon^{-1/\sigma},$$

Discretization in D

Smoothness in D:

$$W = \{v \in V : \Delta v \in L^2(D)\} \subset V.$$

Norms:

$$\|v\|_W := \|v\|_V + |v|_W, \qquad |v|_W = \|\Delta v\|_{L^2(D)}.$$

Note: D convex $\Longrightarrow W = H^2(D) \cap V$.

Theorem 9
Assume $f \in L^2(D)$ and $a(x,z)$ satisfies $\mathbf{UEAC}(r,R)$ for $0 < r \leq R < \infty$.

If for some $0 < p < 1$

$$(\|\psi_j\|_{L^\infty(D)})_{j\geq 1} \in \ell^p(\mathbb{N}) \quad \text{and} \quad (\|\nabla \psi_j\|_{L^\infty(D)})_{j\geq 1} \in \ell^p(\mathbb{N})$$

then

$$(\|u_\nu\|_W)_{\nu\in\mathcal{F}},\ (\|v_\nu\|_W)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})\ .$$

Discretization in D

(Nonadaptive(!)) Finite Element Approximation in D:
D Convex $\Longrightarrow$

$$\inf_{v_h \in V_h} \|w - v_h\|_V \le C M^{-\frac{1}{d}} |w|_W .$$

Non-convex polyhedra D: $W \not\subset H^2(D)$.
Convergence as $M = \dim(V_h) \to \infty$ reduced to

$$\inf_{v_h \in V_h} \|w - v_h\|_V \le C_t M^{-t} |w|_W \quad \text{some} \quad 0 < t < \frac{1}{d} \ .$$

Discretization in D

(Nonadaptive(!)) Finite Element Approximation in D:
Theorem 10:
Assume

- FE spaces $(V_h)_{h>0}$ in $D \subset \mathbb{R}^d$ have approximation order $0 < t \leq 1/d$,
- a satisfies $\mathbf{UEAC}(r,R)$ and $(\|\psi_j\|_{W^{1,\infty}(D)})_{j=1}^{\infty} \in \ell^p(\mathbb{N})$ some $0 < p \leq 1$.

Then

(i) With Λ_N the set of indices corresponding to N largest $\|t_\nu\|_V$, there exist finite element spaces V_ν of dimensions M_ν, $\nu \in \Lambda_N$, such that

$$\sup_{y \in U} \|u(y) - \sum_{\nu \in \Lambda_N} \tilde{t}_\nu y^\nu\|_V \leq C N_{dof}^{-\min\{r,t\}}, \quad r := \frac{1}{p} - 1,$$

where $N_{dof} = \sum_{\nu \in \Lambda_N} M_\nu$ and $C = (\bar{C}_t + \|(\|t_\nu\|_V)\|_{\ell^p(\mathcal{F})})\|(\|t_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

(ii) With Λ_N the set of indices corresponding to N largest $\|u_\nu\|_V$, there exist finite element spaces V_ν of dimensions M_ν, $\nu \in \Lambda_N$, such that

$$\sup_{y\in U} \|u(y) - \sum_{\nu\in\Lambda_N} \tilde{u}_\nu P_\nu(y)\|_V \le C N_{dof}^{-\min\{r,t\}}, \;\; r := \frac{1}{p} - 1,$$

where $N_{dof} = \sum_{\nu\in\Lambda_N} M_\nu$ and $C = (\bar{C}_t + \|(\|u_\nu\|_V)\|_{\ell^p(\mathcal{F})})\|(\|u_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

(iii) With Λ_N the set of indices corresponding to N largest $\|v_\nu\|_V$, there exist finite element spaces V_ν of dimension M_ν, $\nu \in \Lambda_N$, such that

$$\|u - \sum_{\nu\in\Lambda_N} \tilde{v}_\nu L_\nu\|_{L^2(U,V,d\mu)} \le C N_{dof}^{-\min\{r,t\}}, \;\; r := \frac{1}{p} - \frac{1}{2},$$

where $N_{dof} = \sum_{\nu\in\Lambda_N} M_\nu$ and $C = (\bar{C}_t^2 + \|(\|v_\nu\|_V)\|^2_{\ell^p(\mathcal{F})})^{\frac{1}{2}}\|(\|v_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

Here, $\tilde{t}_\nu$, $\tilde{u}_\nu$ and $\tilde{v}_\nu$ are V-projections of t_ν, u_ν and v_ν, respectively, onto V_ν.

Example (R. Andreev)

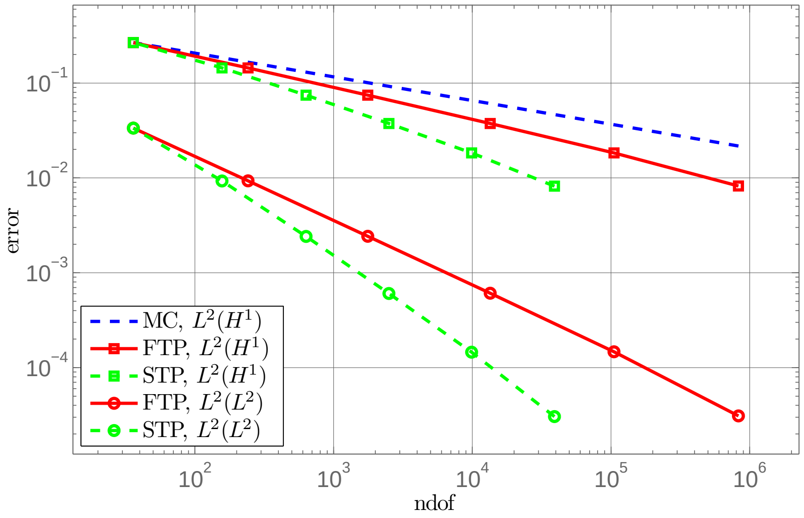
$D = (-1, 1)$, $\mathbb{E}_a(x) = 5 + x$, $C_a(x, x') = \frac{\min\{x,x'\}+1}{2} \in H^{1,1}(D \times D)$.

KL Eigenpairs:

$$\lambda_m = \frac{8}{\pi^2(2m-1)^2}, \qquad \psi_m(x) = \sin((x+1)/\sqrt{2\lambda_m})$$

Algebraic KL decay with rate 2.

Error vs. N_{total}



Conclusion

- gpc: Transform SPDE into parametric, deterministic PDE on $U = (-1,1)^\infty$

- Sparse tensor approximations of
$$B_1(\ell_\infty) \ni y \to u(y,\cdot) \in V$$

- Stochastic Regularity = p-summability of gpc coefficients

- Sparse tensorization of D and Ω Galerkin discretizations: Dimension independent convergence rates in terms of N_{total}.

- requires *mixed regularity* in *both*, D and U: if
$$\{\|\psi_j\|_{W^{1,\infty}(D)}\}_{j=1}^{\infty} \in \ell^p(\mathbb{N}) \quad \text{then} \quad \{\|u_\nu\|_W : \nu \in \mathcal{F}\} \in \ell^p(\mathcal{F})$$

- L^2 (mean square) and L^∞ (worst case) convergence rates.

- Adaptivity essential.

- Diffusion problems in physical dimension $d = 2$, with $\lambda_m \sim m^{-2.2}$ and supp$\Lambda \subset \{1, ..., 1500\}$ on PC (R. Andreev, M. Bieri and CS (SISC 2010)) (superior to Smolyak construction).

- Sparse collocation on input adapted ‘lattices’ of points in $U = (-1, 1)^\infty$ (M. Bieri Dissertation (2009) and SISC (2010))

- Lognormal Case (C.J. Gittelson ($M3AS$ 2010))

- Other PDEs: Heat and Wave Eqn. (C.S. & V. Hoang 2010)

- Fully Adaptive, Convergent Algorithm (Cohen, DeVore & C.S.)

References

N. Wiener (Am. J. Math. 1938), Cameron and Martin (Ann. Math. 1947)

G. E. Karniadakis et al. ($\sim$ 2000 -),

Babuška, Nobile, Tempone etal. (SINUM 2004, 2006, 2008) (CMAME 2005)

P. Frauenfelder, R.A. Todor and CS (CMAME 2005)

R.A. Todor (SINUM 2005 and Diss. ETH 2005)

C.S. and R.A. Todor (JCP 2006 and IMAJNA 2007)

R. Andreev, M. Bieri and CS (SISC 2010)

M. Bieri (SINUM 2010)

V. H. Hoang and C.S. (2010)

A. Cohen, R. DeVore and C.S. (2009) (2010)

www.sam.math.ethz.ch/ reports