Whitham's Modulation Equations and Stability of Periodic Wave Solutions of the Korteweg-de Vries-Kuramoto-Sivashinsky Equation

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ABSTRACT. We study the spectral stability of periodic wave trains of the Korteweg-de Vries-Kuramoto-Sivashinsky equation which are, among many other applications, often used to describe the evolution of a thin liquid film flowing down an inclined ramp. More precisely, we show that the formal slow modulation approximation resulting in the Whitham system accurately describes the spectral stability to side-band perturbations. Here, we use a direct Bloch expansion method and spectral perturbation analysis instead of Evans function computations. We first establish, in our context, the now usual connection between first-order expansion of eigenvalues bifurcating from the origin (both eigenvalue 0 and Floquet parameter 0) and the first-order Whitham's modulation system: the hyperbolicity of such a system provides a necessary condition of spectral stability. Under a condition of strict hyperbolicity, we show that eigenvalues are indeed analytic in the neighborhood of the origin and that their expansion up to second order is connected to a viscous correction of the Whitham's equations. This, in turn, provides new stability criteria. Finally, we study the Korteweg-de Vries limit: in this case, the domain of validity of the previous expansion shrinks to nothing and a new modulation theory is needed. The new modulation system consists of the Korteweg-de Vries modulation equations supplemented with a source term: relaxation limit in such a system provides, in turn, some stability criteria.

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1. INTRODUCTION

Coherent structures such as solitary waves, fronts, or periodic traveling waves usually play an essential role as elementary processes in nonlinear phenomena. It is both usual and useful to try first to analyze the behavior of these elementary structures with canonical models for pattern formation [8]. Here, for such a canonical equation, we try to relate side-band stability of periodic traveling waves with modulation-averaged equations.

We focus our attention on a scalar equation of the form

$$\partial_t u + 6u\,\partial_x u + \delta_1\,\partial_x^2 u + \delta_2\,\partial_x^3 u + \delta_3\,\partial_x^4 u = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+^*,$$

where δ_1, δ_2, and δ_3 are some constant real numbers. This kind of equation arises in many situations as a simplified asymptotic equation. For this purpose, when $\delta_1 < 0$, it is often sufficient to set $\delta_2 = \delta_3 = 0$, that is, to consider a viscous Burgers' equation. In the limit case $\delta_1 = 0$, if $\delta_2 \neq 0$, for some purposes one may also set $\delta_3 = 0$ and work with the Korteweg-de Vries equation (KdV). However, for well-posedness issues, when $\delta_1 < 0$, one cannot stop before a fourth-order term[1] and stop there only if $\delta_3 > 0$. It is this latter case we are interested in here. For this reason, the equation we study incorporates nonlinearity, dispersion, and (insofar as we consider it with respect to constant states) dissipative instability with respect to low frequency perturbations, and stability to high frequencies.

We now perform some scaling transformations to make the structure of the equation clearer. First, up to changing (x,u) into $(-x,-u)$, we may assume $\delta_2 \geq 0$. Then, if δ_1, δ_2 and δ_3 are positive, up to changes $t = \delta_2(\delta_1/\delta_3)^{3/2}t$, $x = (\delta_1/\delta_3)^{1/2}x$, $u = \delta_2^{-1}(\delta_1/\delta_3)^{-1}u$, and $\delta = \delta_2^{-1}(\delta_1/\delta_3)^{-1/2}\delta_1$, the above equation may be recast as

$$\partial_t u + 6u\,\partial_x u + \partial_x^3 u + \delta(\partial_x^2 u + \partial_x^4 u) = 0. \tag{1.1}$$

Note that we have ruled out the relevant case $\delta_2 = 0$, $\delta_1 > 0$, $\delta_3 > 0$, corresponding to the Kuramoto-Sivashinsky equation. Yet, in reference to this equation, we call equation (1.1) the Korteweg-de Vries-Kuramoto-Sivashinsky equation (KdV-KS). The above scaling was intended to shed light on the crucial role of the above δ_1 parameter. Moreover, with pattern formation in mind, we have also ensured that, about any constant state, the linearly most unstable mode has frequency ± 1. Setting $\delta = 0$, the most common form of the Korteweg-de Vries is recovered.[2] It should be clear now that the equation enters naturally in the description of weakly-nonlinear large-scale waves above a threshold where all constant states become unstable to low-frequency perturbations, a threshold corresponding to $\delta = 0$.

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[1] In some situations, one may argue then that a term $D\partial_x^2(u^2)$ should be taken into account [1], but we disregard this term here.

[2] This is the reason why we have chosen not to eliminate the 6 factor.

Though the Kuramoto-Sivashinsky equation was first derived to study chemical reactions and flame stability [17, 18, 22, 23], and though equation (1.1) (being of canonical nature) has been widely used to describe plasma instabilities, flame front propagation, and turbulence in reaction-diffusion systems, for a short time we now focus the discussion on a personal interest of the authors, namely, the evolution of nonlinear waves in fluid mechanics. A detailed description of these applications may be found in [6]; here, we say only a few words about the threshold of stability. When using free-surface incompressible Navier-Stokes equations to analyze the evolution of a thin fluid flow down an incline plane of a given slope, a critical Reynolds number appears, above which, flows parallel to the incline become unstable. Above but close to this critical Reynolds number, equation (1.1) may be used to describe the dynamics, with δ being proportional to the deviation of the Reynolds number from its critical value [26]. When we model the same shallow-flow situation with the Saint-Venant equations instead, δ then corresponds to the deviation of the Froude number from its critical value [27]. We remark that the classic Kuramoto-Sivashinsky equation, not considered here, would instead correspond to modeling of a *vertical* film flow [6, 24].

Related to the instability of constant states is the fact that, for any fixed $\delta > 0$, a family of periodic traveling waves bifurcates from constant states through a Hopf bifurcation. This family is enclosed by a family of solitary waves (see [2, 7] for a detailed description of phase portraits[3]). Since, by nature, all constant states are spectrally unstable, all solitary waves have unstable essential spectrum, and small and large periodic waves are expected to be unstable. Yet periodic waves of intermediate periods or some arrays of solitary waves are not *a priori* forbidden from being stable. For discussion of the latter case, see [15, 20]. As for the former, note that numerical studies [2, 7] indeed show that, for any δ, even up to the Kuramoto-Sivashinsky equation, in parameter space a full band of spectrally stable periodic wave trains does exist. Note also that, under precise assumptions of diffusive spectral stability, one may go from spectral stability to nonlinear stability under localized perturbation [2] (see also announcement of this work in [3]).

To be more precise, let us say that the spectral stability we discussed above is under arbitrary bounded perturbations. Being related to co-periodic stability, the stability we discuss in the rest of this work—side-band stability—is of weaker nature. Many numerical works and even experiments have been focused on the even weaker requirement of stability under co-periodic perturbations. Yet, to be fully significant from a realistic point of view, these studies should be at least extended to an analysis of stability under close-to-coperiodic perturbations,[4] or, in the words

[3] We note, by the way, that our analysis is intended to deal with the family of periodic wave trains that persist for all values of δ, and not with those that exist close enough to the Kuramoto-Sivashinsky equation (i.e., for δ large enough).

[4] That is, perturbations given in a co-moving frame making the wave both stationary and periodic of period one, by $x \mapsto e^{i\xi x}\check{u}(x)$ with $\check{u}$ co-periodic, square-integrable on $[0, 1]$, and ξ *small*. Recall that Floquet theory tells us that, relaxing the smallness of ξ to $\xi \in [-\pi, \pi]$, the full spectrum is attained with such perturbations. We call ξ a Floquet or a Bloch parameter.

of the analysis below, by an analysis of the low-Floquet stability (see the definition of the Bloch transform in (2.5), where the Floquet parameter is ξ). This is the well-known issue of *side-band* stability/instability. Determining side-band stability should indicate for which waves stability to co-periodic perturbations may be observed, whereas side-band instability yields, in any case, spectral instability. Obviously, a side-band analysis is required only where co-periodic spectrum intersects the imaginary axis, thus only when co-periodic stability is marginal; but due to translational and Galilean invariance, the former intersection is always nonempty, 0 being an eigenvalue of algebraic multiplicity larger than or equal to 2.

The present work is restricted to the discussion of this side-band stability. Actually, the role of dynamics of small Floquet perturbations is far greater than that of deciding robustness of co-periodic predictions. For spectrally stable waves, it is expected and observed that the full nonlinear dynamic is driven by the evolution of the small Floquet part of the solution. This is because we expect that, for stable waves, the marginally stable part of the spectrum corresponds to the above mentioned modes, a two-dimensional space of co-periodic perturbations generated from translational and Galilean invariance. It is thus of principal importance to understand how this space bifurcates for low Floquet parameters. More generally, for spectrally stable periodic traveling waves, the general aim (thanks to a suitable parametrization of close-by periodic traveling waves) is to reduce at leading-order the nonlinear long-time dynamic about a periodic traveling wave to the nonlinear dynamic of parameters about a constant state—hence the importance of *phase dynamics* [9] (when only one parameter is involved) or (more generally) of modulation dynamics [25]. Note that the averaging modulation process associates low Floquet modes of the full solution to low Fourier modes of parameters. Therefore, one expects for stable waves that the long-time dynamic will be of *slow* modulation type: that is, in some local scale, the behavior of a periodic traveling wave type, but with parameters evolving on larger scales.

The purpose of deriving modulation-averaged systems (which we shall call Whitham's systems) is to propose approximate equations for the evolution of the parameters. The type of modulation system depends then on the desired degree of accuracy. Note that the dimension of the system is not naively deduced from the dimension of the equation, but is one of the family of periodic traveling waves (considered up to translation invariance). In our case, we will assume the generic situation where the family of periodic traveling is parameterized by phase shift, wave number k, and mean M (see assumption (A') below), so that we are looking for a system describing the slow evolution of (k, M).

Our goal is not to decide side-band stability but to relate it to properties of some modulation-averaged systems, and, when side-band stability is met, to elucidate where one can read both sets of parameters that are crucial for the description of the evolution of (k, M): namely, linear group velocities and diffusion coefficients. For this purpose, we derive on a formal level two kinds of modulation systems: at leading-order, a system involving only first-order derivatives of (k, M); and at second-order, correction of the previous system. We then validate these systems at the spectral level. We prove that the hyperbolicity of the first-order Whitham's system is a necessary condition for side-band stability, and that, when hyperbolicity is met, Fourier eigenvalues of the system yield linear group velocities. Assuming strict hyperbolicity of the first-order system, we then prove that second-order expansions of low-Fourier eigenvalues of the second-order system yield second-order expansions of low-Floquet eigenvalues of the original system. This means that stability with respect to low-frequency perturbations in the second-order modulation system[5] is a necessary condition for side-band stability of the original system, and, when it is fulfilled, it comes with the desired diffusion coefficients. Moreover, when both hyperbolicity and diffusion conditions are satisfied in a strict sense, they imply side-band stability. Stated more briefly, when taken in a strong sense, co-periodic stability plus modulational stability yields side-band stability.

The role of hyperbolicity of the first-order Whitham's system is now well known [21]. Yet, instead of relying on the Evans function calculation [21], we reprove it here by first showing regularity of critical eigenmodes with respect to Floquet parameter, and then directly inspecting low-Floquet expansions. The main advantages of this approach are that we can easily raise the order of approximation, and that this method also provides in a natural way relation among eigenfunctions (not only eigenvalues). Although we do not address the question here but rather postpone it to further work, recall that a final goal is to prove that the full nonlinear evolution may indeed be "reduced" to the evolution of the second-order system mentioned above. For this purpose, description of eigenfunctions is crucial.

For small δ,[6] the above scenario yields poor stability results. Indeed, a close inspection reveals that the fixed-δ low-Floquet expansions we discussed actually involve not only ξ but also ξ/δ, so that the low-Floquet stability mentioned above should be read in the $\delta \to 0$ limit as stability to perturbations corresponding to Floquet parameters ξ such that $|\xi| \leq c \min(\{1, \delta\})$, with c a small fixed constant. But, if we focus on the Korteweg-de Vries equation, there is room for improvement in this small-δ limit. In particular, from the Fenichel-type geometric singular perturbation analysis in [10], we already know that profiles of periodic wave trains of (1.1) perturb for small δ in a regular way from some wave trains of the KdV equation. To show how the ratio δ/ξ enters in the way the small Floquet limit $\xi \to 0$ interplays with the KdV limit $\delta \to 0$, we perform a modulation procedure where this ratio is fixed. This is accomplished by setting $\delta = \varepsilon\bar{\delta}$, where $\bar{\delta}$ is some fixed nonnegative number and ε is the inverse of the characteristic scale on which local parameters evolve, that is, $\varepsilon \sim |\xi|$, where ξ is a Fourier parameter for the Whitham's system, a Floquet parameter for (1.1).

[5]On a formal level, *a priori* many modulation-averaged systems with possibly different high-frequency behaviors may be derived to match the *slow* modulation evolution—hence the relevance for the original system of only a low-frequency Kawashima-type condition: low-frequency hyperbolic modes should be viscously damped by the second-order part of the modulation system.

[6]Recall that the zone δ small is precisely the region we are most interested in.

The new modulation procedure leads to a system of dimension three (and not two), reflecting the dimension of the family of periodic traveling waves for (KdV). Not all periodic wave trains of (KdV) provide traveling waves of (1.1). Indeed, a selection criterion rules out one dimension. Though we derived this system in a slightly different way, when $\bar{\delta} = 0$, we recover the first-order Whitham's system of (KdV) [25]. For $\bar{\delta} \neq 0$, this system is modified by a relaxation term of the form $\bar{\delta} \times$ (selection criterion). Constant solutions of the full Whitham's system correspond to wave trains of (KdV) actually generating periodic waves for (1.1). Moreover, the relaxation limit $\bar{\delta} \to \infty$ limit in the Whitham's system yields back the limit $\delta \to 0$ of the Whitham's system of the fixed-δ modulation.

The relaxation structure suggests that the natural conditions yielding side-band stability for small δ are the following:

- Hyperbolicity of the Whitham's system for (KdV);
- Dissipative nature of the relaxation term;
- Hyperbolicity of the limit $\delta \to 0$ of the Whitham's system for (1.1);
- Intertwining of linear group velocities.

We formulate these subcharacteristic conditions in a precise way in the section dedicated to them. Yet we do not enter into a precise analysis of their role. The first reason for this is that it is not clear to us whether or not a full coherent expansion may be obtained in a full region (ξ, δ) small, and it would be very unsatisfactory to discuss a result for moderate δ/ξ. The second reason is that, nevertheless, we hope in a further work to combine a detailed low-Floquet analysis with regular expansions of moderate-Floquet modes, and energy estimates for high-Floquet modes[7] in order to prove (for small δ) a full diffusive spectral stability for a band of periodic traveling waves. These subcharacteristic conditions should play a key role in the low-Floquet analysis, yet the present discussion does not reveal in what sense this will occur.

The organization of the rest of the paper follows the lines of the previous discussion. In Section 2, we state the main assumptions under which the fixed-δ analysis holds. In particular, we describe how periodic wave trains of (KdV-KS) are parameterized. In Section 3, keeping δ fixed, we discuss side-band stability and modulation-averaged equations. There we examine the regularity of eigenmodes with respect to Floquet parameter ξ, and perform expansions with respect to this parameter. Following this, we derive the Whitham's modulation systems, and show that modulation equations linearized about fixed parameters provide the correct expansion of critical eigenvalues, up to order one for the first-order Whitham's system, and up to second order for the second-order modulation system. In Section 4, to go beyond the restriction ξ/δ small in the KdV limit $\delta \to 0$, we introduce another set of modulation equations, which turns out to be the

[7] Of course, $\xi \in [-\pi, \pi]$ is bounded, but *high* should be understood as high compared with δ.

Whitham's modulation system for (KdV) supplemented by a source term of order $\bar{\delta} \sim \delta/|\xi|$; we then discuss for this system the subcharacteristic conditions deduced from relaxation theory for hyperbolic systems.

2. Preliminaries

Parametrization. We here begin discussing parametrization of periodic traveling waves. The final form of the latter is deferred until the end of the section.

We first look for periodic traveling waves u of (1.1), with speed c and period X, as $u(x,t) = U(x-ct)$, where U is an X-periodic function that satisfies

$$-cU' + 6UU' + U''' + \delta(U'' + U'''') = 0. \tag{2.1}$$

Integrating once introduces a constant of integration q, and turns (2.1) into

$$-cU + 3U^2 + U'' + \delta(U' + U''') = q. \tag{2.2}$$

Denote $U(\cdot\,; b, c, q)$ the maximal solution of (2.2) such that

$$(U, U', U'')(0; b, c, q) = b,$$

and let $O_{c,q,X}$ be the open set of b values such that $U(\cdot\,; b, c, q)$ is defined on $[0, X]$. We may then define $D = \bigcup_{c,q,X} O_{c,q,X} \times \{(c,q,X)\}$ and, on this open subset of $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the Poincaré return map H as

$$\begin{aligned} H : D &\to \mathbb{R}^3 \\ (b, c, q, X) &\mapsto (U, U', U'')(X; b, c, q) - b. \end{aligned}$$

With a slight abuse of notation, $H^{-1}(\{0\})$ represents the set of all periodic traveling waves of (1.1). Fixing $(\bar{b}, \bar{c}, \bar{q}, \bar{X}) \in H^{-1}(\{0\})$, corresponding to some nontrivial wave, we make the following assumption on the (local) structure of $H^{-1}(\{0\})$:

(A) The map H is full rank at $(\bar{b}, \bar{c}, \bar{q}, \bar{X})$.

By the Implicit Function Theorem and translational invariance, condition (A) implies that the set of periodic solutions in the vicinity of the $\bar{X}$-periodic function $\bar{U} = U(\,\cdot\,; \bar{b}, \bar{c}, \bar{q})$ forms a smooth three-dimensional manifold

$$\{(x,t) \mapsto U(x - \alpha - c(\beta)t; \beta) \mid (\alpha, \beta) \in \mathbb{R} \times O\}, \tag{2.3}$$

where O is an open subset of $\mathbb{R}^2$. The question of the parametrization of periodic wave trains is important here in order to derive modulation equations. In view of (2.2), one could wish to look for (X, q) or (c, q) as natural parameters for periodic wave trains. However, these parameterizations may not be admissible; in addition, from an averaging point of view, (X, M) (or (k, M) with k a wavenumber) where $M = \langle U \rangle = X^{-1}\int_0^X U(x)\,\mathrm{d}x$ is the mean value seems a more adequate choice. Moreover, for the present profile equation (2.1), Galilean invariance makes parametrization by M easier. If (U, c) is a solution, so is $(U - M, c - 6M)$ for any $M \in \mathbb{R}$, so that in some sense, as far as parametrization (and not spectral properties) is concerned, the mean value may be factored out. As a consequence, one may search first for periodic wave trains of (2.1) satisfying the additional constraint $\langle U \rangle = X^{-1}\int_0^X U(x)\,\mathrm{d}x = 0$. Denoting by $U^{(0)}(\cdot\,;k)$, $c^{(0)}(k)$ a mean-free solution of (2.1), with period $X = k^{-1}$, one recovers (locally in parameter space and up to translations) the full family of periodic wave trains, now denoted $U(\cdot\,;k, M)$, $c(k, M)$, by setting

$$U(\cdot\,;k, M) = M + U^{(0)}(\cdot\,;k), \quad c(k, M) = 6M + c^{(0)}(k).$$

To make the parametrization of practical use (particularly in combination with the Bloch transform), we still have to normalize wave profiles' periods to one so that, although we do not modify names, $U(\cdot\,;k, M)$ will now denote a periodic function of period 1 and mean M such that a solution to (1.1) is given by $u(x, t) = U(k(x - c(k, M)t); k, M)$. For later references, let us write explicitly that $U(\cdot\,;k, M)$ is thus a solution, of period 1, of

$$k(3U^2 - c(k, M)U)' + k^3U''' + \delta(k^2U'' + k^4U'''') = 0, \quad \langle U \rangle = M \tag{2.4}$$

where $\langle U \rangle = \int_0^1 U$.

For the discussed reasons, we strengthen (A) into the assumption below:

(A') The map H is full rank at $(\bar{b}, \bar{c}, \bar{q}, \bar{X})$, and a parametrization by (k, M) is admissible.

Bloch transform. We will not need much about the Bloch transform, but it underlies our spectral expansions. The Bloch transform rewrites the Fourier transform in view of Floquet's theory. It groups together Fourier modes corresponding to the same Floquet parameter. Explicitly, the Bloch transform $\check{g}$ of a Schwartz-class function g is given by

$$\check{g}(\xi, x) := \sum_{j\in\mathbb{Z}} \hat{g}(\xi + 2j\pi)e^{i2\pi jx}, \tag{2.5}$$

where $\hat{g}(\xi) := (1/(2\pi))\int_{\mathbb{R}} e^{-i\xi x} g(x)\,\mathrm{d}x$ is the Fourier transform of g. The Bloch transform comes with an inverse formula

$$g(x) = \int_{-\pi}^{\pi} e^{i\xi x}\check{g}(\xi, x)\,\mathrm{d}\xi \tag{2.6}$$

and a Parseval equality $\|g\|_{L^2(\mathbb{R})} = (2\pi)^{1/2}\,\|\check{g}\|_{L^2([-\pi,\pi]\times[0,1])}$. A consequence is that we may write any $g \in L^2(\mathbb{R})$ as a "superposition" of functions $x \mapsto$

$e^{i\xi x}\check{g}(\xi, x)$ with $\check{g}(\xi,\cdot)$ of period 1. This transform is well behaved with respect to a differential operator with *periodic* coefficients.

Let L be the linear operator arising from the linearization of (1.1) about a wave $\bar{U}$, in a co-moving frame $(\bar{k}(x-\bar{c}t), t) : \partial_t v = L[v]$. Thus L is the differential operator with coefficients of period 1:

$$L[g] = -\bar{k}((6\bar{U}-\bar{c})g)' - \bar{k}^3 g''' - \delta(\bar{k}^2 g'' + \bar{k}^4 g'''').$$

The Bloch transform "diagonalizes" L, in the sense that

$$(Lg)(x) = \int_{-\pi}^{\pi} e^{i\xi x}(L_\xi \check{g}(\xi,\cdot))(x)\,\mathrm{d}\xi \tag{2.7}$$

that is, $(\check{Lg})(\xi, x) = (L_\xi \check{g}(\xi,\cdot))(x)$, where L_ξ represents differential operators with coefficients of period 1, acting on *periodic* functions *via*

$$(L_\xi w)(x) := e^{-i\xi x} L[e^{i\xi\cdot} w(\cdot)](x). \tag{2.8}$$

Thus the Bloch transform reduces L to operations on co-periodic functions.

Since the Bloch operator-valued symbols L_ξ are relatively compact perturbations of the same operator L_0 with domain $H^4_{\mathrm{per}}([0,1])$ compactly embedded in $L^2_{\mathrm{per}}([0,1])$, their spectrum consists entirely of discrete eigenvalues which, further, depend continuously on the Bloch parameter ξ. Then, by standard considerations relying on inverse formula and Parseval identity, we have

$$\sigma_{L^2(\mathbb{R})}(L) = \bigcup_{\xi\in[-\pi,\pi)} \sigma_{L^2_{\mathrm{per}}([0,1])}(L_\xi);$$

see [12] for details. As a result, the spectrum of L, which consists entirely of essential spectrum, may be decomposed into countably many continuous curves $\lambda(\cdot)$ such that, for $\xi\in[-\pi,\pi)$, $\lambda(\xi)$ is an eigenvalue of L_ξ.

Structure of the kernel. In what follows, we focus only on low Bloch numbers. As a starting point, we need to determine the spectrum of L_0. The purpose of the following lemma is to decide what can be deduced from assumption (A').

Lemma 2.1. *The following assertions are equivalent:*

(1) *Assumption* (A') *is fulfilled.*
(2) *The value* 0 *is an eigenvalue of* L_0 *of geometric multiplicity* 1 *and algebraic multiplicity* 2*. Moreover, the generalized* 0*-eigenspace is determined by*

$$\bar{U}' \in \mathrm{Ker}(L_0), \quad 1 \in \mathrm{Ker}(L_0^2)\setminus \mathrm{Ker}(L_0), \quad 1 \in \mathrm{Ker}(L_0^*)$$

where L_0^* *denotes the formal adjoint of* L_0*.*

One should not be confused by the fact that constant functions appear in two different places for two different reasons. The relation to the adjoint stems from the fact that we deal with a conservation law. The other reason is related to $\partial_M \bar{U} = 1$.

As a preliminary remark, note that, from the fact that L_0 has compact resolvents, one deduces that L_0 is a Fredholm operator[8] of index 0, and that for any positive integer k, dimensions of $\mathrm{Ker}(L_0^k)$ and $\mathrm{Ker}((L_0^*)^k)$ coincide.

Proof. Let us first assume a (k, M) parametrization. From translational invariance, one readily deduces (differentiating the profile equation with respect to the α of (2.3), having in mind that c and k do not depend on α) that $\bar{U}'$ lies in the null space of L_0. We claim $\mathrm{Ker}(L_0) = \mathrm{span}(\bar{U}')$.

Let φ be a 1-periodic function satisfying $L_0\varphi = 0$. The equation yields that φ is necessarily smooth. Then, there is a q_φ such that, with obvious notation,

$$\mathrm{d}H(\bar{U}(0), \bar{U}'(0), \bar{U}''(0), \bar{c}, \bar{q}, \bar{X}) \cdot (\varphi(0), \varphi'(0), \varphi''(0), 0, q_\varphi, 0) = (0, 0, 0).$$

This means that there exist $(\alpha_\varphi, k_\varphi, M_\varphi)$ such that

$$\begin{cases} \varphi = \alpha_\varphi \bar{U}' + k_\varphi \, \partial_k \bar{U} + M_\varphi \, \partial_M \bar{U}, \\ 0 = k_\varphi \, \partial_k \bar{c} + M_\varphi \, \partial_M \bar{c}, \\ q_\varphi = k_\varphi \, \partial_k \bar{q} + M_\varphi \, \partial_M \bar{q}, \\ 0 = -k_\varphi \, \dfrac{1}{\bar{k}^2}. \end{cases}$$

Thus $k_\varphi = 0$. However, differentiating the profile equation with respect to M yields $L_0[\partial_M \bar{U}] = -\bar{k} \, \partial_M \bar{c} \bar{U}'$. Since $\partial_M c = 6$ and the wave is nontrivial, this gives $M_\varphi = 0$, and our claim is proved.

Next, since $\partial_M \bar{U} = 1$, we have already proved $L_0[1] = -6\bar{k}\bar{U}'$. Thus 0 is necessarily an eigenvalue of algebraic multiplicity larger than 1. To see that it is exactly of algebraic multiplicity 2, it remains to prove that 1 does not belong to the range of L_0. Yet, since (1.1) is a conservation law, the constant function 1 lies in the kernel of L_0^*. Of course, $\langle 1, 1 \rangle_{L^2([0,1])} = 1 \neq 0$, hence the result. Note that $\langle 1, \partial_M \bar{U} \rangle_{L^2([0,1])} = \partial_M \bar{M} = 1$ would still stand, even for equations for which $\partial_M \bar{U}$ is not explicitly known.

Now let us assume the announced structure of the kernels. This implies that the kernel of $\mathrm{d}H(\bar{U}(0), \bar{U}'(0), \bar{U}''(0), \bar{c}, \bar{q}, \bar{X})_{|\mathbb{R}^3 \times \{0\} \times \mathbb{R} \times \{0\}}$ is of dimension at most 1, and thus its range is of dimension at least 3. This gives assumption (A), but also the possibility of a parametrization of type (2.3) with $\beta = (c, X)$, and hence also with $\beta = (c, k)$. Now observe that, with this parametrization, $\partial_c M = \frac{1}{6} \neq 0$. A more robust argument involving Evans function computations may also replace this simple observation. This completes the proof. ◻

[8] Such is true when it is considered as a bounded operator defined on its domain, as is usual for unbounded operators.

Note that in the previous proof, scalar products always refer to the canonical scalar product on $L^2_{\text{per}}([0,1])$. From now on, this will be denoted by $\langle\cdot,\cdot\rangle$.

3. Modulational Stability

We inspect now how the eigenvalues of L_ξ may bifurcate from $(\lambda,\xi)=(0,0)$, and we then relate this with Whitham's averaged equations for small-Floquet perturbations of periodic wave trains. We first prove the regularity of the two eigenvalues bifurcating from the origin $(\lambda,\xi)=(0,0)$, and compute an expansion up to order 2 with respect to ξ. Then, we show that, up to first order, they are given by the dispersion relations of classical Whitham's modulations equations and, up to second order, are related to a viscous correction of Whitham's equations.

Regularity. Recall that assumption (A') induces the presence of a Jordan block at $\lambda=0$ for the operator L_0. A careful analysis of the regularity is thus needed. This is the purpose of the next lemma, inspired by a corresponding result in [13].

Lemma 3.1. *Assume* (A').

Then there exist $\xi_0\in\,]0,\pi[$, $\varepsilon_0>0$, *and two curves,* $j=1,2$, $\lambda_j:\,]-\xi_0,\xi_0[\,\to B(0,\varepsilon_0)$ *such that, when* $|\xi|<\xi_0$,

$$\sigma(L_\xi)\cap B(0,\varepsilon_0)=\{\lambda_1(\xi),\lambda_2(\xi)\}.$$

Moreover, these two critical curves are differentiable at 0. *Thus they expand as*

$$\lambda_j(\xi)\overset{0}{=}i\bar{k}\xi\lambda_{0,j}+o(\xi),\quad j=1,2. \tag{3.1}$$

Assume, moreover,

(B) $\lambda_{0,1}\neq\lambda_{0,2}$.

Then the curves λ_j *are analytic in a neighborhood of* 0. *Up to a change of* ξ_0, *there exist, for* $0<|\xi|<\xi_0$, *dual right and left eigenfunctions* $(q_j(\xi,\cdot))_{j=1,2}$ *and* $(\tilde{q}_j(\xi,\cdot))_{j=1,2}$ *of* L_ξ *associated with* $\lambda_j(\xi)$, *of the following form* ($j=1,2$):

$$\begin{aligned} q_j(\xi,\cdot)&=(i\bar{k}\xi)^{-1}\beta_{j,1}(\xi)\,v_1(\xi,\cdot)+\beta_{j,2}(\xi)\,v_2(\xi,\cdot),\\ \tilde{q}_j(\xi,\cdot)&=i\bar{k}\xi\tilde{\beta}_{j,1}(\xi)\tilde{v}_1(\xi,\cdot)+\tilde{\beta}_{j,2}(\xi)\tilde{v}_2(\xi,\cdot),\end{aligned}$$

where

- $j=1,2$, $v_j:\,]-\xi_0,\xi_0[\,\to L^2_{\text{per}}([0,1])$, *and* $\tilde{v}_j:\,]-\xi_0,\xi_0[\,\to L^2_{\text{per}}([0,1])$ *are analytic functions such that* $(v_j(\xi,\cdot))_{j=1,2}$ *and* $(\tilde{v}_j(\xi,\cdot))_{j=1,2}$ *are dual bases of the total eigenspace of* L_ξ *associated with spectrum* $\sigma(L_\xi)\cap B(0,\varepsilon_0)$, *and* $v_1(0,\cdot)=\bar{U}'$, $v_2(0,\cdot)\equiv 1$, *and* $\tilde{v}_2(0,\cdot)\equiv 1$ *when* $|\xi|<\xi_0$;
- $j=1,2$, $k=1,2$, $\beta_{j,k}:\,]-\xi_0,\xi_0[\,\to\mathbb{C}$, *and* $\tilde{\beta}_{j,k}:\,]-\xi_0,\xi_0[\,\to\mathbb{C}$ *are analytic and*

$$\beta_{j,1}(0)\neq 0,\quad j=1,2.$$

Proof. Since 0 is separated from the rest of the spectrum of L_0, standard spectral theory for perturbations by relatively compact operators (see [14]) provides ξ_0, ε_0, and *continuous* λ_1, λ_2 such that, for $|\xi| < \xi_0$, $\sigma(L_\xi) \cap B(0,\varepsilon_0) = \{\lambda_1(\xi), \lambda_2(\xi)\}$.

Moreover, this also yields analytic dual right and left spectral projectors associated to the spectrum in $B(0,\varepsilon_0)$. Analytic dual bases of the right and left eigenspaces may then be obtained by dual bases for spectral spaces of the spectrum of L_0 in $B(0,\varepsilon_0)$. We may choose such bases in the form $(\bar{U}', 1)$ and $(*, 1)$, and obtain in this way the (v_j) and $(\tilde{v}_j)$ of the lemma.

We are thus left with the spectral analysis of

$$M_\xi = [\langle \tilde{v}_j(\xi,\cdot), L_\xi v_k(\xi,\cdot)\rangle_{L^2([0,1])}]_{j,k},$$

a 2×2 matrix perturbation problem. We still have $M_0 = \begin{pmatrix} 0 & -6\bar{k} \\ 0 & 0 \end{pmatrix}$, but we will scale M_ξ to blow up the double eigenvalue.

To do so, let us expand

$$L_\xi = L_0 + i\bar{k}\xi L^{(1)} + (i\bar{k}\xi)^2 L^{(2)} + (i\bar{k}\xi)^3 L^{(3)} + (i\bar{k}\xi)^4 L^{(4)}. \tag{3.2}$$

Specifically,

$$L^{(1)} = -(6\bar{U} - \bar{c}) - 3\bar{k}^2\,\partial_x^2 - \delta(2\bar{k}\,\partial_x + 4\bar{k}^3\,\partial_x^3). \tag{3.3}$$

Then, notice that $\langle \tilde{v}_2(0,\cdot), L^{(1)} v_1(0,\cdot)\rangle = 0$. This may be seen either by direct inspection or by differentiating profile equation (2.4) with respect to k (see (3.7)). Therefore,

$$M_\xi \overset{0}{=} \begin{pmatrix} 0 & -6\bar{k} \\ 0 & 0 \end{pmatrix} + i\bar{k}\xi \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} + \mathcal{O}(|\xi|^2).$$

As a consequence, the scaling

$$\check{M}_\xi := (i\bar{k}\xi)^{-1} S(\xi) M_\xi S(\xi)^{-1}, \quad S(\xi) := \begin{pmatrix} i\bar{k}\xi & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

preserves smoothness in ξ. Since the eigenvalues $m_j(\xi)$ of $\check{M}_\xi$ are $(i\bar{k}\xi)^{-1}\lambda_j(\xi)$, their continuity provides the missing differentiability at 0. As $m_j(0) = \lambda_{0,j}$, under assumption (B), eigenmodes of $\check{M}_\xi$ are analytic in ξ in a neighborhood of 0. Undoing the scaling finishes the proof, except for the nonvanishing condition on $\beta_{j,1}(0)$, $j = 1, 2$.

The only thing left is thus to explain why a right eigenvector of $\check{M}_0$ cannot have 0 as first component. This follows from the structure $\check{M}_0 = \begin{pmatrix} * & -6\bar{k} \\ * & * \end{pmatrix}$. □

Expansion of eigenvalues. Our next step is to obtain information about expansion of critical eigenmodes in a form suitable for comparison with modulation-averaged equations. We assume (A') and (B).

In terms of the previous lemma, for $j = 1, 2$, we set

$$\check{u}_j(\xi, \cdot) = i\bar{k}\xi q(\xi, \cdot), \quad k_{0,j} = \beta_{j,1}(0),$$

and are looking for terms in expansions

$$\begin{aligned}\lambda_j(\xi) &= i\bar{k}\xi\lambda_{0,j} + (i\bar{k}\xi)^2\lambda_{1,j} + (i\bar{k}\xi)^3\lambda_{2,j} + \mathcal{O}(\xi^4),\\ \check{u}_j(\xi, \cdot) &= k_{0,j}\bar{U}' + (i\bar{k}\xi)\check{u}_{1,j}(\xi, \cdot) + (i\bar{k}\xi)^2\check{u}_{2,j}(\xi, \cdot)\\ &\quad + (i\bar{k}\xi)^3\check{u}_{3,j}(\xi, \cdot) + \mathcal{O}(\xi^4),\end{aligned}$$

by inspection of

$$L_\xi\check{u}_j(\xi, \cdot) = \lambda_j(\xi)\check{u}_j(\xi, \cdot). \tag{3.5}$$

For the sake of simplicity, we drop the j-dependence in the following computations.

Recall the notation of (3.2). The first nontrivial step is to write

$$L_0[\check{u}_1] = -k_0L^{(1)}\bar{U}' + k_0\lambda_0\bar{U}'. \tag{3.6}$$

To simplify the equation, we use

$$L_0\,\partial_k\bar{U} + \bar{k}\,\partial_k\bar{c}\bar{U}' + L^{(1)}\bar{U}' = 0 \tag{3.7}$$

(obtained by differentiating profile equation (2.4) with respect to k), and the already known

$$L_0\,\partial_M\bar{U} + \bar{k}\,\partial_M\bar{c}\bar{U}' = 0 \tag{3.8}$$

(obtained by differentiating profile equation (2.4) with respect to M). This yields, for any choice of M_0,

$$L_0(\check{u}_1 - \mathrm{d}\bar{U}[k_0, M_0]) + (\lambda_0k_0 + \bar{k}\,\mathrm{d}\bar{c}[k_0, M_0])\bar{U}' = 0 \tag{3.9}$$

(where d applied to profiles, phase velocity, and time frequency stands for the total derivative with respect to parameters (k, M), and where $\bar{\cdot}$ denotes, as before, evaluation in $(\bar{k}, \bar{M})$). We choose M_0 so that

$$\lambda_0k_0 + \bar{k}\,\mathrm{d}\bar{c}[k_0, M_0] = 0. \tag{3.10}$$

This is possible, and determines M_0, since $\partial_M \bar{c} = 6 \neq 0$.

To go beyond this level of description, we need to make some normalizing choices. To prevent confusion, we temporarily mark again the j-dependence. For $j = 1, 2$, since $k_{0,j} \neq 0$, we may impose, up to changing ξ_0 again, for $|\xi| < \xi_0$,

$$\langle \tilde{v}_1(0,\cdot), \check{u}_j(\xi,\cdot)\rangle = k_{0,j}. \tag{3.11}$$

Moreover, up to changing parametrization by a k-dependent shift, we may also impose

(N) $\quad \langle \tilde{v}_1(0,\cdot), \partial_k \bar{U}\rangle = 0.$

With these choices, since $\langle \tilde{v}_1(0,\cdot), \partial_M \bar{U}\rangle = \langle \tilde{v}_1(0,\cdot), v_2(0,\cdot)\rangle = 0$ also holds, we have that equations (3.9) and (3.10) imply

$$\check{u}_1 = \mathrm{d}\bar{U}[k_0, M_0]. \tag{3.12}$$

Incidentally, notice that, indeed, $M_0 = \langle \check{u}_1\rangle$.

The next step in the expansion of (3.5) is to write

$$L_0[\check{u}_2] = -L^{(1)}(\mathrm{d}\bar{U}[k_0, M_0]) - k_0 L^{(2)}\bar{U}' + \lambda_0\,\mathrm{d}\bar{U}[k_0, M_0] + k_0\lambda_1\bar{U}'. \tag{3.13}$$

The solvability condition is found by taking the scalar product with $\tilde{v}_2(0,\cdot) \equiv 1$, and reduces to

$$\lambda_0 M_0 + \mathrm{d}[\overline{\langle 3U^2\rangle}]\,[k_0, M_0] - \bar{c}M_0 = 0 \tag{3.14}$$

(recall (3.3)). Note that (3.10) and (3.14) form an eigenvalue problem in λ_0 and (k_0, M_0):

$$\lambda_0 \begin{pmatrix} k_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} -\bar{k}\,\partial_k\bar{c} & -\bar{k}\,\partial_M\bar{c} \\ -\partial_k[\overline{\langle 3U^2\rangle}] & -\partial_M[\overline{\langle 3U^2\rangle}] + \bar{c} \end{pmatrix} \begin{pmatrix} k_0 \\ M_0 \end{pmatrix}. \tag{3.15}$$

Notice that, since $-\bar{k}\,\partial_M\bar{c} \neq 0$, once (λ_0, k_0) is fixed, M_0 is determined by the previous relation.

Since solvability conditions have been met, we may introduce, for any $\lambda \in \mathbb{C}$, $\check{f}^k(\lambda;\cdot)$, and $\check{f}^M(\lambda;\cdot)$, the solutions of

$$L_0(\check{f}^k(\lambda;\cdot)) = -L^{(1)}(\partial_k\bar{U}) - \partial_k\,[\overline{\langle 3U^2\rangle}] - L^{(2)}\bar{U}' + \lambda\,\partial_k\bar{U}, \tag{3.16}$$

$$L_0(\check{f}^M(\lambda;\cdot)) = -L^{(1)}(\partial_M\bar{U}) - \partial_M[\overline{\langle 3U^2\rangle}] + \bar{c} + \lambda\,\partial_M\bar{U} - \lambda, \tag{3.17}$$

orthogonal to $\tilde{v}_1(0,\cdot)$. In view of (3.14) and (3.8), this enables us to rewrite (3.13) in the compact form

$$L_0(\check{u}_2 - \partial_M\bar{U}\check{M}_1 - \check{f}(\lambda_0)[k_0, M_0]) - (\lambda_1 k_0 + \bar{k}\,\partial_M\bar{c}\check{M}_1)\bar{U}' = 0 \tag{3.18}$$

for any choice of $\tilde{M}_1$. Choosing $\tilde{M}_1$ to solve $\lambda_1 k_0 + \bar{k}\,\partial_M \bar{c}\tilde{M}_1 = 0$, equation (3.18) reduces to

$$\check{u}_2 = \partial_M \bar{U} \tilde{M}_1 + \check{f}(\lambda_0)[k_0, M_0], \tag{3.19}$$

thanks to normalization (3.11). Notice that the absence of a k_1 parameter is also due to our normalization choices, and that *a priori* $\tilde{M}_1$ is not $\langle \check{u}_2 \rangle$. Let us set $M_1 = \langle \check{u}_2 \rangle$. Then

$$M_1 = \tilde{M}_1 + \langle \check{f}(\lambda_0) \rangle [k_0, M_0], \tag{3.20}$$

and

$$\lambda_1 k_0 + \bar{k}\,\partial_M \bar{c} M_1 = \bar{k}\,\partial_M \bar{c} \langle \check{f}(\lambda_0) \rangle [k_0, M_0]. \tag{3.21}$$

In the expansion of (3.5), we now come to

$$\begin{aligned} L_0[\check{u}_3] = {} & -L^{(1)}\check{u}_2 - L^{(2)}(\mathrm{d}\bar{U}[k_0, M_0]) - k_0 L^{(3)} \bar{U}' \\ & + \lambda_0 \check{u}_2 + \lambda_1 \,\mathrm{d}\bar{U}[k_0, M_0] + k_0 \lambda_2 \bar{U}'. \end{aligned} \tag{3.22}$$

The solvability condition for this equation yields

$$\begin{aligned} & \lambda_1 M_0 + \lambda_0 M_1 + (\partial_M [\overline{\langle 3U^2 \rangle}] - \bar{c}) M_1 \\ & \quad - (\partial_M [\overline{\langle 3U^2 \rangle}] - \bar{c}) \langle \check{f}(\lambda_0) \rangle [k_0, M_0] \\ & \quad = -\langle (6\bar{U} - \bar{c}) \check{f}(\lambda_0) \rangle [k_0, M_0] - \delta M_0. \end{aligned} \tag{3.23}$$

We stop here the analysis of the expansions and discuss now to which extent the obtained relations determine the parameters appearing in the expansions. We have that $\lambda_{0,1}$ and $\lambda_{0,2}$ are the eigenvalues of the matrix appearing in (3.15). Pick one $j \in 1, 2$, and drop the j-dependence. We have left some arbitrariness in the choice of k_0, but we can normalize it to 1. Then M_0 is uniquely determined. We still have to explain why this is also true for M_1 and λ_1. Since their coefficients in (3.21) are both nonzero, in view of (3.23), this amounts to proving that we cannot have simultaneously $M_0 = 0$ and $\lambda_0 + \partial_M [\overline{\langle 3U^2 \rangle}] - \bar{c} = 0$. If $M_0 = 0$, from (3.15) we deduce $\lambda_0 = -\bar{k}\,\partial_k \bar{c}$ and $\partial_k [\overline{\langle 3U^2 \rangle}] = 0$; thus, $\{\lambda_{0,1}, \lambda_{0,2}\} = \{-\bar{k}\,\partial_k \bar{c}, -\partial_M [\overline{\langle 3U^2 \rangle}] + \bar{c}\}$ and

$$\lambda_0 = -\bar{k}\,\partial_k \bar{c} \neq -\partial_M [\overline{\langle 3U^2 \rangle}] + \bar{c}.$$

Therefore, k_1 and M_1 are uniquely determined.

The process described here may be carried out by induction up to any order. We have chosen to stop at the first order that may be nontrivial and stable, that is, at an order at which the expansions of $\check{u}_1(\xi, \cdot)$ and $\check{u}_2(\xi, \cdot)$ are not collinear and the expansions of eigenvalues may have a negative real part (both for nonzero Floquet exponents).

Modulation systems. We now derive formally the modulation-averaged equations to which we will compare the critical low-Floquet expansions.

Although we call Whitham's systems the obtained equations, we will not develop the approach proposed in [25], as it is designed for Lagrangian systems. Instead, we follow a nonlinear WKB approach close to the one adopted in [21] by Serre. Let us stress that both [25] and [21] are only concerned with the formal derivation of a *first-order* modulation system.

First order. We make the derivation as general as possible and therefore *a priori* not restricted to perturbation of a given wave; thus we are back to the original frame, instead of a co-moving one. We are looking for a formal expansion of a solution u of equation (1.1) according to the two-scale *ansatz*

$$u(x,t) = U\left(\frac{\phi(\varepsilon x, \varepsilon t)}{\varepsilon}; \varepsilon x, \varepsilon t\right), \tag{3.24}$$

where

$$U(y,X,T) = \sum_j \varepsilon^j U_j(y;X,T) \quad \text{and} \quad \phi(X,T) = \sum_j \varepsilon^j \phi_j(X,T), \tag{3.25}$$

with all functions of period 1 in the y-variable. We insert the *ansatz* (3.24), (3.25) into (1.1), and collect terms of the same order in ε.

At first, this yields, with $\Omega_0 = \partial_T \phi_0$ and $k_0 = \partial_X \phi_0$,

$$\Omega_0\, \partial_y U_0 + k_0\, \partial_y(3U_0^2) + k_0^3\, \partial_y^3 U_0 + \delta(k_0^2\, \partial_y^2 U_0 + k_0^4\, \partial_y^4 U_0) = 0. \tag{3.26}$$

Assuming k_0 and $M_0 = \int_0^1 U_0(y;\cdot,\cdot)\,\mathrm{d}y$ are valued in the open set covered by assumption (A'), we solve (3.26) with

$$\begin{aligned} \Omega_0(X,T) &= -k_0(X,T)c(k_0(X,T), M_0(X,T)),\\ U_0(y;X,T) &= U(y, k_0(X,T), M_0(X,T)). \end{aligned} \tag{3.27}$$

The compatibility condition $\partial_T\,\partial_X \phi_0 = \partial_X\,\partial_T \phi_0$ yields the first equation of a Whitham's modulation system

$$\partial_T k_0 + \partial_X(k_0 c(k_0, M_0)) = 0. \tag{3.28}$$

We have disregarded in (3.27) the possibility of a phase shift dependent on (X,T), since ϕ_1 already encodes this possibility. We will have to rule out similar problems of uniqueness in the following steps.

As may be guessed from (3.27), many functions or operators of the previous subsections that were there evaluated at $(\bar{k}, \bar{M})$ will appear here evaluated at $(k_0(X,T), M_0(X,T))$, and for operators they act here on the variable y only. When no confusion is possible, we will use for them the notation of the previous subsections without explicit dependence on $(k_0(X,T), M_0(X,T))$, and with a subscript $\cdot_0$ when needed. If we require that k_0 and M_0 are valued in the open set covered by both assumptions (A') and (B), this dependence in parameters is smooth. With this settled, we fix some of the uniqueness issues of the *ansatz* by adding the condition

$$\langle \tilde{v}_1(0,\cdot), U_j(\cdot\,; X, T)\rangle = 0, \quad j \neq 0, \tag{3.29}$$

(recall Lemma 3.1).

The next step of the identification process gives, with $\Omega_1 = \partial_T \phi_1$ and $k_1 = \partial_X \phi_1$,

$$\begin{aligned}(\Omega_1 + c_0 k_1)U_0' - k_1 L^{(1)} U_0' - L_0 U_1 \\ - L^{(1)}\,\partial_X U_0 - \partial_X k_0 L^{(2)} U_0' + \partial_T U_0 + c_0\,\partial_X U_0 = 0,\end{aligned} \tag{3.30}$$

where $'$ denotes ∂_y. The solvability condition for L_0 reads

$$\partial_T M_0 + \partial_X(\langle 3U^2(\cdot\,; k_0, M_0)\rangle) = 0. \tag{3.31}$$

For concision's sake, we denote the averaged flux as $F(k, M) = \langle 3U^2(\cdot\,; k_0, M_0)\rangle$.

The *first-order Whitham's modulation system* (3.28), (3.31) is then written:

$$\begin{cases}\partial_T k_0 + \partial_X(k_0 c(k_0, M_0)) = 0,\\ \partial_T M_0 + \partial_X(F(k_0, M_0)) = 0.\end{cases} \tag{3.32}$$

Before going on with the derivation process, we write down the eigenvalue problem corresponding to (3.32). Linearizing (3.32) about $(\bar{k}, \bar{M})$ in the frame $(\bar{k}(X - \bar{c}T), T)$ yields

$$\begin{cases}\partial_T k + \bar{k}\,\partial_X(\bar{k}\,\mathrm{d}\bar{c}(k, M)) = 0,\\ \partial_T M + \bar{k}\,\partial_X(\mathrm{d}\bar{F}(k, M) - \bar{c}M) = 0.\end{cases} \tag{3.33}$$

Hence, applying Fourier transform, we receive the eigenvalue problem

$$\lambda(\xi)\begin{pmatrix}\hat{k}(\xi)\\ \hat{M}(\xi)\end{pmatrix} = i\bar{k}\xi\begin{pmatrix}-\bar{k}\,\partial_k\bar{c} & -\bar{k}\,\partial_M\bar{c}\\ -\partial_k\bar{F} & -\partial_M\bar{F} + \bar{c}\end{pmatrix}\begin{pmatrix}\hat{k}(\xi)\\ \hat{M}(\xi)\end{pmatrix}. \tag{3.34}$$

Second order. To proceed, for arbitrary $(\bar{k}, \bar{M})$ we introduce $g^k(\bar{k}, \bar{M};\,\cdot)$ and $g^M(k, M;\,\cdot)$, the solutions of

$$\begin{aligned}L_0(g^k(\bar{k}, \bar{M};\,\cdot)) = -L^{(1)}\,\partial_k\bar{U} - \partial_k\bar{F} - L^{(2)}\bar{U}'\\ - \partial_k\bar{U}\bar{k}\,\partial_k\bar{c} - (\partial_M\bar{U} - 1)\,\partial_k\bar{F},\end{aligned} \tag{3.35}$$

$$\begin{aligned}L_0(g^M(\bar{k}, \bar{M};\,\cdot)) = -L^{(1)}\,\partial_M U_0 - \partial_M\bar{F} + \bar{c}\\ - \partial_k\bar{U}\bar{k}\,\partial_M\bar{c} - (\partial_M\bar{U} - 1)[\partial_M\bar{F} - \bar{c}],\end{aligned} \tag{3.36}$$

orthogonal to $\tilde{v}_1(0,\cdot)$.

Then, with (3.32), (3.7), and (3.8), equation (3.30) reads

$$L_0(U_1 - \mathrm{d}U_0(k_1, \tilde{M}_1) - g_0[\partial_X k_0, \partial_X M_0]) \\ = (\Omega_1 + k_0\,\mathrm{d}c_0[k_1, \tilde{M}_1] + c_0 k_1)U_0' \tag{3.37}$$

for any choice of $\tilde{M}_1$. If we choose $\tilde{M}_1$ to get $\Omega_1 + k_0\,\mathrm{d}c_0[k_1, \tilde{M}_1] + c_0 k_1 = 0$, and normalize parametrization according to (N), (3.37) is then reduced to

$$U_1 = \mathrm{d}U_0(k_1, \tilde{M}_1) + g_0[\partial_X k_0, \partial_X M_0]. \tag{3.38}$$

Let us set $M_1 = \langle U_1 \rangle$. Then

$$M_1 = \tilde{M}_1 + \langle g_0 \rangle [\partial_X k_0, \partial_X M_0], \tag{3.39}$$

and compatibility condition $\partial_T k_1 = \partial_X \Omega_1$ yields

$$\partial_T k_1 + \partial_X(k_0\,\mathrm{d}c_0[k_1, M_1] + c_0 k_1) = \partial_X(k_0\,\partial_M c_0 \langle g_0 \rangle [\partial_X k_0, \partial_X M_0]). \tag{3.40}$$

Back to the identification process, we obtain

$$\begin{aligned} &(\Omega_2 + c_0 k_2)U_0' - k_2 L^{(1)} U_0' - L_0 U_1 - L^{(1)}\,\partial_X U_1 - k_1^2 L^{(2)} U_0'' \\ &- k_1\, L^{(1)} U_1' - L^{(2)}\,\partial_X^2 U_0 - \partial_X k_0 L^{(2)} U_1' - \partial_X k_1 L^{(2)} U_0' - 2k_1 L^{(2)}\,\partial_X U_0' \\ &- \partial_X^2 k_0 L^{(3)} U_0' - \partial_X k_0 L^{(3)}\,\partial_X U_0' - 3(\partial_X k_0)^2 L^{(4)} U_0'' + (\Omega_1 + c_0 k_1)U_1' \\ &+ \partial_T U_1 + c_0\,\partial_X U_1 + 6U_1\,\partial_X U_0 + 3k_0(U_1^2)' + 6k_1 U_1 U_0' \\ &+ 6\delta k_1\,\partial_X(k_0^2)U_0''' + 2k_1\,\partial_X k_0 U_0''' = 0, \end{aligned} \tag{3.41}$$

an equation of the form $\partial_T U_1 + \partial_X(6U_0U_1) + \delta\,\partial_X^2 U_0 + \partial_y(\cdots) = 0$, whose solvability condition is

$$\partial_T M_1 + \partial_X(\mathrm{d}F_0[k_1, M_1]) = -\delta\,\partial_X^2 M_0 - \partial_X(\langle 6U_0 g_0 \rangle [\partial_X k_0, \partial_X M_0]) \\ + \partial_X(\partial_M F_0 \langle g_0 \rangle [\partial_X k_0, \partial_X M_0]). \tag{3.42}$$

To write the second-order system in a compact form, let us introduce

$$\begin{aligned} d_{1,1}(k, M) &= k\,\partial_M c(k, M) \langle g^k(k, M) \rangle, \\ d_{1,2}(k, M) &= k\,\partial_M c(k, M) \langle g^M(k, M) \rangle, \\ d_{2,1}(k, M) &= -\langle 6U(k, M) g^k(k, M) \rangle + \partial_M F(k, M) \langle g^k(k, M) \rangle, \\ d_{2,2}(k, M) &= -\delta - \langle 6U(k, M) g^M(k, M) \rangle + \partial_M F(k, M) \langle g^M(k, M) \rangle. \end{aligned}$$

Note that equations (3.28), (3.31), (3.40), and (3.42) are the first equations obtained in the formal expansion of a solution $(\kappa, \mathcal{M})$ of

$$(3.43)\qquad \begin{cases} \partial_t \kappa + \partial_x(\kappa c(\kappa, \mathcal{M})) = \partial_x(d_{1,1}(\kappa, \mathcal{M})\,\partial_x \kappa + d_{1,2}(\kappa, \mathcal{M})\,\partial_x \mathcal{M}), \\ \partial_t \mathcal{M} + \partial_x(F(\kappa, \mathcal{M})) = \partial_x(d_{2,1}(\kappa, \mathcal{M})\,\partial_x \kappa + d_{2,2}(\kappa, \mathcal{M})\,\partial_x \mathcal{M}), \end{cases}$$

according to the low-frequency *ansatz*

$$(3.44)\qquad (\kappa, \mathcal{M})(x,t) = (k, M)(\varepsilon x, \varepsilon t) \quad \text{where } (k, M)(X,T) = \sum_j \varepsilon^j (k_j, M_j)(X,T).$$

We call the system (3.43) a *second-order Whitham's modulation system*. Note that, in contrast with the present situation, going back to physical variables (x,t) would have been of no effect on the shape of system (3.32); hence, we have skipped the unscaling step there. We now write down the eigenvalue problem corresponding to (3.43). Linearizing (3.43) about $(\bar{k}, \bar{M})$ in the frame $(\bar{k}(x - \bar{c}t), t)$ yields

$$(3.45)\qquad \begin{cases} \partial_t k + \bar{k}\,\partial_x(\bar{k}\,\mathrm{d}\bar{c}(k, M)) = \bar{k}\,\partial_x(\bar{d}_{1,1}\bar{k}\,\partial_x k + \bar{d}_{1,2}\bar{k}\,\partial_x M) \\ \partial_t M + \bar{k}\,\partial_x(\mathrm{d}\bar{F}(k, M) - \bar{c}M) = \bar{k}\,\partial_x(\bar{d}_{2,1}\bar{k}\,\partial_x k + \bar{d}_{2,2}\bar{k}\,\partial_x M). \end{cases}$$

Hence, applying Fourier transform, the eigenvalue problem

(3.46)

$$\lambda(\xi)\begin{pmatrix} \hat{k}(\xi) \\ \hat{M}(\xi) \end{pmatrix} = \left[i\bar{k}\xi \begin{pmatrix} -\bar{k}\,\partial_k \bar{c} & -\bar{k}\,\partial_M \bar{c} \\ -\partial_k \bar{F} & -\partial_M \bar{F} + \bar{c} \end{pmatrix} + (i\bar{k}\xi)^2 \begin{pmatrix} \bar{d}_{1,1} & \bar{d}_{1,2} \\ \bar{d}_{1,1} & \bar{d}_{1,2} \end{pmatrix} \right] \begin{pmatrix} \hat{k}(\xi) \\ \hat{M}(\xi) \end{pmatrix}.$$

Statements. We now look back at expansions of critical eigenmodes to validate at the spectral level the derived modulation systems.

Theorem 3.2. *Assume* (A') *and* (B), *and adopt notation of Lemma 3.1. Then:*

(1) *The Fourier eigenvalues of the first-order Whitham's system* (*see system* (3.34)) *are* $i\bar{k}\xi\,\lambda_{0,1}$ *and* $i\bar{k}\xi\,\lambda_{0,2}$. *Moreover, choosing corresponding eigenvectors* $(k_0^{(1)}, M_0^{(1)})$ *and* $(k_0^{(2)}, M_0^{(2)})$, *and normalizing parametrization according to* (N), *we may normalize right eigenfunctions in such a way that, for* $j = 1, 2$,

$$(3.47)\qquad q_j(\xi, \cdot) \overset{0}{=} \frac{1}{i\bar{k}\xi} k_0^{(j)} \bar{U}' + \mathrm{d}\bar{U}(k_0^{(j)}, M_0^{(j)}) + \mathcal{O}(\xi).$$

(2) *Normalize parametrization according to* (N). *In a neighborhood of the origin, the Fourier modes of the second-order Whitham's system* (*see system* (3.46)) *are analytic functions of the Fourier frequency. For small Fourier frequency* ξ, *we*

may label these eigenvalues as $i\bar{k}\xi\mu_1(\xi)$ *and* $i\bar{k}\xi\mu_2(\xi)$ *in such a way that, for* $j = 1, 2$,

$$\lambda_j(\xi) \stackrel{0}{=} i\bar{k}\xi\mu_j(\xi) + \mathcal{O}(\xi^3); \tag{3.48}$$

and we may normalize corresponding eigenmodes

$$(k^{(1)}(\xi), M^{(1)}(\xi)) \text{ and } (k^{(2)}(\xi), M^{(2)}(\xi))$$

and eigenfunctions in such a way that, for $j = 1, 2$,

$$\begin{aligned} \begin{pmatrix} k^{(j)}(\xi) \\ M^{(j)}(\xi) \end{pmatrix} &\stackrel{0}{=} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} + i\bar{k}\xi \begin{pmatrix} 0 \\ M_1^{(j)} \end{pmatrix} + \mathcal{O}(\xi^2), \\ q_j(\xi, \cdot) &\stackrel{0}{=} \frac{1}{i\bar{k}\xi} k^{(j)_0}\bar{U}' + \mathrm{d}\bar{U}(k^{(j)_0}, M_0^{(j)}) \\ &\quad + i\bar{k}\xi\, \partial_M \bar{U}(M_1^{(j)} - \langle \bar{g} \rangle [k_0^{(j)}, M_0^{(j)}]) \\ &\quad + i\bar{k}\xi \bar{g}[k_0^{(j)}, M_0^{(j)}] + \mathcal{O}(\xi^2), \end{aligned}$$

where $\bar{g}$ *is obtained* via *solutions of* (3.35), (3.36) *orthogonal to* $\tilde{v}_1(0, \cdot)$.

Proof. We first remark that the singularity at $\xi = 0$ in (3.34) and (3.46) is artificial, as may be seen by dividing by $i\bar{k}\xi$ and setting $\mu = \lambda/(i\bar{k}\xi)$. In this way, (3.34) is made independent of ξ, and in (3.46) we are looking for $\mu_j(\xi)$.

The first point follows from a direct comparison of (3.15) and (3.34) and the possibility of prescribing k_0 in (3.12).

Problem (3.46) is a low-frequency perturbation of (3.34). To prove the second point, we fix j and expand $\mu_j(\xi)$ and $(k^{(j)}(\xi), M^{(j)}(\xi))$ according to the fact that $(i\bar{k}\xi\mu_j(\xi), k^{(j)}(\xi), M^{(j)}(\xi))$ solves (3.46) and $\mu_j(0) = \lambda_{0,j}$. Since $\mu_1(0) \neq \mu_2(0)$, expansions are regular in a neighborhood of 0. Since $k^{(j)}(0)$ is necessarily nonzero, we may normalize eigenvectors to impose $k^{(j)}(\xi) = k_0^{(j)}$; thus,

$$\begin{aligned} \mu_j(0) &\stackrel{0}{=} \lambda_{0,j} + i\bar{k}\xi\mu_{1,j} + \mathcal{O}(\xi^2), \\ \begin{pmatrix} k^{(j)}(\xi) \\ M^{(j)}(\xi) \end{pmatrix} &\stackrel{0}{=} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} + i\bar{k}\xi \begin{pmatrix} 0 \\ M_1^{(j)} \end{pmatrix} + \mathcal{O}(\xi^2). \end{aligned}$$

This leads to

$$\begin{aligned} \lambda_{0,j} \begin{pmatrix} 0 \\ M_1^{(j)} \end{pmatrix} + \begin{bmatrix} \bar{k}\,\partial_k \bar{c} & \bar{k}\,\partial_M \bar{c} \\ \partial_k \bar{F} & \partial_M \bar{F} - \bar{c} \end{bmatrix} \begin{pmatrix} 0 \\ M_1^{(j)} \end{pmatrix} + \mu_{1,j} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} \\ = \begin{bmatrix} \bar{d}_{1,1} & \bar{d}_{1,2} \\ \bar{d}_{2,1} & \bar{d}_{2,2} \end{bmatrix} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix}. \end{aligned} \tag{3.49}$$

Yet, using (3.15) to rewrite $\lambda_{0,j}k_0^{(j)}$ and $\lambda_{0,j}M_0^{(j)}$, we obtain

$$\check{f}(\lambda_{0,j})[k_0^{(j)}, M_0^{(j)}] = g(\bar{k}, \bar{M})[k_0^{(j)}, M_0^{(j)}],$$

yielding

$$\begin{aligned}
\bar{d}_{1,1}k_0^{(j)} + \bar{d}_{1,2}M_0^{(j)} &= \bar{k}\,\partial_M\bar{c}\langle\check{f}(\lambda_{0,j})\rangle[k_0^{(j)}, M_0^{(j)}],\\
\bar{d}_{2,1}k_0^{(j)} + \bar{d}_{2,2}M_0^{(j)} &= (\partial_M\bar{F} - \bar{c})\langle\check{f}(\lambda_{0,j})\rangle[k_0^{(j)}, M_0^{(j)}]\\
&\qquad - \langle(6\bar{U} - \bar{c})\check{f}(\lambda_{0,j})\rangle[k_0^{(j)}, M_0^{(j)}] - \delta M_0^{(j)}.
\end{aligned}$$

Therefore, $(\mu_{1,j}, M_1^{(j)})$ solves (3.21) and (3.23), and hence by uniqueness of the solution, the theorem is proved, recalling (3.19). □

Having Lemma 3.1 in mind, one may ask whether something remains from Theorem 3.2 if assumption (B) is removed. The answer is positive; assuming (A') only, it may be proved that the characteristic speeds of the first-order Whitham's system are the $\lambda_{0,j}$ values appearing in (3.1).

Proposition 3.3. *Assume* (A'), *and adopt notation of Lemma 3.1.*

Then the Fourier eigenvalues of the first-order Whitham's system (*see system* (3.34)) *are* $i\bar{k}\xi\lambda_{0,1}$ *and* $i\bar{k}\xi\lambda_{0,2}$.

Proof. Coming back to the proof of Lemma 3.1, we know that $\lambda_{0,1}$ and $\lambda_{0,2}$ are the eigenvalues of

$$\check{M}_0 = \begin{pmatrix}
\begin{aligned}&\tfrac{1}{i\bar{k}}\langle\tilde{v}_1(0), L_0\,\partial_\xi v_1(0)\rangle\\ &\qquad + \langle\tilde{v}_1(0), L^{(1)}v_1(0)\rangle\end{aligned} & \langle\tilde{v}_1(0), L_0 v_2(0)\rangle \\
\begin{aligned}&\tfrac{1}{(i\bar{k})^2}\langle\partial_\xi\tilde{v}_2(0), L_0\,\partial_\xi v_1(0)\rangle\\ &\qquad + \tfrac{1}{i\bar{k}}\langle\tilde{v}_2(0), L^{(1)}\partial_\xi v_1(0)\rangle\\ &\qquad + \tfrac{1}{i\bar{k}}\langle\partial_\xi\tilde{v}_2(0), L^{(1)}v_1(0)\rangle\\ &\qquad + \langle\tilde{v}_2(0), L^{(2)}v_1(0)\rangle\end{aligned} & \begin{aligned}&\tfrac{1}{i\bar{k}}\langle\partial_\xi\tilde{v}_2(0), L_0 v_2(0)\rangle\\ &\qquad + \langle\tilde{v}_2(0), L^{(1)}v_2(0)\rangle\end{aligned}
\end{pmatrix}.$$

Yet, once (for $(j,\ell) \neq (1,2)$) the equations

$$\langle\partial_\xi\tilde{v}_j(0), v_\ell(0)\rangle = \langle\tilde{v}_j(0), \partial_\xi v_\ell(0)\rangle = 0 \quad \text{and} \quad \partial_\xi v_1(0) = i\bar{k}\,\partial_k\bar{U}, \tag{3.50}$$

are proved, a direct examination of $\check{M}_0$ (using (3.7)) shows that

$$\check{M}_0 = \begin{bmatrix} -\bar{k}\,\partial_k\bar{c} & -\bar{k}\,\partial_M\bar{c} \\ -\partial_k\bar{F} & -\partial_M\bar{F} + \bar{c} \end{bmatrix}.$$

Hence comes the result. We are left now with the proof of (3.50).

First, since expanding the duality relation $\langle \tilde{v}_2(\xi), v_1(\xi)\rangle = 0$ yields

$$\langle \partial_\xi \tilde{v}_2(0), v_1(0)\rangle + \langle \tilde{v}_2(0), \partial_\xi v_1(0)\rangle = 0,$$

we may replace $\tilde{v}_2(\xi)$ with $\tilde{v}_2(\xi) - \langle \partial_\xi \tilde{v}_2(0), v_1(0)\rangle \xi \tilde{v}_1(\xi)$ and $v_1(\xi)$ with $v_1(\xi) - \langle \tilde{v}_2(0), \partial_\xi v_1(0)\rangle \xi v_2(\xi)$. Thus we may assume the first part of the claim is satisfied when $(j,\ell) = (2,1)$. Now fix ℓ, and normalize $v_\ell(\xi)$ according to $\langle \tilde{v}_\ell(0), v_\ell(\xi)\rangle = 1$. This yields $\langle \tilde{v}_\ell(0), \partial_\xi v_\ell(0)\rangle = 0$. Yet, expanding the duality relation $\langle \tilde{v}_\ell(\xi), v_\ell(\xi)\rangle = 1$ at first order in ξ gives $\langle \partial_\xi \tilde{v}_\ell(0), v_\ell(0)\rangle + \langle \tilde{v}_\ell(0), \partial_\xi v_\ell(0)\rangle = 0$. From this, the first part of the claim follows when $j = \ell$.

Now, let us denote

$$\Pi(\xi) = \langle \tilde{v}_1(\xi), \cdot\rangle v_1(\xi) + \langle \tilde{v}_2(\xi), \cdot\rangle v_2(\xi).$$

Expanding $L_\xi v_1(\xi) = \Pi(\xi) L_\xi v_1(\xi)$ at first order in ξ, and using (3.7), provide

$$L_0[\partial_\xi v_1(0) - i\bar{k}\,\partial_k \bar{U}] = \Pi(0) L_0[\partial_\xi v_1(0) - i\bar{k}\,\partial_k \bar{U}].$$

Therefore, there exist α and β such that

$$L_0[\partial_\xi v_1(0) - i\bar{k}\,\partial_k \bar{U}] = \alpha v_1(0) + \beta v_2(0).$$

Since $\tilde{v}_2(0)$ lies in the left kernel of L_0, we have $\beta = 0$. Normalizing parametrization according to (N), we deduce that

$$\partial_\xi v_1(0) = i\bar{k}\,\partial_k \bar{U} + \frac{\alpha}{-\bar{k}\,\partial_M \bar{c}}\,\partial_M \bar{U}.$$

Then, the first part of the claim implies $\alpha = 0$, and this finishes the proof. □

Actually, the previous proposition holds even if we assume only (A) instead of the stronger (A'). This requires us to write the first-order Whitham's system in an intrinsic way, not depending on the choice of the parametrization. The main achievement of [21] is precisely the proof that, for general conservation laws, a similar proposition holds if we assume only the non-degeneracy of the Poincaré return map (in the sense of (A)). There it is proved by a direct inspection of an Evans function.

We now state some easy consequences of the previous statements. We start with necessary conditions for side-band stability. Again, the first part of the next corollary (A') may be replaced with (A) (see [21]). In the second part, we use a nonstandard terminology for Kawashima's conditions. Kawashima's conditions are a set dealing with stability of constant states for quasilinear (degenerate) hyperbolic-parabolic systems (see [16]). Besides structural assumptions, the key conditions are: (1) there is a symmetrization of the hyperbolic part that makes the viscous part nonnegative (usually required *via* the existence of a dissipative convex entropy); (2) no hyperbolic mode lies in the kernel of the viscous part. According to the consequences of these conditions, we say that the weak Kawashima's condition is satisfied at a point if there the action of the viscous part on hyperbolic modes is nonpositive, and that the strong Kawashima's condition holds if this action is negative. To make this explicit for the second-order Whitham's system, let us complete the $((k_0^{(1)}, M_0^{(1)}), (k_0^{(2)}, M_0^{(2)}))$ appearing in Theorem 3.2 with a dual basis of left eigenvectors $((\tilde{k}_0^{(1)}, \tilde{M}_0^{(1)}), (\tilde{k}_0^{(2)}, \tilde{M}_0^{(2)}))$. Then, the weak Kawashima's condition at $(\bar{k}, \bar{M})$ is written

$$\mathrm{Re}\left(\left\langle \begin{pmatrix} \tilde{k}_0^{(j)} \\ \tilde{M}_0^{(j)} \end{pmatrix}, \begin{bmatrix} \bar{d}_{1,1} & \bar{d}_{1,2} \\ \bar{d}_{2,1} & \bar{d}_{2,2} \end{bmatrix} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} \right\rangle_{\mathbb{R}^2}\right) \geq 0, \quad j = 1, 2,$$

and the strong condition reads

$$\mathrm{Re}\left(\left\langle \begin{pmatrix} \tilde{k}_0^{(j)} \\ \tilde{M}_0^{(j)} \end{pmatrix}, \begin{bmatrix} \bar{d}_{1,1} & \bar{d}_{1,2} \\ \bar{d}_{2,1} & \bar{d}_{2,2} \end{bmatrix} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} \right\rangle_{\mathbb{R}^2}\right) > 0, \quad j = 1, 2.$$

Note that, if the first-order Whitham's system is strictly hyperbolic, the real parts may be omitted in the previous inequalities because the above scalar products are already real, and, even in the weakly hyperbolic case, we may choose eigenvectors to make them so.

Corollary 3.4. *Assume* (A').

(1) *Assume the first-order Whitham's system is not weakly hyperbolic at* $(\bar{k}, \bar{M})$.
Then there exists $\varepsilon \in \,]0, \pi[$ *such that, for all* ξ *such that* $0 < \xi < \varepsilon$ *or for all* ξ *such that* $-\varepsilon < \xi < 0$,

$$\sigma(L_\xi) \cap \{\lambda \mid \mathrm{Re}(\lambda) > 0\} \neq \emptyset.$$

(2) *Assume the first-order Whitham's system is strictly hyperbolic, but that the second-order Whitham's system violates at* $(\bar{k}, \bar{M})$ *the weak Kawashima's condition.*
Then there exists $\varepsilon \in \,]0, \pi[$ *such that, for all* ξ *such that* $0 < |\xi| < \varepsilon$,

$$\sigma(L_\xi) \cap \{\lambda \mid \mathrm{Re}(\lambda) > 0\} \neq \emptyset.$$

Proof. Thanks to Proposition 3.3, the first condition tells us that $\lambda_{0,1} \notin \mathbb{R}$ and $\lambda_{0,2} \notin \mathbb{R}$. The result follows then from (3.1).

Thanks to Proposition 3.3, the second condition implies $\lambda_{0,1} \in \mathbb{R}$, $\lambda_{0,2} \in \mathbb{R}$, and thus (B) is satisfied. Thus through Theorem 3.2, the second condition also yields $\mathrm{Re}(\lambda_1''(0)) > 0$ or $\mathrm{Re}(\lambda_2''(0)) > 0$, by taking scalar products with $(\tilde{k}_0^{(j)}, \tilde{M}_0^{(j)})$ in equality (3.49). Hence the result follows. ◻

We now make precise the claim that strong co-periodic stability plus strong modulational stability yields side-band stability.

Corollary 3.5. *Assume* (A') *and the following conditions:*

(1) *There exists* $\eta > 0$ *such that* $\sigma(L_0) \subset \{0\} \cup \{\lambda \mid \mathrm{Re}(\lambda) \le -\eta\}$.

(2) *The first-order Whitham's system is strictly hyperbolic and the second-order Whitham's system satisfies the strong Kawashima's condition.*

Then there exists $\varepsilon \in \,]0, \pi[$ *such that, for all* ξ *such that* $0 < |\xi| < \varepsilon$,

$$\sigma(L_\xi) \subset \{\lambda \mid \mathrm{Re}(\lambda) < 0\}.$$

Proof. Under these assumptions, (B) holds, $\mathrm{Re}(\lambda_1'(0)) = 0$, $\mathrm{Re}(\lambda_2'(0)) = 0$, $\mathrm{Re}(\lambda_1''(0)) < 0$, and $\mathrm{Re}(\lambda_2''(0)) < 0$. Therefore a perturbation argument provides the result. □

We stop here the derivation of corollaries about spectral instability or stability criteria from our main results. Let us make a final comment about the fact that the coefficients of the second-order averaged system share an unsatisfactory property with those of many effective approximate systems whose derivation involves homogenization: namely, that they are written in an intricate way. It may then seem that these criteria are of no practical use. Yet let us mention that for the Kuramoto-Sivashinsky equation, the seminal work [11] already proposed a second-order modulation system, derived in a formal way, and that therein a numerical study of the first- and second-order necessary conditions for side-band stability was then performed. Up to the fact that such an approach does not capture all kinds of spectral instabilities, their conclusions are in good agreement with numerical studies of full stability [2, 7].

Besides stability criteria, the main motivation of Theorem 3.2 is the will to provide foundations for a nonlinear validation of the slow modulation *ansatz* and the corresponding second-order Whitham's system as a description of the asymptotic behavior near the wave. In this respect, key relations are (3.48) and (3.47). Their particular role is the counterpart of the fact that many quasilinear hyperbolic-parabolic systems are asymptotically equivalent near constant states. Indeed, at the linear level, asymptotic equivalence of two such systems is characterized by the fact that they share the same hyperbolic modes and the same second-order low-frequency expansions for their eigenvalues (see [19]). Note that, in the mentioned expansions for the second-order Whitham's system, we find linear group velocities which are the characteristic speeds of the hyperbolic part ($\lambda_{0,j} - \bar{c}$ with the above notation), as well as diffusion coefficients, that is,

$$\left\langle \begin{pmatrix} \tilde{k}_0^{(j)} \\ \tilde{M}_0^{(j)} \end{pmatrix}, \begin{bmatrix} \bar{d}_{1,1} & \bar{d}_{1,2} \\ \bar{d}_{2,1} & \bar{d}_{2,2} \end{bmatrix} \begin{pmatrix} k_0^{(j)} \\ M_0^{(j)} \end{pmatrix} \right\rangle_{\mathbb{R}^2},$$

with the above notation.

4. The KdV Limit

Recall that the authors' interest in the (KdV-KS) equation lies essentially in the δ-small region. Yet a quick look at formulas of the previous section suggests that a result such as Corollary 3.5 could only show stability for perturbations corresponding to Floquet parameters of size $o(\delta)$. Even at the formal level, something new is needed to try to capture the behavior corresponding to a full (ξ, δ)-small region.

Our goal now is precisely to provide another modulation procedure, and to infer from it stability conditions that are the natural candidates to decide stability in the full (ξ, δ)-small region.

Preliminaries. The δ-small limit is a singular perturbation of the KdV equation. Therefore, the very first step of our discussion is to obtain a parametrization of a family of periodic traveling waves of (KdV) that provides for (KdV) an analogue of condition (A'). This may then be related to the structure of the generalized co-periodic kernels of the operator describing the KdV evolution linearized about a given wave (in a way similar to Lemma 2.1). One should also prove using geometric singular perturbation analysis that wave profiles of (KdV-KS) emerge smoothly from wave profiles of (KdV). This is a nontrivial task, as may be guessed from the fact that dimensions of respective sets of periodic traveling waves do not match.

As the focus of the present section is on the formal derivation, we do not formulate precise statements answering these questions. The reader is referred to [10], where a wave profile perturbation result is proved. One may also read [5], where the full Bloch spectrum of the KdV waves is investigated in details (including explicit resolvent formulas). This latter work relies heavily on the integrability of the KdV equation.

Notably, the KdV waves we will deal with are explicit, being cnoidal waves. Yet we restrain from giving explicit formulas, and choose a parametrization convenient for the slow modulation discussion. We denote by $U_{\mathrm{cn}}(\,\cdot\,; k, M, e)$ the solution, of period 1, of

$$k(3U^2 - c_{\mathrm{cn}}(k, M, e)U)' + k^3 U''' = 0, \quad \langle U \rangle = M, \quad \frac{1}{2}\langle U^2 \rangle = e. \tag{4.1}$$

Waves of (KdV) are recovered through

$$u(x, t) = U_{\mathrm{cn}}(k(x + \phi - c_{\mathrm{cn}}(k, M, e)t); k, M, e)$$

(where ϕ is an arbitrary phase shift). As for (KdV-KS), arbitrary profiles may be recovered from zero-mean profiles, since we can split profiles along

$$\begin{aligned} U_{\mathrm{cn}}(\cdot\,; k, M, e) &= M + U_{\mathrm{cn}}^{(0)}(\cdot\,; k, e - M^2/2), \\ c_{\mathrm{cn}}(k, M, e) &= 6M + c_{\mathrm{cn}}^{(0)}(k, e - M^2/2). \end{aligned}$$

On wave profiles for (KdV-KS), we mark now the δ-dependence. They expand as

$$\begin{aligned} U^{\delta}(\cdot\,; k, M) &= U_0(\cdot\,; k, M) + \mathcal{O}(\delta), \\ c^{\delta}(k, M) &= c_0(k, M) + \mathcal{O}(\delta^2), \end{aligned}$$

where

$$\begin{aligned} U_0(\cdot\,; k, M) &= U_{\mathrm{cn}}(\cdot\,; k, M, M^2/2 + E(k)), \\ c_0(k, M) &= c_{\mathrm{cn}}(k, M, M^2/2 + E(k)), \end{aligned}$$

where E is some function described implicitly below [10]. Again, note that not all KdV waves generate KdV-KS waves; the relation $e = M^2/2 + E(k)$ implements this selection principle.

Fix some wave parameters $(\bar{k}, \bar{M}, \bar{e})$ (*a priori*, $\bar{e}$ is arbitrary) corresponding to a wave $\bar{U}_{\mathrm{cn}}$ with speed $\bar{c}_{\mathrm{cn}}$, and introduce the KdV operator

$$\mathbb{L}[g] = -\bar{k}((6\bar{U}_{\mathrm{cn}} - \bar{c}_{\mathrm{cn}})g)' - \bar{k}^3 g'''.$$

Denote $\mathbb{L}_{\xi}$ the corresponding Bloch symbol, which expands as

$$\mathbb{L}_{\xi} = \mathbb{L}_0 + i\bar{k}\xi\mathbb{L}^{(1)} + (i\bar{k}\xi)^2\mathbb{L}^{(2)} + (i\bar{k}\xi)^3\mathbb{L}^{(3)}.$$

Then, the value 0 is an eigenvalue of $\mathbb{L}_0$ of geometric multiplicity 2 and algebraic multiplicity 3. Moreover,

$$\mathbb{L}_0\bar{U}_{\mathrm{cn}}' = 0, \quad \mathbb{L}_0\,\partial_M\bar{U}_{\mathrm{cn}} = -\bar{k}\,\partial_M\bar{c}_{\mathrm{cn}}\bar{U}_{\mathrm{cn}}', \quad \mathbb{L}_0\,\partial_e\bar{U}_{\mathrm{cn}} = -\bar{k}\,\partial_e\bar{c}_{\mathrm{cn}}\bar{U}_{\mathrm{cn}}',$$

and

$$\mathbb{L}_0^* 1 = 0, \quad \mathbb{L}_0^*\bar{U}_{\mathrm{cn}} = 0$$

(where $\mathbb{L}_0^*$ denotes the formal adjoint of $\mathbb{L}_0$), with pairing relations

$$\begin{aligned} &\langle \bar{U}_{\mathrm{cn}}' \rangle = \langle \partial_e\bar{U}_{\mathrm{cn}} \rangle = 0, &&\langle \bar{U}_{\mathrm{cn}}, \bar{U}_{\mathrm{cn}}' \rangle = \langle \bar{U}_{\mathrm{cn}}, \partial_M\bar{U}_{\mathrm{cn}} \rangle = 0, \\ &\langle \partial_M\bar{U}_{\mathrm{cn}} \rangle = 1, &&\langle \bar{U}_{\mathrm{cn}}, \partial_e\bar{U}_{\mathrm{cn}} \rangle = 1. \end{aligned}$$

Note also that $\mathbb{L}_0\,\partial_k\bar{U}_{\mathrm{cn}} + \bar{k}\,\partial_k\bar{c}_{\mathrm{cn}}\bar{U}' + \mathbb{L}^{(1)}\bar{U}_{\mathrm{cn}}' = 0$.

Now fix some wave parameters $(\bar{k}, \bar{M})$. To discuss further the spectral problem we are considering, let us mark the δ dependence on the operator L:

$$L^{\delta}[g] = -\bar{k}((6\bar{U}^{\delta} - \bar{c}^{\delta})g)' - \bar{k}^3 g''' - \delta(\bar{k}^2 g'' + \bar{k}^4 g'''').$$

Denote L_{ξ}^{δ} the corresponding Bloch symbol. Our focus is thus on the critical spectral problems $L_{\xi}^{\delta}q = \lambda q$ when (δ, ξ) is small. There are at least two natural asymptotic limits to investigate.

First, if we set $\bar{e} = \bar{M}^2/2 + E(\bar{k})$, $L^0_\xi = \mathbb{L}_\xi$, the spectrum of $\mathbb{L}_\xi$ is known [5], and we may indeed expect to expand the spectrum of $L^{\bar{\delta}|\xi|}_\xi$ when $(\xi, \bar{\delta})$ is small. A relevant first-order modulation system should there be that of the KdV equation. In this area, formal expansions have already been performed and numerically evaluated [1]. They are in agreement with full stability numerical investigations [2,7].

Second, the spectrum of L^δ_0 is known, and we may also expect to expand the spectrum of $L^\delta_{\delta\bar{\xi}}$ when $(\bar{\xi}, \delta)$ is small. Here, the relevant first-order modulation system is the limit when $\delta \to 0$ of the first-order Whitham's system of the previous section.

We insist on the fact that the dimensions of the two situations are different and that there is *a priori* no trivial way to relate them. Relying on a formal slow-modulation *ansatz*, we propose now intermediate modulation systems that fill this gap. They correspond to the situation where one fixes a $\bar{\delta}$ and expands the spectrum of $L^{\bar{\delta}|\xi|}_\xi$ in ξ small. Then, the limit $\bar{\delta} \to 0$ is related to the first zone, and the singular limit $\bar{\delta} \to \infty$ brings us into the second area.

A modulation system. Let us fix $\bar{\delta}$. We insert the *ansatz* ((3.24), (3.25)) in

$$\partial_t u + \partial_x(3u^2) + \partial_x^3 u + \bar{\delta}\varepsilon(\partial_x^2 u + \partial_x^4 u) = 0.$$

First, identification yields the following (with $\Omega_0 = \partial_T \phi_0$ and $k_0 = \partial_X \phi_0$):

$$\Omega_0\, \partial_y U_0 + k_0\, \partial_y(3U_0^2) + k_0^3\, \partial_y^3 U_0 = 0, \tag{4.2}$$

solved with

$$\begin{aligned} \Omega_0(X,T) &= -k_0(X,T) c_{\mathrm{cn}}(k_0(X,T), M_0(X,T), e_0(X,T)),\\ U_0(y;X,T) &= U_{\mathrm{cn}}(y, k_0(X,T), M_0(X,T), e_0(X,T)). \end{aligned} \tag{4.3}$$

Next, setting $\Omega_1 = \partial_T \phi_1$ and $k_1 = \partial_X \phi_1$ gives

$$\begin{aligned} (\Omega_1 + c_{\mathrm{cn},0} k_1) U_0' - k_1 \mathbb{L}^{(1)} U_0' - \mathbb{L}_0 U_1 - \mathbb{L}^{(1)}\, \partial_X U_0 \\ - \partial_X k_0 \mathbb{L}^{(2)} U_0' + \partial_T U_0 + c_{\mathrm{cn},0}\, \partial_X U_0 = -\bar{\delta}(k_0^2 U_0'' + k_0^4 U_0''''). \end{aligned} \tag{4.4}$$

Solvability conditions are obtained by taking scalar products with 1 and U_0.

The compatibility equality $\partial_T k_0 = \partial_X \Omega_0$, together with solvability conditions, composes the modulation system

$$\begin{cases} \partial_t k_0 + \partial_x(k_0 c_{\mathrm{cn}}(k_0, M_0, e_0)) = 0, \\ \partial_t M_0 + \partial_x(6e_0) = 0, \\ \partial_t e_0 + \partial_x(Q(k_0, M_0, e_0)) = -\bar{\delta} R(k_0, M_0, e_0), \end{cases} \tag{4.5}$$

where we have used the following definitions, for a general $(\bar{k}, \bar{M}, \bar{e})$:

$$Q(\bar{k}, \bar{M}, \bar{e}) = 2\langle \bar{U}^3 \rangle - 3\bar{k}\bar{e},$$
$$R(\bar{k}, \bar{M}, \bar{e}) = \bar{k}^4\langle (\bar{U}'')^2 \rangle - \bar{k}^2\langle (\bar{U}')^2 \rangle.$$

One may easily check that $R(\bar{k}, \bar{M}, \bar{e}) = R(\bar{k}, 0, \bar{e} - \bar{M}^2/2)$.

As a particular case, setting $\bar{\delta} = 0$ recovers the first-order Whitham's system for KdV waves [25]. At the opposite, KdV-KS waves correspond to constant solutions in the limit $\bar{\delta} \to \infty$. Therefore, one reads on (4.5) the above mentioned selection principle $R(\bar{k}, \bar{M}, \bar{e}) = 0$, also written $\bar{e} = \bar{M}^2/2 + E(\bar{k})$. Going further, one may look at the relaxed version of system (4.5):

$$\begin{cases} \partial_t k_0 + \partial_x(k_0 c_{\mathrm{cn}}(k_0, M_0, M_0^2/2 + E(k_0))) = 0', \\ \partial_t M_0 + \partial_x(3M_0^2 + 6E(k_0)) = 0, \end{cases} \tag{4.6}$$

which coincides with the limit $\delta \to 0$ of system (3.32).

Subcharacteristic conditions. Going onward, and exploiting system (4.5), we arrive at a supposition about the following set of subcharacteristic conditions:

- (S1) The system without relaxation ((4.5) with $\bar{\delta} = 0$) is strictly hyperbolic;
- (S2) Relaxation is dissipative; that is $\partial_e R(\bar{k}, \bar{M}, \bar{M}^2/2 + E(\bar{k})) > 0$;
- (S3) The relaxed system (4.6) is strictly hyperbolic;
- (S4) Previous characteristic speeds intertwine; that is, once characteristic speeds of (4.5) with $\bar{\delta} = 0$ are ordered as $\alpha_1 < \alpha_2 < \alpha_3$ and characteristic speeds of (4.6) are ordered as $\beta_1 < \beta_2$, then $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$.

Namely, we guess that this set is a natural candidate to provide sufficient conditions for side-band stability uniformly in δ small. By replacing everywhere "strictly" with "weakly", and all strict inequalities with large inequalities, we receive a natural candidate for a set of necessary conditions.

It may seem that these conditions are far from considerations on Kawashima's condition of the previous section. This, however, is not so: first, the subcharacteristic conditions are a reformulation of Kawashima's condition for (degenerate) quasilinear hyperbolic systems with relaxation; second, relaxation limit theory also provides a viscous correction to (4.6) *via* Chapman-Enskog expansions. This relaxed second-order modulation system does provide second-order terms for eigenvalues expansions, and, under subcharacteristic conditions, these second-order corrections have negative real parts. The reader is referred to [4] for a detailed description of the role of Kawashima's condition, and for a general validation of the Chapman-Enskog expansion in the context of (degenerate) quasilinear hyperbolic systems with relaxation.

5. Conclusion

We have studied the prescription power of modulation-averaged systems on the spectral stability of periodic wave trains. For its own interest and its canonical nature, we focused on the Korteweg-de Vries-Kuramoto-Sivashinsky equation. Yet, except for the KdV limit part, the arguments used are both general and robust.

We have first shown how to obtain modulation-averaged equations approximating evolution of local parameters in a slow modulation regime. We then related eigenmodes of the original equation with those of the modulation systems in the respective low-Floquet and low-Fourier regimes. Relations on eigenvalues are readily converted into stability criteria, whereas we hope that relations on eigenfunctions and eigenvectors could serve as the main spectral lemma in a nonlinear justification of the slow modulation *ansatz*. Note that numerical studies show that spectrally stable wave trains (in the diffusive sense discussed here) do exist [2, 7], and this spectral stability may be converted in a nonlinear stability under localized perturbations [2]. However, up till now, nonlinear results are not precise enough to validate the full modulation scenario.

The universal nature of (KdV-KS) stems from the fact that it can be derived in a $\delta \to 0$ regime from many similar situations. Yet expansions discussed previously are not uniform with respect to δ. In the last part of this paper, we tried to explain how to fix this in the slow modulation *ansatz*. This revealed some subcharacteristic conditions we believe crucial for understanding the side-band stability in the $\delta \to 0$ limit. We do not expect to also deduce from modulation theory expansion of eigenfunctions uniformly in δ. Yet there is some hope of proving that modulation predictions are again correct at the eigenvalue level by working on dispersion relations, either from Evans functions (as in [21]) or from critical Bloch matrices (as in the proofs of Lemma 3.1 and Proposition 3.3). Since the spectrum of periodic traveling waves of (KdV) is explicitly known [5], we even expect to be able to combine low-Floquet arguments validating the subcharacteristic conditions with high-Floquet energy estimates, and with examination of regular expansions of intermediate-Floquet eigenvalues. Performing such combinations will allow us to obtain for small δ an analytic proof of diffusive spectral stability for a band of periodic traveling waves of (KdV-KS).

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References

[1] Doron E. Bar and Alexander A. Nepomnyashchy, *Stability of periodic waves governed by the modified Kawahara equation*, Phys. D **86** (1995), no. 4, 586–602. http://dx.doi.org/10.1016/0167-2789(95)00174-3. MR1353179 (96m:35018)

[2] Blake Barker, Matthew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun, *Non linear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation*, in preparation.

[3] ———, *Stability of periodic Kuramoto-Sivashinsky waves*, Appl. Math. Lett. **25** (2012), no. 5, 824–829. http://dx.doi.org/10.1016/j.aml.2011.10.026. MR2888080

[4] STEFANO BIANCHINI, BERNARD HANOUZET, and ROBERTO NATALINI, *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Comm. Pure Appl. Math. **60** (2007), no. 11, 1559–1622. http://dx.doi.org/10.1002/cpa.20195. MR2349349 (2010i:35227)

[5] NATE BOTTMAN and BERNARD DECONINCK, *KdV cnoidal waves are spectrally stable*, Discrete Contin. Dyn. Syst. **25** (2009), no. 4, 1163–1180. http://dx.doi.org/10.3934/dcds.2009.25.1163. MR2552133 (2011h:35239)

[6] HSUEH-CHIA CHANG and EVGENY A. DEMEKHIN, *Complex Wave Dynamics on Thin Films*, Studies in Interface Science, vol. 14, Elsevier Science B.V., Amsterdam, 2002. MR2362445 (2008i:76084)

[7] HSUEH-CHIA CHANG, EVGENY A. DEMEKHIN, and DMITY I. KOPELEVICH, *Laminarizing effects of dispersion in an active-dissipative nonlinear medium*, Phys. D **63** (1993), no. 3–4, 299–320. http://dx.doi.org/10.1016/0167-2789(93)90113-F.

[8] MICHAEL C. CROSS and PIERRE C. HOHENBERG, *Pattern formation out of equilibrium*, Rev. Mod. Phys. **65** (1993), 851–1112. http://dx.doi.org/10.1103/RevModPhys.65.851.

[9] ARJEN DOELMAN, BJÖRN SANDSTEDE, ARND SCHEEL, and GUIDO SCHNEIDER, *The dynamics of modulated wave trains*, Mem. Amer. Math. Soc. **199** (2009), no. 934, viii+105. MR2507940 (2011d:35238)

[10] NICHOLAS M. ERCOLANI, DAVID W. MCLAUGHLIN, and HEINZ ROITNER, *Attractors and transients for a perturbed periodic KdV equation: a nonlinear spectral analysis*, J. Nonlinear Sci. **3** (1993), no. 4, 477-539. http://dx.doi.org/10.1007/BF02429875. MR1246704 (94j:35155)

[11] URIEL FRISCH, ZHEN-SU SHE, and OLIVIER THUAL, *Viscoelastic behaviour of cellular solutions to the Kuramoto-Sivashinsky model*, J. Fluid Mech. **168** (1986), 221–240. http://dx.doi.org/10.1017/S0022112086000356. MR861201 (87j:80015)

[12] ROBERT A. GARDNER, *On the structure of the spectra of periodic travelling waves*, J. Math. Pures Appl. (9) **72** (1993), no. 5, 415–439. MR1239098 (94j:35072)

[13] MATTHEW A. JOHNSON and KEVIN ZUMBRUN, *Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case*, J. Differential Equations **249** (2010), no. 5, 1213–1240. http://dx.doi.org/10.1016/j.jde.2010.04.015. MR2652171 (2011k:35014)

[14] TOSIO KATO, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition. MR1335452 (96a:47025)

[15] TAKUJI KAWAHARA and SADAYOSHI TOH, *Pulse interactions in an unstable dissipative-dispersive nonlinear system*, Phys. Fluids **31** (1988), no. 8, 2103–2111. http://dx.doi.org/10.1063/1.866610. MR951327 (89f:76029)

[16] SCHUICHI KAWASHIMA, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics*, PhD thesis, Kyoto University, 1983.

[17] YOSHIKI KURAMOTO, *Chemical Oscillations, Waves, and Turbulence*, Springer Series in Synergetics, vol. 19, Springer-Verlag, Berlin, 1984. http://dx.doi.org/10.1007/978-3-642-69689-3. MR762432 (87e:92054)

[18] YOSHIKI KURAMOTO and TOSHIO TSUZUKI, *On the formation of dissipative structures in reaction-diffusion systems*, Progress of Theoretical Physics **54** (1975), no. 3. http://dx.doi.org/10.1143/PTP.54.687.

[19] TAI-PING LIU and YANNI ZENG, *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws*, Mem. Amer. Math. Soc. **125** (1997), no. 599, viii+120. MR1357824 (97g:35107)

[20] ROBERT L. PEGO, GUIDO SCHNEIDER, and HANNES UECKER, *Long-time persistence of Korteweg-de Vries solitons as transient dynamics in a model of inclined film flow*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 1, 133–146. http://dx.doi.org/10.1017/S0308210505001113. MR2359776 (2009b:35365)

[21] DENIS SERRE, *Spectral stability of periodic solutions of viscous conservation laws: large wavelength analysis*, Comm. Partial Differential Equations **30** (2005), no. 1-3, 259–282. http://dx.doi.org/10.1081/PDE-200044492. MR2131054 (2006f:35178)

[22] GREGORY I. SIVASHINSKY, *Nonlinear analysis of hydrodynamic instability in laminar flames I: Derivation of basic equations*, Acta Astronaut. **4** (1977), no. 11-12, 1177–1206. MR0502829 (58 #19741)

[23] ______, *Instabilities, pattern formation, and turbulence in flames*, Annual Review of Fluid Mechanics **15** (Jan. 1983), no. 1, 179–199. http://dx.doi.org/10.1146/annurev.fl.15.010183.001143.

[24] GREGORY I. SIVASHINSKY and D. M. MICHELSON, *On irregular wavy flow of a liquid film down a vertical plane*, Progress of Theoretical Physics **63** (1980), no. 6, 2112–2114. http://dx.doi.org/10.1143/PTP.63.2112.

[25] GERALD B. WHITHAM, *Linear and Nonlinear Waves*, Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics. MR0483954 (58 #3905)

[26] HITAI AUNG WIN, *Model equation of surface waves of viscous fluid down an inclined plane*, J. Math. Kyoto Univ. **33** (1993), no. 3, 803–824. MR1239094 (95e:35171)

[27] JUN YU and YI YANG, *Evolution of small periodic disturbances into roll waves in channel flow with internal dissipation*, Stud. Appl. Math. **111** (2003), no. 1, 1–27. http://dx.doi.org/10.1111/1467-9590.t01-2-00225. MR1985993 (2004d:76039)

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