Stability of periodic waves: roll-waves over thin films

Miguel Rodrigues

Institut Camille Jordan
Université Lyon 1

Short course II
IHP, March 2015

Jointly with Blake Barker (Brown), Mathew Johnson (Kansas), Pascal Noble (Toulouse) and Kevin Zumbrun (Indiana).
Outline.

1 Models
   • General picture
   • The St-Venant system
   • The KdV-KS equation

2 Overview of nonlinear theory

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
Outline.

1 Models
   - General picture
     - The St-Venant system
     - The KdV-KS equation

2 Overview of nonlinear theory

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
Roll-waves in shallow flows.

In a channel.  

Courtesy of Neil Balmforth.

The Saint-Venant system (SV)

\[ h_t + (hu)_x = 0, \]

\[ (hu)_t + \left( hu^2 + \frac{h^2}{2F^2} \right)_x = h - |u|u + \nu (hu_x)_x. \]

\( F > 2 \), primary instability.
Roll-waves in shallow flows.

In a channel.

Courtesy of Neil Balmforth.

The Saint-Venant system (SV)

\[
\begin{align*}
ht & + (hu)_x = 0, \\
(hu)_t & + \left( hu^2 + \frac{h^2}{2F^2} \right)_x \\
& = h - |u|u + \nu (hu_x)_x.
\end{align*}
\]

\( F > 2 \), primary instability.

The Korteweg-de Vries/Kuramoto-Sivashinsky equation (KdV-KS)

\[
Us + \left( \frac{1}{2} U^2 \right)_Y + U_{YYY} + \delta(U_{YY} + U_{YYY}) = 0.
\]

Near threshold \( \delta \sim \sqrt{F - 2} \).
Outline.

1 Models
   • General picture
   • The St-Venant system
   • The KdV-KS equation

2 Overview of nonlinear theory

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
Nondimensionalization.

Starting from

\[
\begin{align*}
\partial_t \tilde{h} + \partial_x (\tilde{h} \tilde{u}) &= 0, \\
\partial_t (\tilde{h} \tilde{u}) + \partial_x (\tilde{h} \tilde{u}^2 + g \cos(\theta) \frac{1}{2} \tilde{h}^2) &= g \sin(\theta) \tilde{h} - C_f |\tilde{u}| \tilde{u} + \nu \partial_x (\tilde{h} \partial_x \tilde{u}),
\end{align*}
\]

Froude number.

\[
F = \sqrt{\frac{\tan(\theta)}{C_f}}.
\]
Scaling (SV).

Scale $a$ as

$$a = \frac{\tilde{a}}{a}.$$

$$x = \nu^{2/3} (g \sin \theta)^{-1/3} \quad \quad t = \nu^{1/3} (g \sin \theta)^{-2/3}$$

$$u = \nu^{1/3} (g \sin \theta)^{1/3} \quad \quad h = C_f \nu^{2/3} (g \sin \theta)^{-1/3}$$

$\nu$ is arbitrary.

We could choose $\nu = 1$ through $\nu = \tilde{\nu}$. 

L.M. Rodrigues (Lyon 1) About periodic waves IHP 2015 3 / 48
Mass-Lagrangian formulation.

Replace $x$ with $y$ such that

$$dy = h \, dx - hu \, dt,$$

and set $\tau = 1/h$.

The Saint-Venant system (SV) becomes

$$\partial_t \tau - \partial_y u = 0,$$

$$\partial_t u + \partial_y \left( \frac{\tau^{-2}}{2F^2} \right) = 1 - \tau |u|u + \nu \partial_y (\tau^{-2} \partial_y u).$$
Mass-Lagrangian formulation.

Replace $x$ with $y$ such that

\[ dy = h \, dx - hu \, dt, \]

and set $\tau = 1/h$.

The Saint-Venant system (SV) becomes

\[
\begin{align*}
\partial_t \tau & - \partial_y u = 0, \\
\partial_t u & + \partial_y \left( \frac{\tau^{-2}}{2F^2} \right) = 1 - \tau |u|u + \nu \partial_y (\tau^{-2} \partial_y u).
\end{align*}
\]

Spectral equivalence respecting Floquet decomposition

Outline.

1 Models
   - General picture
   - The St-Venant system
   - The KdV-KS equation

2 Overview of nonlinear theory

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
A canonical equation.

Start for example with a scalar conservation law

\[ \partial_t u + \partial_x (f(u; \epsilon)) + \partial_x^2 (g(u; \epsilon)) + \partial_x^3 (h(u; \epsilon)) + \cdots = 0. \]
A canonical equation.

Start for example with a scalar conservation law

\[
\partial_t u + \partial_x(f(u; \epsilon)) + \partial_x^2(g(u; \epsilon)) + \partial_x^3(h(u; \epsilon)) + \cdots = 0.
\]

Take a constant \( u \) such that \( g'(u; \epsilon) = O(\epsilon) \).
A canonical equation.

Start for example with a scalar conservation law

\[ \partial_t u + \partial_x (f(u; \epsilon)) + \partial_x^2 (g(u; \epsilon)) + \partial_x^3 (h(u; \epsilon)) + \cdots = 0. \]

Take a constant \( u \) such that \( g'(u; \epsilon) = O(\epsilon) \).

A KdV-like small-amplitude long-wave scaling

\[ v = \frac{u - u}{\epsilon} \quad \text{and} \quad (\xi, \tau) = \left( \frac{1}{\epsilon^2} (x - f'(u; \epsilon) t), \epsilon^\frac{3}{2} t \right) \]
A canonical equation.

Start for example with a scalar conservation law

\[ \partial_t u + \partial_x (f(u; \epsilon)) + \partial_x^2 (g(u; \epsilon)) + \partial_x^3 (h(u; \epsilon)) + \cdots = 0. \]

Take a constant \( u \) such that \( g'(u; \epsilon) = O(\epsilon) \).

A KdV-like small-amplitude long-wave scaling

\[ \nu = \frac{u - u}{\epsilon} \quad \text{and} \quad (\xi, \tau) = \left( \epsilon^{\frac{1}{2}} (x - f'(u; \epsilon) t), \epsilon^{\frac{3}{2}} t \right) \]

leads modulo \( O(\epsilon) \) to

\[
\begin{align*}
\partial_\tau \nu + \partial_\xi \left( \frac{1}{2} f''(u; \epsilon) \nu^2 \right) + \partial_\xi^3 (h'(u; \epsilon) \nu) \\
= -\epsilon^{\frac{1}{2}} \left( \partial_\xi^2 \left( \frac{g'(u; \epsilon)}{\epsilon} \nu \right) + \partial_\xi^2 \left( \frac{1}{2} g''(u; \epsilon) \nu^2 \right) + \partial_\xi^4 (i'(u; \epsilon) \nu) \right).
\end{align*}
\]
Weak low-frequency instability in a parabolic equation.

\[ \partial_t u + \alpha_1 \partial_x (u^2) + \delta_1 \partial_x^3 u + \delta_2 \partial_x^2 u + \alpha_2 \partial_x^2 (u^2) + \delta_3 \partial_x^4 u = 0. \]

Nonlinearity, \( \alpha_1 \neq 0 \),
dispersion, \( \delta_1 \neq 0 \),
Weak low-frequency instability in a parabolic equation.

\[ \partial_t u + \alpha_1 \partial_x (u^2) + \delta_1 \partial_x^3 u + \delta_2 \partial_x^2 u + \alpha_2 \partial_x^2 (u^2) + \delta_3 \partial_x^4 u = 0. \]

Nonlinearity, \( \alpha_1 \neq 0 \),
dispersion, \( \delta_1 \neq 0 \),
low-frequency instability of 0, \( \delta_2 > 0 \)
parabolicity, \( \delta_3 > 0 \).
Weak low-frequency instability in a parabolic equation.

\[
\partial_t u + \alpha_1 \partial_x (u^2) + \delta_1 \partial_x^3 u + \delta_2 \partial_x^2 u + \alpha_2 \partial_x^2 (u^2) + \delta_3 \partial_x^4 u = 0.
\]

Nonlinearity, \( \alpha_1 \neq 0 \),
dispersion, \( \delta_1 \neq 0 \),
low-frequency instability of \( 0 \), \( \delta_2 > 0 \)
parabolicity, \( \delta_3 > 0 \).

One may scale to

\[
\partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) + \partial_x^3 u + \delta \left( \partial_x^2 u + R \partial_x^2 (u^2) + \partial_x^4 u \right) = 0.
\]

with \( \delta > 0 \) and \( R \in \mathbb{R} \).
Most unstable mode for \( 0 \) has typical length 1.
From (SV) : scaling.

### Constant background

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$</td>
<td>$u_0 = \tau_0^{-1/2}$ velocity, $c_0 = \frac{1}{2} \tau_0^{-3/2}$ sound speed at $F = 2$.</td>
</tr>
</tbody>
</table>
From (SV) : scaling.

Constant background

\[ \tau_0, \quad u_0 = \tau_0^{-1/2} \text{ velocity,} \quad c_0 = \frac{1}{2} \tau_0^{-3/2} \text{ sound speed at } F = 2. \]

Small-amplitude long-wave scaling with \( \tilde{\delta} = \sqrt{F - 2} \)

\[
\tilde{\tau} = 3 \tilde{\delta}^{-2} \left( \frac{\tau}{\tau_0} - 1 \right), \quad \tilde{u} = 6 \tilde{\delta}^{-2} \left( \frac{u}{u_0} - 1 \right),
\]

\[
S = \frac{\tilde{\delta}^3 t}{4 \tau_0^{1/4} \nu^{1/2}}, \quad Y = \frac{\tau_0^{5/4} \tilde{\delta} (x - c_0 t)}{\nu^{1/2}}.
\]
Derivation of (KdV-KS) from (SV).

Set $\tilde{w} = 2\tilde{\delta}^{-2}(\tilde{u} + \tilde{\tau})$

$\partial_S \tilde{\tau} - \partial_Y \tilde{w} = 0$

$\frac{\tilde{\delta}^3}{8\tau_0^{1/4} \nu^{1/2}} \partial_S \tilde{w} + \frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y \left( \tilde{\tau} + \frac{1}{2} \tilde{\tau}^2 - \tilde{w} + \tilde{\delta}^2 f(\tilde{\tau}, \delta) \right)$

$= \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau} + \tilde{\delta}^2 g(\tilde{\tau}, \tilde{w}, \delta) + \frac{1}{2} \tilde{\delta}^2 \partial_{YY} \tilde{w}$

$+ \tilde{\delta}^2 \partial_Y \left( r(\tilde{\tau}, \delta) \partial_Y (\delta^2 \tilde{w} - c_0 \tilde{\tau}) \right)$

for some smooth $f, g, r$. 
Derivation of (KdV-KS) from (SV).

Set $\tilde{w} = 2\tilde{\delta}^{-2}(\tilde{u} + \tilde{\tau})$

$$\partial_S \tilde{\tau} - \partial_Y \tilde{w} = 0$$

$$\frac{\tilde{\delta}^3}{8\tau_0^{1/4} \nu^{1/2}} \partial_S \tilde{w} + \frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y \left( \tilde{\tau} + \frac{1}{2} \tilde{\tau}^2 - \tilde{w} + \tilde{\delta}^2 f(\tilde{\tau}, \delta) \right)$$

$$= \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau} + \tilde{\delta}^2 g(\tilde{\tau}, \tilde{w}, \delta) + \frac{1}{2} \tilde{\delta}^2 \partial_{YY} \tilde{w}$$

$$+ \tilde{\delta}^2 \partial_Y \left( r(\tilde{\tau}, \delta) \partial_Y (\delta^2 \tilde{w} - c_0 \tilde{\tau}) \right)$$

for some smooth $f, g, r$.

Modulo $O(\tilde{\delta}^2)$

$$\partial_S \tilde{\tau} - \partial_Y \tilde{w} = 0,$$

$$\frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y \left( \tilde{\tau} + \frac{1}{2} \tilde{\tau}^2 - \tilde{w} \right) = \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau}. $$
Derivation of (KdV-KS) from (SV).

Set \( \tilde{w} = 2\tilde{\delta}^{-2}(\tilde{u} + \tilde{\tau}) \)

\[
\begin{align*}
\partial_S \tilde{\tau} - \partial_Y \tilde{w} &= 0 \\
\frac{\tilde{\delta}^3}{8\tau_0^{1/4} \nu^{1/2}} \partial_S \tilde{w} + \frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y \left( \tilde{\tau} + \frac{1}{2} \tilde{\tau}^2 - \tilde{w} + \tilde{\delta}^2 f(\tilde{\tau}, \delta) \right) \\
&= \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau} + \tilde{\delta}^2 g(\tilde{\tau}, \tilde{w}, \delta) + \frac{1}{2} \tilde{\delta}^2 \partial_{YY} \tilde{w} \\
&\quad + \tilde{\delta}^2 \partial_Y \left( r(\tilde{\tau}, \delta) \partial_Y (\delta^2 \tilde{w} - c_0 \tilde{\tau}) \right)
\end{align*}
\]

for some smooth \( f, g, r \).

Modulo \( \mathcal{O}(\tilde{\delta}^2) \)

\[
\begin{align*}
\partial_S \tilde{\tau} - \partial_Y \tilde{w} &= 0, \\
\frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y (\tilde{\tau} + \partial_{YY} \tilde{\tau}) &= \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau}.
\end{align*}
\]
Derivation of \((\text{KdV-KS})\) from \((\text{SV})\).

Set \(\tilde{w} = 2\tilde{\delta}^{-2}(\tilde{u} + \tilde{\tau})\)

\[
\partial_S \tilde{\tau} - \partial_Y \tilde{w} = 0
\]

\[
\frac{\tilde{\delta}^3}{8\tau_0^{1/4} \nu^{1/2}} \partial_S \tilde{w} + \frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y \left( \tilde{\tau} + \frac{1}{2} \tilde{\tau}^2 - \tilde{w} + \tilde{\delta}^2 f(\tilde{\tau}, \delta) \right)
\]

\[
= \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau} + \tilde{\delta}^2 g(\tilde{\tau}, \tilde{w}, \delta) + \frac{1}{2} \tilde{\delta}^2 \partial_{YY} \tilde{w}
\]

\[
+ \tilde{\delta}^2 \partial_Y \left( r(\tilde{\tau}, \delta) \partial_Y (\delta^2 \tilde{w} - c_0 \tilde{\tau}) \right)
\]

for some smooth \(f, g, r\).

Modulo \(O(\tilde{\delta}^2)\)

\[
\partial_S \tilde{\tau} - \partial_Y \tilde{w} = 0,
\]

\[
\frac{\tilde{\delta}}{2\tau_0^{1/4} \nu^{1/2}} \partial_Y (\tilde{\tau} + \partial_{YY} \tilde{\tau}) = \frac{1}{2} \tilde{\tau}^2 - \tilde{w} - \partial_{YY} \tilde{\tau}.
\]
Outline.

1 Models

2 Overview of nonlinear theory
   - Results
   - Diffusive spectral stability
   - Whitham’s averaging

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
Outline.

1. Models

2. Overview of nonlinear theory
   - Results
     - Diffusive spectral stability
     - Whitham’s averaging

3. Survey of spectral numerical studies

4. Near threshold of primary instability

5. Conclusion
From the spectrum to the nonlinear dynamics.


A diffusively spectrally stable periodic traveling wave is nonlinearly stable.


About a stable wave the dynamics is well-described by the averaging theory for slowly modulated wavetrains.
Direct simulation: space-time diagram.

About a stable wave of (KdV-KS).


Peaks.

Troughs.

Theory.
Outline.

1 Models

2 Overview of nonlinear theory
   - Results
   - Diffusive spectral stability
   - Whitham’s averaging

3 Survey of spectral numerical studies

4 Near threshold of primary instability

5 Conclusion
Model equation.

$t > 0$ time, $x \in \mathbb{R}$ space, $U(x, t) \in \mathbb{R}^d$ unknown.

$$U_t + (f(U))_x = U_{xx}.$$
Model equation.

\( t > 0 \) time, \( x \in \mathbb{R} \) space, \( U(x, t) \in \mathbb{R}^d \) unknown.

\[
U_t + (f(U))_x = U_{xx}.
\]

Traveling waves. \( U(t, x) = U(k(x - ct)) \) moving with velocity \( c \).

Profile \( U \) of period 1, solution to

\[
-kc U_x + k(f(U))_x = k^2 U_{xx}.
\]
Model equation.

\( t > 0 \) time, \( x \in \mathbb{R} \) space, \( U(x, t) \in \mathbb{R}^d \) unknown.

\[
U_t + (f(U))_x = U_{xx}.
\]

Traveling waves. \( U(t, x) = U(k(x - ct)) \) moving with velocity \( c \).

Profile \( U \) of period 1, solution to \( -k c U_x + k(f(U))_x = k^2 U_{xx} \).

Linear generator

\[
L := k^2 \partial_x^2 + k c \partial_x - k \partial_x df(U).
\]

Integral transform?
Model equation.

\( t > 0 \) time, \( x \in \mathbb{R} \) space, \( U(x, t) \in \mathbb{R}^d \) unknown.

\[
U_t + (f(U))_x = U_{xx}.
\]

Traveling waves. \( U(t, x) = U(k(x - ct)) \) moving with velocity \( c \).

Profile \( U \) of period 1, solution to
\[
-kc U_x + k(f(U))_x = k^2 U_{xx}.
\]

Linear generator

\[
L := k^2 \partial_x^2 + kc \partial_x - k \partial_x df(U)
\]

with Bloch symbols

\[
L_\xi := k^2 (\partial_x + i\xi)^2 + k(\partial_x + i\xi)(c - df(U)), \quad \xi \in [-\pi, \pi].
\]

Each \( L_\xi \) acts on functions of period 1.

\( \triangleright \) Integral transform?
Diffusive spectral stability.

### Spectral parametrization

\[
\sigma(L) = \bigcup_{\xi \in [-\pi, \pi]} \sigma_{\text{per}}(L\xi).
\]

(D1) Critical spectrum reduced to \{0\}.

\[
\sigma(L) \subset \{\lambda \mid \text{Re}\lambda < 0\} \cup \{0\}.
\]

(D2) \(\exists \theta > 0, \forall \xi \in [-\pi, \pi],\)

\[
\sigma_{\text{per}}(L\xi) \subset \{\lambda \mid \text{Re}\lambda \leq -\theta|\xi|^2\}.
\]

(D3) \(\lambda = 0\) of multiplicity \(d + 1\) for \(L_0\) (minimal dimension).

(H) Distinct group velocities.

---

Spectrum of a stable wave of (SV).

Outline.

1. Models

2. Overview of nonlinear theory
   - Results
   - Diffusive spectral stability
   - Whitham’s averaging

3. Survey of spectral numerical studies

4. Near threshold of primary instability

5. Conclusion
Two-scale expansion.

Up to translation, parametrization \( U = U^{(M,k)} \), \( c = c(M,k) \), \( \omega = -kc \), with

\[
k \text{ wavenumber}, \quad M = \langle U^{(M,k)} \rangle,
\]

where \( \langle \cdot \rangle \) denotes averaging over a period.
Two-scale expansion.

Up to translation, parametrization \( U = U^{(M,k)} \), \( c = c(M, k) \), \( \omega = -k c \), with

\[ k \text{ wavenumber}, \quad M = \langle U^{(M,k)} \rangle, \]

where \( \langle \cdot \rangle \) denotes averaging over a period.

Fast-oscillatory/slow ansatz.

\[
U(t, x) = (U_0 + \varepsilon U_1) \left( \frac{\varepsilon t}{T}, \frac{\varepsilon x}{X}, \frac{\Psi(\varepsilon t, \varepsilon x)}{\varepsilon} \right) + o(\varepsilon),
\]

with \( U_0 \) and \( U_1 \) of period 1 in \( \theta \).
At first order: slowly modulated wavetrains.

\[
U(t, x) = U^{(M, \kappa)}(\varepsilon t, \varepsilon x) \left( \frac{\Psi(\varepsilon t, \varepsilon x)}{\varepsilon} \right) + \mathcal{O}(\varepsilon),
\]

with

\[
\kappa = \Psi_X \quad \text{local wavenumber},
\]

\[
\omega = \Psi_T \quad \text{local time frequency},
\]

\[
c \quad \text{local phase velocity},
\]

satisfying a nonlinear dispersion relation

\[
\Psi_T = \omega(M, \kappa).
\]
First-order averaged equations.

First-order Whitham’s system

\[
(W)_0 \quad \begin{cases} 
\kappa_T = \left(\omega(M, \kappa)\right)_X, \\
M_T = -\left(F(M, \kappa)\right)_X,
\end{cases}
\]

where

\[F(M, \kappa) = \langle f(U(M, \kappa)) \rangle.\]
Slow modulation (at second order).

\[ U(t, x) \sim U^{(M, \kappa)}(t, x) \left( \psi(t, x) \right) \]

with \( \kappa = \Psi_x \)

\[
\begin{aligned}
(W) & \quad \left\{ \begin{array}{l}
M_t + (F(M, \kappa))_x = (d_{11}(M, \kappa)M_x + d_{12}(M, \kappa)\kappa_x)_x, \\
\kappa_t - (\omega(M, \kappa))_x = (d_{21}(M, \kappa)M_x + d_{22}(M, \kappa)\kappa_x)_x,
\end{array} \right.
\end{aligned}
\]

and

\[
(W)_{phase} \quad \psi_t - \omega(M, \kappa) = d_{21}(M, \kappa)M_x + d_{22}(M, \kappa)\kappa_x.
\]
Outline.

1 Models

2 Overview of nonlinear theory

3 Survey of spectral numerical studies
   - Numerical methods
   - The KdV-KS model
   - The Saint Venant model

4 Near threshold of primary instability

5 Conclusion
Outline.

1 Models

2 Overview of nonlinear theory

3 Survey of spectral numerical studies
   • Numerical methods
     • The KdV-KS model
     • The Saint Venant model

4 Near threshold of primary instability

5 Conclusion
Hill’s method & Evans’ function.

- Hill’s method:
  for each $\xi$, project $L_\xi$ on a finite number of Fourier modes (Galerkin) to obtain a matrix.
  Compute the spectrum.

Matlab library by Blake Barker, Jeffrey Humpherys, Kevin Zumbrun.
http://impact.byu.edu/stablab/
Hill’s method & Evans’ function.

- **Hill’s method**: for each $\xi$, project $L_\xi$ on a finite number of Fourier modes (Galerkin) to obtain a matrix. Compute the spectrum.

- **Evans’ function**.
  Write $LV = \lambda V$ as a differential equation $Y' = A(\lambda, \cdot)Y$. Solution operator $\Phi_\lambda$. Then, $X$ being the period,

  \[ \lambda \in \sigma_{\text{per}}(L_\xi) \Leftrightarrow \det(\Phi_\lambda(X, 0) - e^{i\xi}X) = 0. \]

  Count roots of $D(\cdot, \xi)$ (an analytic function).

Matlab library by Blake Barker, Jeffrey Humpherys, Kevin Zumbrun.
http://impact.byu.edu/stablab/
Outline.

1 Models

2 Overview of nonlinear theory

3 Survey of spectral numerical studies
   - Numerical methods
   - The KdV-KS model
   - The Saint Venant model

4 Near threshold of primary instability

5 Conclusion
A full numerical study.

\[ U_t + \left( \frac{1}{2} U^2 \right)_x + \varepsilon U_{xxx} + \delta (U_{xx} + U_{xxxx}) = 0. \]

with \( \varepsilon^2 + \delta^2 = 1. \)

Stability diagram.


Analytic result in the KdV limit \((\varepsilon, \delta) \rightarrow (1, 0)\).

Outline.

1 Models

2 Overview of nonlinear theory

3 Survey of spectral numerical studies
   - Numerical methods
   - The KdV-KS model
   - The Saint Venant model

4 Near threshold of primary instability

5 Conclusion
Near threshold: transition.

Analytic result in the KdV limit.

Green lines: near-KdV theory.

Blue/dark lines: best-fit from large Froude numerics.

Discharge rate: \( q \equiv -c \tau(\cdot) + u(\cdot) \).

\( q = q_0 F, \quad q_0 = 0.4, \quad \nu = 0.1. \)
A full numerical study.


Partial analytic result in the infinite-Froude limit: no stable wave.

Numerical observation: asymptotically

\[
\begin{align*}
\log(X) &= -0.692 \log(F) + 3.46 \log(q) + 0.3 \quad \text{lower stability}, \\
\log(X) &= -0.791 \log(F) + 1.73 \log(q) + 3.92 \quad \text{upper stability}.
\end{align*}
\]
An intermediate case: $F = \sqrt{6}$, $\nu = 0.1$, $q = 1.5745$.


Superimposed phase portraits.  

Phase speed vs. period.

Lower, typical, upper.  

Diammond, circle.  

Upper: $X = 20.4$.  

---
Spectral transition: from small-amplitude.
Spectral transition: to large-period.
Convective instability of solitary waves.


Time-evolution snapshots.
Bumps vs. flat parts : a stable case.


Time-evolution snapshots.
Bumps vs. flat parts: an unstable case.


Time-evolution snapshots.
Outline.

1 Models

2 Overview of nonlinear theory

3 Survey of spectral numerical studies

4 Near threshold of primary instability
   - The KdV-KS case
   - The Saint-Venant case

5 Conclusion
Outline.

1. Models
2. Overview of nonlinear theory
3. Survey of spectral numerical studies
4. Near threshold of primary instability
   - The KdV-KS case
   - The Saint-Venant case
5. Conclusion
Wave parametrization for (KdV).

Up to translation, profile and phase velocity

$$U_{KdV}^{(k,M,P)}(\cdot), \quad c_{KdV}(k, M, P),$$

given in terms of $k$, $M$ and $P := \langle \frac{1}{2} U^2 \rangle$ averaged impulse.
Wave parametrization for (KdV).

Up to translation, profile and phase velocity

$$U_{KdV}^{(k,M,P)}(\cdot), \quad c_{KdV}(k, M, P),$$

given in terms of $k$, $M$ and $P := \langle \frac{1}{2} U^2 \rangle$ averaged impulse.

**Theorem (Spektor, ZEFT 1988.)**

Cnoidal waves are spectrally stable

$$\sigma(L_{KdV}) \subset i\mathbb{R}.$$

Bottman-Deconinck *DCDS* 2009.
Near KdV waves.

Selection criterion \( R(k, M, P) = 0 \)

\[
R(k, M, P) := \langle ( (U_{KdV}^{(k,M,P)})' )^2 \rangle - \langle ( (U_{KdV}^{(k,M,P)})'' )^2 \rangle
\]

gives \( P \) in terms of \( (k, M) \), \( P = P_*(k, M) \).
Near KdV waves.

Selection criterion \( R(k, M, P) = 0 \)

\[
R(k, M, P) := \langle ( (U_{KdV}^{(k,M,P)})')^2 \rangle - \langle ( (U_{KdV}^{(k,M,P)})'')^2 \rangle
\]

gives \( P \) in terms of \((k, M)\), \( P = P_*(k, M) \).

Theorem (Ercolani-McLaughlin-Roitner, J. Nonlinear Sc. 1993.)

Any selected \( U_{KdV}^{(M,P,k)} \) generates a family of near-KdV waves \( (U_\delta^{(M,k)})_{\delta} \).

Geometric singular perturbation and normal forms.
A stability index.

(Formal) expansion: if \( \lambda_0 \in \sigma(L_{KdV, \xi}) \) is simple then

\[
\lambda(\xi, \lambda_0, \delta) \xrightarrow{\delta \to 0} \lambda_0 + \delta \lambda_1(\xi, \lambda_0) + O(\delta^2)
\]

with an explicit corrector \( \lambda_1 \).
A stability index.

(Formal) expansion: if $\lambda_0 \in \sigma(L_{KdV,\xi})$ is simple then

$$\lambda(\xi, \lambda_0, \delta) \xrightarrow{\delta \to 0} \lambda_0 + \delta \lambda_1(\xi, \lambda_0) + \mathcal{O}(\delta^2)$$

with an explicit corrector $\lambda_1$.

For non degenerate selected KdV waves, define

$$\text{Ind} := \sup_{\lambda_0 \in \sigma(L_{KdV,\xi}) \setminus \{0\}} \xi \in [-\pi, \pi] \ \text{Re} \left( \lambda_1(\xi, \lambda_0) \right).$$

Expected persistence of stability when $\text{Ind} < 0$. 
Persistence of stability.

**Numerical observation.** (Bar-Nepomnyashchy, *Phys. D* 1995.)

\[ \text{Ind} < 0 \text{ when } X \text{ lies in} \]

\[ (X_1, X_2) \quad \text{with } X_1 \approx 8.49 \quad \text{and} \quad X_2 \approx 26.17. \]
Persistence of stability.

**Numerical observation.** (Bar-Nepomnyashchy, *Phys. D* 1995.)

\[ \text{Ind} < 0 \text{ when } X \text{ lies in} \]

\[ (X_1, X_2) \quad \text{with } X_1 \approx 8.49 \quad \text{and} \quad X_2 \approx 26.17. \]

**Numerical proof** (Barker, *JDE* 2014.)
Persistence of stability.

**Numerical observation.** (Bar-Nepomnyashchy, *Phys. D* 1995.)

\[ \text{Ind} < 0 \text{ when } X \text{ lies in} \]

\[ (X_1, X_2) \quad \text{with } X_1 \approx 8.49 \quad \text{and} \quad X_2 \approx 26.17. \]

**Numerical proof** (Barker, *JDE* 2014.)


Any non-degenerate selected KdV wave such that \( \text{Ind} < 0 \) yields a family of diffusively spectrally stable waves for (KdV-KS).
Outline of the proof.

**Step 1** By parabolic energy estimates, for any positive $\eta$ nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq -\eta \text{ and } |\text{Re}(\lambda)| + \delta^{\frac{3}{4}} |\text{Im}(\lambda)| \geq C \}.$$

Thus any unstable $\lambda$ satisfies $|\lambda| = O(\delta^{-\frac{3}{4}})$ hence $\delta |\lambda| = o(1)$. 
Outline of the proof.

**Step 1** By **parabolic energy estimates**, for any positive $\eta$ nothing in

$$\{ \lambda \mid \Re(\lambda) \geq -\eta \text{ and } |\Re(\lambda)| + \delta^{\frac{3}{4}} |\Im(\lambda)| \geq C \}.$$

**Step 2** By **approximate Floquet reduction and diagonalization** nothing in

$$\{ \lambda \mid \Re(\lambda) \geq 0 \text{ and } \delta^{-1} |\Re(\lambda)| + |\Im(\lambda)| \geq C \}.$$

Outline of the proof.

**Step 1** By **parabolic energy estimates**, for any positive $\eta$ nothing in

$$\{ \lambda \mid \Re(\lambda) \geq -\eta \text{ and } |\Re(\lambda)| + \delta^{\frac{3}{4}}|\Im(\lambda)| \geq C \}.$$  

**Step 2** By **approximate Floquet reduction and diagonalization** nothing in

$$\{ \lambda \mid \Re(\lambda) \geq 0 \text{ and } \delta^{-1}|\Re(\lambda)| + |\Im(\lambda)| \geq C \}.$$  

**Step 3** For any positive $\epsilon$ and $C$, **geometric singular perturbation à la Fenichel** leads to a **regular spectral problem** when $\epsilon \leq |\lambda| \leq C$.  

---

**L.M. Rodrigues (Lyon 1)**

**About periodic waves**

**IHP 2015**
Outline of the proof.

**Step 1** By **parabolic energy estimates**, for any positive $\eta$ nothing in

$$\{ \lambda | \Re(\lambda) \geq -\eta \text{ and } |\Re(\lambda)| + \delta^{\frac{3}{4}} |\Im(\lambda)| \geq C \}.$$  

**Step 2** By **approximate Floquet reduction and diagonalization** nothing in

$$\{ \lambda | \Re(\lambda) \geq 0 \text{ and } \delta^{-1} |\Re(\lambda)| + |\Im(\lambda)| \geq C \}.$$  

**Step 3** For any positive $\epsilon$ and $C$, **geometric singular perturbation à la Fenichel** leads to a **regular spectral problem** when $\epsilon \leq |\lambda| \leq C$.  

**Step 4** **Critical part** proving and using strict subcharacteristic conditions.
Averaging in a singular limit.

Fix $\bar{\delta} = \delta/\epsilon$ (Noble-Rodrigues, *Indiana Univ. Math. J.* 2013.)

\[
(W)_0 \begin{cases}
\kappa_T - (\omega_{KdV}(\kappa, M, P))_X = 0 \\
M_T + P_X = 0 \\
P_T + (F_{KdV}(\kappa, M, P))_X = \bar{\delta} R(\kappa, M, P)
\end{cases}
\]
Averaging in a singular limit.

Fix \( \bar{\delta} = \delta/\epsilon \) (Noble-Rodrigues, \textit{Indiana Univ. Math. J.} 2013.)

\[
(W)_0 \quad \begin{cases} 
\kappa_T - (\omega_{KdV}(\kappa, M, P))_X & = 0 \\
M_T + P_X & = 0 \\
P_T + (F_{KdV}(\kappa, M, P))_X & = 0
\end{cases}
\]

The Whitham system for (KdV), \( \bar{\delta} = 0 \).
Linear group velocities \( \alpha_1^0 < \alpha_2^0 < \alpha_3^0 \).
Averaging in a singular limit.

Fix $\bar{\delta} = \delta / \epsilon$ (Noble-Rodrigues, *Indiana Univ. Math. J.* 2013.)

\[
(W)_0 \begin{cases}
\kappa_T - (\omega_{KdV}(\kappa, M, P))_X &= 0 \\
M_T + P_X &= 0 \\
P_T + (F_{KdV}(\kappa, M, P))_X &= \bar{\delta} R(\kappa, M, P)
\end{cases}
\]

The Whitham system for (KdV), $\bar{\delta} = 0$.

Linear group velocities $\alpha_0^0 < \alpha_2^0 < \alpha_3^0$.

Linear relaxation coefficient $\gamma = -\partial_P R(k, M, P_*(k, M))$. 
Averaging in a singular limit.

Fix $\bar{\delta} = \delta/\epsilon$ (Noble-Rodrigues, *Indiana Univ. Math. J.* 2013.)

$$\begin{align*}
(W)_0 = & \begin{cases} 
\kappa_T - (\omega_{\text{KdV}}(\kappa, \mathcal{M}, \mathcal{P}))_X = 0 \\
\mathcal{M}_T + \mathcal{P}_X = 0 \\
0 = R(\kappa, \mathcal{M}, \mathcal{P})
\end{cases}
\end{align*}$$

The Whitham system for (KdV), $\bar{\delta} = 0$.
Linear group velocities $\alpha^0_1 < \alpha^0_2 < \alpha^0_3$.

Linear relaxation coefficient $\gamma = -\partial_P R(k, M, P_*(k, M))$.

Limit $\delta \to 0$ of the Whitham system for (KdV-KS), $\bar{\delta} \to \infty$.
Relaxed linear group velocities $\beta^0_1, \beta^0_2$. 
Subcharacteristic conditions.

\[(S1) \quad \beta_1^0, \beta_2^0 \in \mathbb{R} \text{ and } \beta_1^0 \neq \beta_2^0; \]
Subcharacteristic conditions.

\((S1)\) \(\beta_1^0, \beta_2^0 \in \mathbb{R}\) and \(\beta_1^0 < \beta_2^0\);
Subcharacteristic conditions.

(S1) $\beta_1^0, \beta_2^0 \in \mathbb{R}$ and $\beta_1^0 < \beta_2^0$;
(S2) $\alpha_1^0 < \beta_1^0 < \alpha_2^0 < \beta_2^0 < \alpha_3^0$;
Subcharacteristic conditions.

(S1) $\beta_1^0, \beta_2^0 \in \mathbb{R}$ and $\beta_1^0 < \beta_2^0$;
(S2) $\alpha_1^0 < \beta_1^0 < \alpha_2^0 < \beta_2^0 < \alpha_3^0$;
(S3) $\gamma > 0$. 

Modulo a nonvanishing factor $D_\delta(\lambda, \xi)$ expands when $(\xi, \lambda, \delta) \to (0, 0, 0)$ as

$$
\prod_{j=1}^3 (\lambda - i \alpha_j(\xi) \xi) + \gamma \delta^2 \prod_{k=1}^3 (\lambda - i \beta_0^k \xi) + O(\delta^2 (|\lambda|^2 + |\xi|^2) + \delta^3 (|\lambda| + |\xi|))
$$

with $\alpha_j(0) = \alpha_{0j}, \ j = 1, 2, 3$. Inspired by Serre, CPDE 2005.

---

L.M. Rodrigues (Lyon 1)

About periodic waves

IHP 2015 33 / 48
Subcharacteristic conditions.

(S1) \( \beta_1^0, \beta_2^0 \in \mathbb{R} \) and \( \beta_1^0 < \beta_2^0 \);
(S2) \( \alpha_1^0 < \beta_1^0 < \alpha_2^0 < \beta_2^0 < \alpha_3^0 \);
(S3) \( \gamma > 0. \)

Direct inspection

**Subcharacteristic conditions.**

\((S1)\) \(\beta_1^0, \beta_2^0 \in \mathbb{R} \text{ and } \beta_1^0 < \beta_2^0;\)

\((S2)\) \(\alpha_1^0 < \beta_1^0 < \alpha_2^0 < \beta_2^0 < \alpha_3^0;\)

\((S3)\) \(\gamma > 0.\)

Modulo a nonvanishing factor \(D_\delta(\lambda, \xi)\) expands when \((\xi, \lambda, \delta) \to (0, 0, 0)\) as

\[
\prod_{j=1}^{3} (\lambda - i\alpha_j(\xi)\xi) + \gamma\delta \prod_{k=1}^{2} (\lambda - i\beta_k^0\xi) + O(\delta^2(\lambda^2 + |\xi|^2) + \delta^3(\lambda + |\xi|))
\]

with \(\alpha_j(0) = \alpha_j^0, j = 1, 2, 3.\) Inspired by Serre, \textit{CPDE} 2005.
When \(|(\xi, \delta/\xi)| \to 0\)

\[ k = 1, 2, 3, \quad \lambda_k(\xi, \delta) = i\alpha_k(\xi)\xi + \delta A_k + O(\delta\xi) \]
High-frequency expansion.

When \(|(\xi, \delta/\xi)| \to 0\)

\[ k = 1, 2, 3, \quad \lambda_k(\xi, \delta) = i\alpha_k(\xi) \xi + \delta A_k + \mathcal{O}(\delta \xi) \]

where

\[ k = 1, 2, 3, \quad A_k = -\frac{\gamma \prod_{j=1}^{2}(\alpha_k^0 - \beta_j^0)}{\prod_{j \neq k}(\alpha_k^0 - \alpha_j^0)}. \]
Low-frequency expansion.

As $|(\delta, \xi/\delta)| \to 0$

$$j = 1, 2, \quad \lambda_j(\xi, \delta) = \ i \beta_j^0 \xi + B_j \frac{\xi^2}{\delta} + \mathcal{O} \left( \delta |\xi| + \frac{|\xi|^3}{\delta^2} \right)$$

and

$$\lambda_3(\xi, \delta) = B_3 \delta + o(\delta)$$
Low-frequency expansion.

As $|(\delta, \xi/\delta)| \to 0$

$$ j = 1, 2, \quad \lambda_j(\xi, \delta) = \imath \beta_j^0 \xi + B_j \frac{\xi^2}{\delta} + \mathcal{O}\left(\delta|\xi| + \frac{|\xi|^3}{\delta^2}\right) $$

and

$$ \lambda_3(\xi, \delta) = B_3 \delta + o(\delta) $$

where

$$ j = 1, 2, \quad B_j = \frac{\prod_{k=1}^{3} (\beta_j^0 - \alpha_k^0)}{\gamma \prod_{k \neq j} (\beta_j^0 - \beta_k^0)} $$

and

$$ B_3 = -\gamma. $$
Outline.

1. Models

2. Overview of nonlinear theory

3. Survey of spectral numerical studies

4. Near threshold of primary instability
   - The KdV-KS case
   - The Saint-Venant case

5. Conclusion
Near KdV waves.

Reminder: \((\tau, u, y, t, c)\) to \((\tilde{\tau}, \tilde{u}, Y, S, \tilde{c})\).

Set

\[
\delta_F = \frac{\sqrt{F - 2}}{2\tau_0^{1/4} \nu^{1/2}}.
\]
Near KdV waves.

Reminder: \((\tau, u, y, t, c)\) to \((\tilde{\tau}, \tilde{u}, Y, S, \tilde{c})\).

Set

\[
\delta_F = \frac{\sqrt{F - 2}}{2\tau_0^{1/4} \nu^{1/2}}.
\]

Theorem (Barker-Jonhson-Noble-LMR-Zumbrun, preprint 2015.)

Any selected \(U^{(M,P,k)}_{\text{KdV}}\) generates a family of near-KdV waves \((U_F^{(M,k)})_F\) with

\[
\tilde{\tau}_F = -U_{\delta_F} + O((\delta_F)^2) \quad \text{and} \quad \tilde{c}_F = c_{\delta_F} + O((\delta_F)^2).
\]

Phase plane analysis.
Persistence of stability.

With the same index $\text{Ind}$!


Any non-degenerate selected KdV wave such that $\text{Ind} < 0$ yields a family of diffusively spectrally stable waves for (SV).
Persistence of stability.

With the same index $\text{Ind}$!


Any non-degenerate selected KdV wave such that $\text{Ind} < 0$ yields a family of diffusively spectrally stable waves for (SV).

**Framework**: spatial dynamics, eigenvalue problem seen as

$$Z' = A(\lambda, \cdot)Z \quad \text{and} \quad Z(X_F) = e^{i\xi X_F} Z(0)$$

for some real $\xi \in [-\pi/X_F, \pi/X_F)$. 
Step 1: large eigenvalues.

For positive $C$ large enough and $\eta$ small enough, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq -\eta \text{ and } |\lambda| \geq C \}.$$
Step 1: large eigenvalues.

For positive $C$ large enough and $\eta$ small enough, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq -\eta \text{ and } |\lambda| \geq C \}.$$


$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\lambda, \cdot)W$ with

$$B(\lambda, \cdot) = \begin{pmatrix}
\frac{\lambda}{c} + \theta_0(\cdot) & 0 & 0 \\
0 & \bar{\tau}_F \sqrt{\frac{\lambda}{\nu}} & 0 \\
0 & 0 & -\bar{\tau}_F \sqrt{\frac{\lambda}{\nu}}
\end{pmatrix} + \begin{pmatrix}
O(\lambda^{-1}) & O(1) \\
O(1) & O(1)
\end{pmatrix}$$

with

$$\lim_{F \to 2} \theta_0(x) = \frac{\tau_0^{1/2}}{2\nu} \quad \text{and} \quad \lim_{F \to 2} \bar{\tau}_F(x) = \tau_0.$$
Step 2 : intermediate eigenvalues.

For any positive fixed $C$ and $\epsilon$, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq 0 \text{ and } \epsilon \leq |\lambda| \leq C \}.$$ 

Consistent splitting.
Step 3: small eigenvalues.

For positive $C$ large enough and $\eta$ small enough, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq 0, \ |\lambda| \leq \epsilon \text{ and } \text{Re}(\lambda)|\lambda|^{1/3} + |\text{Im}(\lambda)||\lambda|^{2/3} \geq C(\delta_F)^4 \}.$$
Step 3: small eigenvalues.

For positive $C$ large enough and $\eta$ small enough, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq 0, \ |\lambda| \leq \epsilon \text{ and } \text{Re}(\lambda)|\lambda|^{1/3} + |\text{Im}(\lambda)||\lambda|^{2/3} \geq C(\delta_F)^4 \}.$$  

Approximate diagonalization.

$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\lambda, \cdot)W$ with

$$B(\lambda, \cdot) = \text{diag}(\mu_0, \mu_1, \mu_{-1}) + \mathcal{O}(|\lambda| + (\delta_F)^2 + (\delta_F)^4 |\lambda|^{-1})$$

with $j = 1, 0, -1$,

$$\mu_j = -K_0 \omega^j \lambda^{1/3} - K'_0 \bar{\omega}^j (\lambda^{2/3} + (\delta_F)^2 \lambda^{-1/3} a).$$

for some $K_0 > 0$, $K'_0 > 0$, $a \in \mathbb{R}$ and with $\omega = e^{2i\pi/3}$.
Step 3: small eigenvalues.

For positive $C$ large enough and $\eta$ small enough, nothing in

$$\{ \lambda \mid \text{Re}(\lambda) \geq 0, \; |\lambda| \leq \epsilon \; \text{ and } \; \text{Re}(\lambda) |\lambda|^{1/3} + |\text{Im}(\lambda)| |\lambda|^{2/3} \geq C (\delta_F)^4 \}.$$

Approximate diagonalization.

$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\lambda, \cdot)W$ with

$$B(\lambda, \cdot) = \text{diag}(\mu_0, \mu_1, \mu_{-1}) + O(|\lambda| + (\delta_F)^2 + (\delta_F)^4 |\lambda|^{-1})$$

with $j = 1, 0, -1$,

$$|\text{Re}(\mu_j)| \gtrsim \frac{\text{Re}(\lambda)}{|\lambda|^{2/3}} + \frac{|\text{Im}(\lambda)|}{|\lambda|^{1/3}}$$

when $\text{Re}(\lambda) \geq 0$ and $\lambda$ and $\delta^2 |\lambda|^{-2/3}$ are sufficiently small.
Scaling eigenvalues.

Scale now profiles and

\[ \lambda = \frac{(\sqrt{F - 2})^3}{4\tau_0^{1/4} \nu^{1/2}} \Lambda. \]
Scaling eigenvalues.

Scale now profiles and

$$\lambda = \frac{(\sqrt{F-2})^3}{4\tau_0^{1/4} \nu^{1/2}} \Lambda.$$ 

Any unstable $\Lambda$ satisfies $\text{Re}(\Lambda) = \mathcal{O}(1)$ and $|\Lambda| = \mathcal{O}(\delta_F^{-3/5})$ hence $\delta_F |\Lambda| = o(1)$. 

L.M. Rodrigues (Lyon 1)

About periodic waves

IHP 2015 41 / 48
Step 4 : large scaled eigenvalues.

For $C$ large enough, nothing in

$$\{ \Lambda | \text{Re}(\Lambda) \geq 0, \ |\Lambda| \geq C \text{ and } \frac{\text{Re}(\Lambda)}{\delta |\Lambda|^{2/3}} + |\Lambda|^{2/3} \geq C \}.$$
Step 4: large scaled eigenvalues.

For $C$ large enough, nothing in

$$\{ \Lambda \mid \text{Re}(\Lambda) \geq 0, \ |\Lambda| \geq C \text{ and } \frac{\text{Re}(\Lambda)}{\delta |\Lambda|^{2/3}} + |\Lambda|^{2/3} \geq C \}.$$

KdV Floquet reduction then approximate diagonalization.

$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\Lambda, \cdot)W$ with

$$B(\Lambda, \cdot) = \text{diag}(\mu_0, \mu_1, \mu_{-1}) + O(|\Lambda|^{5/3} (\delta_F)^4 + \delta_F)$$

with $j = 1, 0, -1,$

$$\mu_j = \mu_j^{\text{KdV}} + K_0 \check{\omega}^j \Lambda^{2/3} \delta_F$$

for some $K_0 > 0$ and with $\omega = e^{2i\pi/3}.$
Step 4: large scaled eigenvalues.

For $C$ large enough, nothing in

$$\{ \Lambda \mid \text{Re}(\Lambda) \geq 0, \quad |\Lambda| \geq C \quad \text{and} \quad \frac{\text{Re}(\Lambda)}{\delta |\Lambda|^{2/3}} + |\Lambda|^{2/3} \geq C \}.$$ 

KdV Floquet reduction then approximate diagonalization.

$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\Lambda, \cdot)W$ with

$$B(\Lambda, \cdot) = \text{diag}(\mu_0, \mu_1, \mu_{-1}) + O(|\Lambda|^{5/3} (\delta_F)^4 + \delta_F)$$

with $j = 1, 0, -1,$

$$\mu_j = \Lambda^{1/3} (\omega^j + O(|\Lambda|^{-2/3})) + K_0 \bar{\omega}^j \Lambda^{2/3} \delta_F$$

for some $K_0 > 0$ and with $\omega = e^{2i\pi/3}.$
Step 4: large scaled eigenvalues.

For $C$ large enough, nothing in

$$\{ \Lambda \mid \text{Re}(\Lambda) \geq 0, \ |\Lambda| \geq C \ \text{and} \ \frac{\text{Re}(\Lambda)}{\delta |\Lambda|^{2/3}} + |\Lambda|^{2/3} \geq C \}.$$  

KdV Floquet reduction then approximate diagonalization.

$W(\cdot) = P(\cdot)Z(\cdot)$ leads to $W' = B(\Lambda, \cdot)W$ with

$$B(\Lambda, \cdot) = \text{diag}(\mu_0, \mu_1, \mu_{-1}) + \mathcal{O}(|\Lambda|^{5/3}(\delta_F)^4 + \delta_F)$$

with $j = 1, 0, -1$,

$$|\text{Re}(\mu_j)| \gtrsim \text{Re}(\Lambda)|\Lambda|^{-2/3} + \delta_F |\Lambda|^{2/3}$$

when $\text{Re}(\Lambda) \geq 0$, $\Lambda$ is large, and $\delta_F |\Lambda|^{1/3}$ is sufficiently small.
Step 5 : connection to (KdV-KS).

Modulo a nonvanishing factor $D_{\delta F}(\Lambda, \xi)$ expands when $F \to 2$ as

$$D_F(\Lambda, \xi) + \mathcal{O}(\delta^2(|\lambda|^2 + |\xi|^2) + \delta^3(|\lambda| + |\xi|))$$

uniformly on compact sets of $(\Lambda, \xi)$. 
Step 5: connection to (KdV-KS).

Modulo a nonvanishing factor $D_{\delta F}(\Lambda, \xi)$ expands when $F \to 2$ as

\[ D_F(\Lambda, \xi) + O(\delta^2(|\lambda|^2 + |\xi|^2) + \delta^3(|\lambda| + |\xi|)) \]

uniformly on compact sets of $(\Lambda, \xi)$.

Sufficently to deal with

1. intermediate $\Lambda$ by regular perturbation;

2. small $\Lambda$ using subcharacteristic conditions.
Outline.

1. Models
2. Overview of nonlinear theory
3. Survey of spectral numerical studies
4. Near threshold of primary instability
5. Conclusion
Main references.

Main references.


With Pascal Noble


plus Mathew Johnson and Kevin Zumbrun


plus Blake Barker

Main references continued.

With Blake Barker, Mathew Johnson and Kevin Zumbrun


plus Pascal Noble

Other references.

By Hsueh-Chia Chang and Evgeny A. Demekhin

**Thin films** *Complex wave dynamics on thin films* (2002).

By Serafim Kalliadasis, Christian Ruyer-Quil, Benoît Scheid and Manuel G. Velarde

**Thin films** *Falling liquid films* (2012).

By Michael C. Cross and Pierre C. Hohenberg

Open problems.

Nonlinear theory.
- Description of instabilities.
- Inviscid theory, either dispersive or hyperbolic.
- Free-surface formulation.

Spectral studies.
- Mechanism for stability boundaries.
- Capillary-viscous roll-waves.

Beyond.
- Secondary instabilities, multiD periodic waves.
- Injection problem, boundary value stability.
Extra references: related topics.

By Todd Kapitula and Keith Promislow


With Sylvie Benzoni-Gavage and Colin Mietka

Integral transform.

Bloch-wave decomposition

\[ g(x) = \int_{-\pi}^{\pi} e^{i\xi x} \hat{g}(\xi, x) \, d\xi, \]

where \( \xi \) is a Floquet exponent

\[ \hat{g}(\xi, x + 1) = \hat{g}(\xi, x), \]
\[ e^{i\xi(x+1)} \hat{g}(\xi, x + 1) = e^{i\xi} e^{i\xi x} \hat{g}(\xi, x). \]

From Fourier decomposition

\[ g(x) = \int_{\mathbb{R}} e^{i\xi x} \hat{g}(\xi) \, d\xi. \]

Floquet-Bloch transform

\[ \hat{g}(\xi, x) := \sum_{j \in \mathbb{Z}} e^{i2j\pi x} \hat{g}(\xi + 2j\pi) = \sum_{k \in \mathbb{Z}} e^{-i\xi(x+k)} g(x + k). \]