Some PDE methods in mean field games theory
Cours FSMP

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We recalled the notion of *renormalized solution* for second order problems. This notion, originally introduced by Di Perna and Lions [1] for transport equations, was later extensively used for second order problems with data belonging to L^1, following [2]. In fact, in case of L^1 data, the usual H^1 energy setting is no more available for the solution, but remains valid for its truncations. However, compared with the transport equation, one needs to add a condition in the formulation in order to recover some crucial information on the set where the solution blows-up.

Let us recall briefly this definition for the case of L^1-data. First we define, for $k > 0$, the truncation function

$$T_k(s) := \min(k, \max(s, -k)) . \tag{1}$$

A function $v \in L^1(Q_T)$ is a renormalized solution to

$$\begin{cases} -v_t - \Delta v = f & \text{in } Q_T, \\ v(T) = v_T & \text{in } \Omega, \end{cases} \tag{2}$$

with, for example, Dirichlet conditions $v_{|_\Sigma} = 0$, if

$$(i) \qquad T_k(v) \in L^2((0,T); H_0^1(\Omega)) \qquad \forall k > 0$$

$$(ii) \qquad \lim_{n\to\infty} \frac{1}{n} \iint_{\{(t,x)\,: n<|v(t,x)|<2n\}} |Dv|^2 \, dxdt = 0$$

$$(iii) \qquad \begin{cases} -\partial_t S(v) - \Delta S(v) = S'(v) f - S''(v)|Dv|^2 & \text{in } Q_T \\ S(v)(T) = S(v_T) & \text{in } \Omega \end{cases}$$

for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support.

The regularity required for the truncations and the fact that S' has compact support give sense to the renormalized equation (iii) as usual, namely, $S(v) \in L^2((0,T); H_0^1(\Omega))$ and

$$\int_0^T\!\!\int_\Omega S(v)\partial_t\varphi \, dxdt + \int_0^T\!\!\int_\Omega DS(v) D\varphi \, dxdt =$$
$$\int_0^T\!\!\int_\Omega S'(v) f \varphi \, dxdt - \int_0^T\!\!\int_\Omega S''(v)|Dv|^2 \varphi \, dxdt + \int_\Omega S(v_T)\varphi(T) \, dx$$

for every $\varphi \in C_c^\infty((0,T] \times \Omega)$.

Notice that condition (ii) preserves, asymptotically, the character of weak solution. Precisely, we define the following sequence of auxiliary functions:

$$S_n(r) = n\,S\left(\frac{r}{n}\right)\,,\ \text{where } S(r) = \textstyle\int_0^r S'(r)dr, \quad S'(r) = \begin{cases} 1 & \text{if } |s| \le 1 \\ 2-|s| & \text{if } 1 < |s| \le 2 \\ 0 & \text{if } |s| > 2 \end{cases} \tag{3}$$

so that S_n converges to the identity locally uniformly as $n \to \infty$. Since $S_n'' = -\frac{1}{n}\mathrm{sign}(r)\,\chi_{\{n<|r|<2n\}}$, if we take $S = S_n$ in (iii) and let $n \to \infty$, thanks to (ii) we recover the weak formulation.

If $\Omega = \mathbb{T}^N$, or if Neumann boundary conditions are required, the same definition extends replacing $H^1_0(\Omega)$ with $H^1(\Omega)$, and test functions are allowed to be in $C^\infty_c((0,T] \times \overline{\Omega})$.

The above renormalized setting extends to the study of the Fokker-Planck equation

$$\begin{cases} \partial_t m - \Delta m - \mathrm{div}(m\,b) = 0\,, & \text{in } (0,T)\times\Omega, \\ m(0) = m_0 & \text{in } \Omega \end{cases} \tag{4}$$

whenever $b \in L^2(m)$, i.e. $m|b|^2 \in L^1(Q_T)$. This context for the study of (4) was extensively discussed in our previous lesson (26/03/2015).

A function $m \in L^1(Q_T)_+$ is a renormalized solution to (4), complemented for example with Dirichlet conditions $m_{|_\Sigma} = 0$, if

$$T_k(m) \in L^2((0,T); H^1_0(\Omega)) \qquad \forall k > 0 \tag{5}$$

$$\lim_{n\to\infty} \frac{1}{n} \iint_{\{n<|m|<2n\}} |Dm|^2\,dxdt = 0 \tag{6}$$

$$\begin{cases} \partial_t S(m) - \Delta S(m) - \mathrm{div}(S'(m)m\,b) + S''(m)|Dm|^2 + S''(m)mbDm = 0 & \text{in } Q_T, \\ S(m)(0) = S(m_0) & \text{in } \Omega, \end{cases} \tag{7}$$

for every $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support.

Two main remarks are immediately in order. First of all, when we use (7) with $S = S_n$, defined in (3), we notice that

$$\|S_n''(m)|Dm|^2\|_{L^1} + \|S_n''(m)mbDm\|_{L^1} \le \frac{C}{n} \iint_{\{n<|m|<2n\}} |Dm|^2\,dxdt + C \iint_{\{n<|m|<2n\}} \frac{m^2}{n}|b|^2\,dxdt \to 0$$

thanks to (6) and since $m|b|^2 \in L^1(Q_T)$. Therefore, a renormalized solution satisfies

$$\partial_t S_n(m) - \Delta S_n(m) - \mathrm{div}(S_n'(m)m\,b) = \omega_n \quad \text{in } Q_T, \text{ and } \omega_n \overset{L^1(Q_T)}{\to} 0. \tag{8}$$

This implies that any bounded function in $L^2(0,T;H^1_0(\Omega)$ can be used as test function and in particular allows one to work with integral estimates, going through the renormalized formulation. Secondly, the renormalized formulation implies that $S(m) \in L^2(0,T;H^1_0(\Omega)$ with $S(m)_t \in L^2(0,T;H^{-1}(\Omega)$, which implies that $S(m) \in C^0([0,T];L^1(\Omega)$ (see [3]). This will readily bring the continuity of m into L^1.

Finally, the rest of the lesson is devoted to prove the existence and uniqueness of weak solutions to the mean field games system

$$\begin{cases} -\partial_t u - \nu\Delta u + H(t,x,Du) = F(t,x,m)\,, & (t,x) \in (0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \operatorname{div}(mH_p(t,x,Du)) = 0\,, & (t,x) \in (0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x,m(T))\,, & x \in \Omega \end{cases}$$

under general conditions on the functions f, G and H, and possibly different boundary conditions. The results presented are those contained in the work [3].

REFERENCES

[1] R. J. Di Perna and P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. math. **98** (1989), 511–547.

[2] P.-L. Lions and F. Murat, *On renormalized solutions for nonlinear elliptic equations*, unpublished.

[3] A. Porretta, *Weak solutions to Fokker-Planck equations and mean field games*, Arch. Rational Mech. Anal. 216 (2015), 1-62.