STABILITY OF LARGE-AMPLITUDE SHOCK WAVES OF COMPRESSIBLE NAVIER–STOKES EQUATIONS

KEVIN ZUMBRUN
Mathematics Department, Indiana University
Bloomington, IN 47405-4301, USA
e-mail: kzumbrun@indiana.edu


Abstract. We summarize recent progress on one- and multi-dimensional stability of viscous shock wave solutions of compressible Navier–Stokes equations and related symmetrizable hyperbolic–parabolic systems, with an emphasis on the large-amplitude regime where transition from stability to instability may be expected to occur. The main result is the establishment of rigorous necessary and sufficient conditions for linearized and nonlinear planar viscous stability, agreeing in one dimension and separated in multi-dimensions by a codimension one set, that both extend and sharpen the formal conditions of structural and dynamical stability found in classical physical literature. The sufficient condition in multi-dimensions is new, and represents the main mathematical contribution of this article. The sufficient condition for stability is always satisfied for sufficiently small-amplitude shocks, while the necessary condition is known to fail under certain circumstances for sufficiently large-amplitude shocks; both are readily evaluable numerically. The precise conditions under and the nature in which transition from stability to instability occurs are outstanding open questions in the theory.

Preface.

Five years ago, we and coauthors introduced in the pair of papers [GZ] and [ZH] a new, “dynamical systems” approach to stability of viscous shock wave solutions based on Evans function and inverse Laplace transform techniques, suitable for the treatment of large-amplitude and or strongly nonlinear waves such as arise in the regime where physical transition from stability to instability may be expected to occur. Previous results for systems (see, e.g., [MN.1, KMN, Go.1–2, L.1–3, SzX, LZ.1–2] and references therein) had concerned only sufficient conditions for stability, and had been confined almost exclusively to the small-amplitude case.\(^1\) The philosophy of [GZ, ZH] was, rather, to determine useful necessary and sufficient conditions for stability in terms of the Evans function, which could then be investigated either

\(^1\)The single exception is the partial, zero-mass stability analysis of [MN.1].
numerically [Br.1–2, BrZ, KL] or analytically [MN.1, KMN, Go.1–2, Z.5, HuZ, Hu.1–3, Li, FréS.2, PZ] in cases of interest.$^2$

This approach has proven to be extremely fruitful. In particular, the method has been generalized in [ZS, Z.3, HoZ.1–2] and [God.1, Z.3–4, MaZ.1–5], respectively, to multi-dimensions and to real viscosity and relaxation systems, two improvements of great importance in physical applications. Extensions in other directions include generalizations to sonic shocks [H.4–5, HZ.3], boundary layers [GG, R], discrete and semidiscrete models [Se.1, God.2, B-G.3, BHR], combustion [Ly, LyZ.1–2, JLy.1–2], and periodic traveling-wave solutions in phase-transitional models [OZ.1–2]. Most recently, the qualitative connection to inviscid theory pointed out in [ZS] has been made explicit in [MéZ.1, GMWZ.1–4] by the introduction of Kreiss symmetrizer and pseudodifferential techniques, leading to results on existence and stability of curved shock and boundary layers in the inviscid limit.

In this article, we attempt to summarize the developments of the past five years as they pertain to the specific context of compressible Navier–Stokes equations and related hyperbolic–parabolic systems. As the theory continues in a state of rapid development, this survey is to some extent a snapshot of a field in motion. For this reason, we have chosen to emphasize those aspects that appear to be complete, in particular, to expose the basic planar theory and technical tools on which current developments are based. It is our hope and expectation that developments over the next five years will substantially change the complexion of the field, from the technical study of stability conditions as carried out here to their exploitation in understanding phenomena of importance in physical applications.

As the list of references makes clear, the development of this program has been the work of several different groups and individuals, each contributing their unique point of view. We gratefully acknowledge the contribution and companionship of our colleagues and co-authors in this venture. Special thanks go to former graduate students Peter Howard, Len Brin, Myungyun Oh, Jeffrey Humpherys, and Gregory Lyng (IU Bloomington, advisor K. Zumbrun); Pauline Godillon, Pierre Huot, and Frederic Rousset (ENS Lyons, respective advisors D. Serre, S. Benzoni–Gavage, and D. Serre); and Ramon Plaza (NYU, advisor J. Goodman); and former postdoctoral researchers Corrado Mascia and Kristian Jenssen (IU Bloomington), whose interest and innovation have changed the Evans function approach to conservation laws from a handful of papers into an emerging subfield.

Special thanks also to Rob Gardner, Todd Kapitula, and Chris Jones, whose initial contributions were instrumental to the development of the Evans function approach; to Denis Serre, Sylvie Benzoni–Gavage, and Heinrich Freistühler, whose vision and innovation immediately widened the scope of investigations, and whose ideas have contributed still more than is indicated by the record of their joint and individually published articles; to Olivier Guès, Mark Williams, and Guy Métivier, $^2$A similar philosophy may be found in the partial, zero-mass stability analysis of [KK], which was carried out simultaneously to and independently of [GZ, ZH]; that work, however, concerned only sufficient stability conditions.
whose ideas have not only driven the newest developments in the field, but also profoundly affected our view of the old; and to Arthur Azevedo, Constantin Dafermos, Jonathan Goodman, David Hoff, Tai-Ping Liu, Dan Marchesin, Brad Plohr, Keith Promislow, Jeffrey Rauch, Joel Smoller, Anders Szepessy, Bjorn Sandstede, Blake Temple, and Yanni Zeng for their generously offered ideas and continued support and encouragement outside the bounds of collaboration. Thanks to Denis Serre and the anonymous referee for their careful reading of the manuscript and many helpful corrections. Finally, thanks to my colleagues in Bloomington for their friendship, aid, and stimulating mathematical company, and to my family for their patience, love, and support.


CONTENTS

Main Sections:

1. Introduction
   1.1. Equations and assumptions
   1.2. Structure and classification of profiles
   1.3. Classical (inviscid) stability analysis
      1.3.1. Structural stability
      1.3.2. Dynamical stability
   1.4. Viscous stability analysis
      1.4.1. Connection with classical theory
      1.4.2. One-dimensional stability
      1.4.3. Multi-dimensional stability

2. Preliminaries
   2.1. The spectral resolution formulae
   2.2. Evans function framework
      2.2.1. The gap/conjugation lemma
      2.2.2. The tracking/reduction lemma
      2.2.3. Formal block-diagonalization
      2.2.4. Splitting of a block-Jordan block
      2.2.5. Approximation of stable/unstable manifolds
   2.3. Hyperbolic–parabolic smoothing
   2.4. Construction of the resolvent
      2.4.1. Domain of consistent splitting
      2.4.2. Basic construction
      2.4.3. Generalized spectral decomposition

3. The Evans function, and its low-frequency limit
   3.1. The Evans function
   3.2. The low-frequency limit
4. One-dimensional stability: the stability index
   4.1. Necessary conditions: the stability index
   4.2. Sufficient conditions
      4.2.1. Linearized estimates
      4.2.2. Linearized stability
      4.2.3. Auxiliary energy estimate
      4.2.4. Nonlinear stability
   4.3. Proof of the linearized bounds
      4.3.1. Low-frequency bounds on the resolvent kernel
      4.3.2. High-frequency bounds on the resolvent kernel
      4.3.3. Extended spectral theory
      4.3.4. Bounds on the Green distribution
      4.3.5. Inner layer dynamics
5. Multi-dimensional stability: the refined Lopatinski condition
   5.1. Necessary conditions
   5.2. Sufficient conditions
      5.2.1. Linearized estimates
      5.2.2. Auxiliary energy estimate
      5.2.3. Nonlinear stability
   5.3. Proof of the linearized estimates
      5.3.1. Low-frequency bounds on the resolvent kernel
      5.3.2. High-frequency bounds
      5.3.3. Bounds on the solution operator
Appendices:
   A. Auxiliary results/calculations
      A.1. Applications to example systems
      A.2. Structure of viscous profiles
      A.3. Asymptotic ODE estimates
      A.4. Expansion of the one-dimensional Fourier symbol
   B. A weak version of Métivier’s Theorem
   C. Evaluation of the Lopatinski determinant for gas dynamics
5.1. INTRODUCTION.

Fluid- and gas-dynamical flow, though familiar in everyday experience, are sufficiently complex as to defy simple description. Reacting flow, plastic–elastic flow in solids, and magnetohydrodynamic flow are still more complicated, and moreover are removed from direct observation in our daily lives. Historically, therefore, much of the progress in understanding these phenomena has come from the study of simple flows occurring in special situations, and their stability and bifurcation.

3Contributed by K. Jenssen and G. Lyng.
For example, one may neglect compressibility of the conducting medium for low Mach number flow, or viscosity for high Reynolds number, or both, in each case obtaining a reduction in the complexity of the governing equations. Alternatively, one may consider solutions with special symmetry, reducing the spatial dimension of solutions: for example, laminar (shear) flows exhibiting local symmetry parallel to the direction of flow, or compressive (shock) flows exhibiting local symmetry normal to the flow.

Once existence of such solutions has been ascertained, there remains the interesting question of stability of the flow/validity of simplifying assumptions; more precisely, one would like to know the parameter range under which such solutions may actually be found in nature and to understand the corresponding bifurcation at the point where stability is lost. These questions comprise the classical subject of hydrodynamic stability; see e.g. [Ke.1–2, Ra.1–6, T, Ch, DR, Lin]. With the advent of powerful and affordable processors, this subject has lost its central place to computational fluid dynamics, which in effect brings the most remote parameter regimes into the realm of our direct experience. However, we would argue that this development makes all the more interesting the rigorous understanding of stability/validity. For, numerical approximation adds another layer of effects between model and experiment that must be carefully separated from actual behavior, and this is an important practical role for rigorous analysis. Likewise, numerical experiments suggest new phenomena for which analysis can yield qualitative understanding.

Stability and the related issue of validity of formal or numerical approximations are difficult questions, and unfortunately analysis has often lagged behind the needs of practical application. However, this area is currently in a period of rapid development, to the extent that practical applications now seem not only realistic but imminent. In this article, we give an account of recent developments in the study of stability and behavior of compressive “shock-type” fronts, with an eye toward both mathematical completeness and eventual physical applications. The latter criterion means that we must consider stability in the regime where transition from stability to instability is likely to occur, and this means in most cases that we must consider simplified flow within the context of the full model, without simplifying idealizations, i.e., as occurring in “real media” and usually in the large-amplitude regime. This requirement leads to substantial technical difficulties, and much of the mathematical interest of the analysis.

Consider a general system of viscous conservation laws

$$\begin{align*}
U_t + \sum_j F_j(U)_{x_j} &= \nu \sum_{j,k} (B_{jk}(U)U_{x_k})_{x_j}, \\
x &\in \mathbb{R}^d; U, F^j \in \mathbb{R}^n; B^{jk} \in \mathbb{R}^{n \times n},
\end{align*}$$

modeling flow in a compressible medium, where $\nu$ is a constant measuring transport effects (e.g. viscosity or heat conduction).
An important class of solutions are planar viscous shock waves

\[ U(x, t) = \bar{U} \left( \frac{x_1 - st}{\nu} \right), \quad \lim_{x_1 \to \pm \infty} \bar{U}(x_1) = U_\pm, \]

satisfying the traveling-wave ordinary differential equation (ODE)

\[ B^{11}(\bar{U})\bar{U}' = F^1(\bar{U}) - F^1(U_-) - s(\bar{U} - U_-), \]

which reduce in the vanishing-viscosity limit \( \nu \to 0 \) to “ideal shocks,” or planar discontinuities between constant states \( U_\pm \) of the corresponding “inviscid” system

\[ U_t + \sum_j F^j(U)_{x_j} = 0. \]

The shape function \( \bar{U} \) is often called a “viscous shock profile”. Planar shock waves are archetypal of more general, curved shock solutions, which have locally planar structure (see [M.1–4], [GW, GMWZ.2–3] for discussions of existence of curved shocks in the respective contexts (1.1), (1.4)). The equations of reacting flow, containing additional reaction terms \( Q(u) \) on the righthand sides of (1.1) and (1.4), likewise feature planar solutions (1.2), known variously as strong or weak detonations, or strong or weak deflagrations, depending on their specific physical roles; see, e.g., [FD, CF, GS, Ly] for details.

The theories of stability of shock and detonation waves are historically intertwined. For physical reasons, the more complicated detonation case seems to have driven the study of stability, at least initially. For, experiments early on indicated several possible types of instabilities that could occur, in particular: galloping longitudinal instabilities in which the steadily traveling planar front is replaced by a time-periodic planar solution moving with time-periodic speed; cellular instabilities, in which it is replaced by a nonplanar traveling wave with approximately periodic transverse spatial structure and different speed; and spinning instabilities, in which it is replaced by a nonplanar, steadily traveling and rotating wave; see, e.g., [FD, Ly, LyZ.1–2] for further discussion. The latter two types are the ones most frequently observed; note that these are transverse instabilities, and thus multidimensional in character. The questions of main physical interest seem to be: 1. the origins/mechanism for instability; 2. location of the transition from stability to instability; 3. understanding/prediction of the associated bifurcation to more complicated solutions; and 4. the effect of front geometry (curvature) on stability.

Shock waves, on the other hand, were apparently thought for a long time to be universally stable. Indeed, the possibility of instability was first predicted analytically, through the normal modes analyses of [Kr, D, Er.1–2], in a regime that was at the time not accessible to experiment; see [BE] for an account of these and other interesting details of the early theory. On the other hand, in the inviscid case \( \nu = 0 \), the stability condition can be explicitly calculated for shock waves, whereas
for detonation waves it cannot. For this and other reasons, the shock case is much better understood; in particular, there exists a rather complete analytical multidimensional stability theory for inviscid shock waves, see [M.1–4, Mé.1–5], whereas the corresponding (ZND) theory for detonation waves is mainly numerical, with sensitive numerical issues limiting conclusions; see [LS] for a fairly recent survey.

As we shall discuss further below, the inviscid shock theory gives partial answers to question 1, above, but does not rule out possible additional mechanisms connected with viscosity or other transport effects. Likewise, it gives only a partial answer to question 2, reducing the location of the transition point to an open region in parameter space, and so question 3 cannot even be addressed. Question 4 inherently belongs to the viscous theory, as there is no effect of shock curvature without finite shock width. In this article, we present a mathematical framework for the study of viscous shock stability, based on the Evans function of [E.1–4, AGJ, PW], that has emerged through the series of papers [GZ, ZH, BSZ.1, ZS, Z.1–3, MaZ.1–5] and references therein, and which features necessary and sufficient conditions for planar viscous stability analogous to but somewhat sharper than the weak and strong (or “uniform”) Lopatinski conditions of the inviscid theory [Kr, M.1–4, Mé.5]. These are sufficient in principle to resolve questions 1–2, and give also a useful starting point for the study of question 3. Question 4 remains for the moment open, and is an important direction for further investigation.

Aside from their intrinsic interest, these results are significant for their bearing on the originally motivating problem of detonation and on related problems associated with nonclassical shocks in phase-transitional or magnetohydrodynamical flows, for which significant stability questions remain even in “typical” regimes occurring frequently in applications. Though we shall not discuss it here, the technical tools we develop in this article can be applied also in these more exotic cases; see [ZH, Z.5, Z.3, HZ.2, BMSZ, FreZ.2, LyZ.2] for discussions in the strictly parabolic case, and [Ly, LyZ.1] in the case of “real”, or physical viscosity.

1.1. Equations and Assumptions.

We begin by identifying an abstract class of equations, generalizing the Kawashima class of [Kaw], that isolates those qualities relevant to the investigation of shock stability in compressible flow. The Navier–Stokes equations of compressible gas- or magnetohydrodynamics (MHD) in dimension $d$, may be expressed in terms of conserved quantities $U$ in the standard form (1.1), with

\begin{equation}
U = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \quad B^{jk} = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},
\end{equation}

\begin{equation}
u^I \in \mathbb{R}^{n-r}, \quad u^{II} \in \mathbb{R}^r, \quad \text{and}
\end{equation}

\begin{equation}
\text{Re} \sigma \sum \xi_j \xi_k b_2^{jk} \geq \theta |\xi|^2,
\end{equation}

$\theta > 0$, for all $\xi \in \mathbb{R}^d$; for details, see Appendix A.1. Since in this work we are interested in time-asymptotic stability for a fixed, finite viscosity rather than the
vanishing-viscosity limit, we set $\nu = 1$ from here forward, and suppress the parameter $\nu$.

Near states of stable thermodynamic equilibrium, where there exists an associated convex entropy $\eta$, they can be written, alternatively, in terms of the “entropy variable” $W := d\eta(U)$, in symmetric hyperbolic–parabolic form

\begin{equation}
U(W)_t + \sum_j \tilde{F}_j(W)_{x_j} = \sum_{j,k} \tilde{B}^{jk}W_{x_k})_{x_j}, \quad W = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix},
\end{equation}

where

\begin{equation}
d\tilde{F}_j = dF_j(\partial U/\partial W), \quad \tilde{B}^{jk} = B^{jk}(\partial U/\partial W) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}_j^k \end{pmatrix}
\end{equation}

are symmetric, and $(\partial U/\partial W)$ and $\sum \xi_j \xi_k \tilde{b}^{jk}$ are (uniformly) symmetric positive definite,

\begin{equation}
\sum \xi_j \xi_k \tilde{b}^{jk} \geq \theta |\xi|^2, \quad \theta > 0,
\end{equation}

for all $\xi \in \mathbb{R}^d$; see [Kaw], and references therein, or Appendix A.1. This fundamental observation generalizes the corresponding observation of Godunov [G] (see also [Fri,M,Bo]) in the inviscid setting $B^{jk} \equiv 0$.

In either setting, symmetric form (1.7)–(1.9) has the important property that its structure guarantees local existence under small perturbations of equilibrium, as may be established by standard energy estimates [Fri,Kaw]. The viscous equations, moreover, satisfy the further condition of genuine coupling

\begin{equation}
\text{No eigenvector of } \sum_j \xi_j dF_j \text{ lies in the kernel of } \sum \xi_j \xi_k B^{jk},
\end{equation}

for all nonzero $\xi \in \mathbb{R}^d$ (equivalently, $\sum_j \xi_j d\tilde{F}_j$, $\sum \xi_j \xi_k \tilde{B}^{jk}$), which implies that the system is “dissipative” in a certain sense; in particular, it implies, together with form (1.7), global existence and decay under small perturbations of equilibrium, via a series of clever energy estimates revealing “hyperbolic compensation” for absent parabolic terms [Kaw]. We refer to equations of form (1.7) satisfying (1.8)–(1.10) as “Kawashima class”.

Here, we generalize these ideas to the situation of a viscous shock solution (1.2) of (1.1) consisting of a smooth shock profile connecting two different thermodynamically stable equilibria, and small perturbations thereof. Specifically, we assume that, by some invertible change of coordinates $U \rightarrow W(U)$, possibly but not necessarily connected with a global convex entropy, followed if necessary by multiplication on the left by a nonsingular matrix function $S(W)$, equations (1.1) may be written in the quasilinear, partially symmetric hyperbolic-parabolic form

\begin{equation}
\tilde{A}^0 W_t + \sum_j \tilde{A}^j W_{x_j} = \sum_{j,k} (\tilde{B}^{jk}W_{x_k})_{x_j} + G, \quad W = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix},
\end{equation}

where
\( w^I \in \mathbb{R}^{n-r}, w^{II} \in \mathbb{R}^r, x \in \mathbb{R}^d, t \in \mathbb{R} \), where, defining \( W_\pm := W(U_\pm) \):

(A1) \( \tilde{A}^j(W_\pm), \tilde{A}_i^j := \tilde{A}_{i1}^j, \tilde{A}^0 \) are symmetric, \( \tilde{A}^0 > 0 \).

(A2) No eigenvector of \( \sum \xi_j dF^j(U_\pm) \) lies in the kernel of \( \sum \xi_j \xi_k B^{jk}(U_\pm) \), for all nonzero \( \xi \in \mathbb{R}^d \). (Equivalently, no eigenvector of \( \sum \xi_j \tilde{A}^j(\tilde{A}^0)^{-1}(W_\pm) \) lies in the kernel of \( \sum \xi_j \xi_k \tilde{B}^{jk}(W_\pm) \).

(A3) \( \tilde{B}^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}, \ G = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix} \), with \( \text{Re} \sum \xi_j \xi_k \tilde{b}^{jk}(W) \geq \theta |\xi|^2 \) for some \( \theta > 0 \), for all \( W \) and all \( \xi \in \mathbb{R}^d \), and \( \tilde{g}(W_x, W_x) = \mathcal{O}(|W_x|^2) \).

Here, the coefficients of (1.11) may be expressed in terms of the original equation (1.1), the coordinate change \( U \rightarrow W(U) \), and the approximate symmetrizer \( S(W) \), as

\[
\tilde{A}^0 := S(W)(\partial U/\partial W), \quad \tilde{A}^j := S(W)dF^j(\partial U/\partial W),
\]

\[
\tilde{B}^{jk} := S(W)B^{jk}(\partial U/\partial W), \quad G = -\sum_{jk}(dSW_{x_j})B^{jk}(\partial U/\partial W)W_{x_k}.
\]

Note that, in accord with the general philosophy of [ZH, Z.3] in the strictly parabolic case, the conditions of dissipative symmetric hyperbolic–parabolicity connected with stability of equilibrium states are required only at the endstates of the profile, the conditions along the profile–symmetrizable hyperbolicity, \( d\tilde{F}_{11} \), in the first equation and parabolicity, \( \sum \xi_j \xi_k \tilde{b}^{jk} > 0 \), in the second equation–being concerned with local well-posedness rather than time-asymptotic stability of any intermediate state. This allows for interesting applications to phase-transitional or van der Waals gas dynamics, for which hyperbolicity of the associated ideal system (1.4) may be lost along the profile; see Appendix A.1.

**Remark 1.1.** The differential symbol \( A^\xi := \sum_j \xi_j A^\xi_j \) defined by (A1) governs convection in “hyperbolic” modes, as described in [MaZ.3]; for general equations of form (1.1), this must be replaced by the pseudodifferential symbol \( A^\xi := dF^{\xi} - dF^{\xi}_{11}(b^{\xi\xi}_{22})^{-1}b^{\xi\xi}_{21} \), where \( dF^{\xi} := \sum_j dF^j \xi_j \) and \( b^{\xi\xi} := \sum_j b^{jk}\xi_j \xi_k \). That is, the form (1.11) imposes interesting structure also at the linearized symbolic level.

**Remark 1.2.** Assumptions (A1)–(A3) were introduced in [MaZ.4] under the restriction \( G \equiv 0 \), or equivalently \( S \equiv I \). This is sufficient to treat isentropic van der Waals gas dynamics, as discussed in the example at the end of the introduction of [MaZ.4]; however, it is not clear whether there exists such a coordinate change for the full equations of gas dynamics with van der Waals equation of state. The improved version \( G \neq 0 \), introduced in [GMWZ.4], is much easier to check, and in particular holds for the full van der Waals gas equations under the single (clearly necessary) condition \( T_s > 0 \), where \( T = T(\rho, e) \) is temperature, and \( \rho \) and \( e \) are density and internal energy; see Appendix A.1. Still more general versions of (1.11)
(in particular, including zero-order terms with vanishing $1 - 1$ block) might be accommodated in our analysis; however, so far there seems to be no need in applications. Of course, equations in the Kawashima class, in particular systems possessing a global entropy, are included under (A1)–(A3) as a special case.

Along with the above structural assumptions, we make the technical hypotheses:

(H0) $F^j, B^{jk}, W, S \in C^s$, with $s \geq 2$ in our analysis of necessary conditions for linearized stability, $s \geq 5$ in our analysis of sufficient conditions for linearized stability, and $s \geq s(d) := [d/2] + 5$ in our analysis of nonlinear stability.

(H1) The eigenvalues of $A^1_j$ are (i) distinct from the shock speed $s$; (ii) of common sign relative to $s$; and (iii) of constant multiplicity with respect to $U$.

(H2) The eigenvalues of $dF^1(U_\pm)$ are distinct from $s$.

(H3) Local to $\bar{U}(\cdot)$, solutions of (1.2)–(1.3) form a smooth manifold $\{\bar{U}^\delta(\cdot)\}$, $\delta \in \mathcal{U} \subset \mathbb{R}^\ell$.

Conditions (H0)–(H2) may be checked algebraically, and are generically satisfied for profiles of the equations of gas dynamics and MHD; see Appendix A.1. We remark that (H1)(i) arises naturally as the condition that the traveling-wave ODE be of nondegenerate type, while (H2) is the condition for normal hyperbolicity of $U_\pm$ as rest points of that ODE; see Section 1.2. The “weak transversality” condition (H3) is more difficult to verify; however, it is automatically satisfied for extreme Lax shocks such as arise in gas dynamics; see Remark 2, Section 1.2. More generally, it is implied by but does not imply transversality of the traveling-wave connection $\bar{U}$ as an orbit of (1.3).

In our analysis of sufficient conditions for stability, we make two further hypotheses at the level of the inviscid equations (1.4), analogous to but somewhat stronger than the block structure condition of the inviscid stability theory [Kr, M.1–4, Mé.5]. The first is, simply:

(H4) The eigenvalues of $\sum \xi_j dF^j(U_\pm)$ are of constant multiplicity with respect to $\xi \in \mathbb{R}^d \setminus \{0\}$.

This condition was shown in [Mé.4] to imply block structure, and is satisfied for all physical examples for which the block structure condition has currently been verified. In particular, it holds always for gas dynamics, but fails for MHD; see, respectively, Appendix C of this article and [Je, JeT].

The second concerns the structure of the glancing set of the symbol

\[ P_\pm(\xi, \tau) := i\tau + \sum i\xi_j dF^j(U_\pm) \]  

\[ (1.13) \]  

---

4As mentioned in [MaZ.3], we suspect that $C^3$ should suffice, with further work.

5In our analysis of necessary conditions for stability, we require only (H1)(i). Condition (H1)(iii) can be dropped, at the expense of some detail in the linear estimates [Z.4].

6In particular, $A^1_j = w_j J_{n-\tau}$, where $w_j$ is the $j$th component of fluid velocity, and so (H1) is trivially satisfied. That is, hyperbolic modes experience only passive, scalar convection for these equations.
of the inviscid equations (1.4) at $x_1 = \pm \infty$. By symmetrizability assumption (A1),
together with hypothesis (H4), the characteristic equation $\det P_{\pm}(\xi, \tau) = 0$ has $n$
roots
\begin{equation}
\tau = -a_r(\xi), \quad r = 1, \ldots, n,
\end{equation}
locally analytic and homogeneous degree one in $\xi$, where $a_r$ are the eigenvalues of
$\sum \xi_j dF^j(U_{\pm})$.

**Definition 1.3.** Setting $\xi = (\xi_1, \ldots, \xi_d) =: (\xi_1, \tilde{\xi})$, we define the glancing set of
$P_{\pm}$ as the set of frequencies $(\tilde{\xi}, \tau)$ for which $\tau = a_r(\xi_1, \tilde{\xi})$ and
$(\partial a_r / \partial \xi_1) = 0$ for some real $\xi_1$ and $1 \leq r \leq n$.

**Remark 1.4.** The word “glancing” is used in Definition 1.3 because null bicharacteristics of
$P_{\pm}$ through points $(\xi, \tau)$ with $(\partial a_r / \partial \xi_1) = 0$ are parallel to
$x_1 = 0$.

The significance of the glancing set for our analysis is that it corresponds to the
set of frequencies $(\tilde{\xi}, \tau)$ for which the (rotated) inverse functions $i\xi_1 = \mu_r(\xi, \tau)$
associated with $\tau = a_r(\xi_1, \tilde{\xi})$, corresponding to rates of spatial decay in $x_1$ for the inviscid
resolvent equation, are pure imaginary and have branch singularities of degree $s_r$
equal to the multiplicity of root $\xi_1$. As pointed out by Kreiss [Kr], these are the
frequencies for which resolvent estimates become delicate, and structural assumptions become important (e.g., strict hyperbolicity as in [Kr], or, more generally, block structure as in [M.1–4, M.6.5]).

Our final technical hypothesis is, then:

(H5) The glancing set of $P_{\pm}$ is the union of $k$ (possibly intersecting) smooth
curves $\tau = \eta_q(\tilde{\xi})$, $0 \leq k \leq n$, defined as the locii on which $\tau = -a_q(\xi_1, \tilde{\xi})$ and
$(\partial a_q / \partial \xi_1) = 0$ for some real $\xi_1$, on which root $\xi_1$ has constant multiplicity $s_q$,
defined as the order of the first nonvanishing partial derivative $(\partial s a_q / \partial \xi_1^s)$ with
respect to $\xi_1$, i.e., the associated inverse function $\xi_1^q(\tilde{\xi}, \tau)$ has constant degree of
singularity $s_q$.

Condition (H5) is automatic in dimensions $d = 1, 2$ and in any dimension for
rotationally invariant problems. In one dimension, the glancing set is empty. In the two-dimensional case, the homogeneity of $a_r$ and its derivatives implies that the
ray through $(\tilde{\xi}, \tau)$ is the graph of $\tau(\tilde{\xi})$ and that (H5) holds there. By the Implicit
Function Theorem, (H5) holds also in the case that all branch singularities are
of square root type, degree $s_q = 2$, with $\eta_q$ defined implicitly by the requirement
$(\partial a_q / \partial \xi_1) = 0$ (indeed, $\eta_q$ is in this case analytic). In particular, it holds in the case
that all eigenvalues $a_r(\xi)$ are either linear or else strictly convex/concave in $\xi_1$ for
$\tilde{\xi} \neq 0$.\(^7\) Thus, (H5) is satisfied in all dimensions for the equations of gas dynamics,
for which the characteristic eigenvalues $a_r$ are linear combinations of $(\xi, \eta)$ and $|\xi, \eta|
(see Appendix C), hence clearly linear or else strictly concave/convex for $\tilde{\xi} \neq 0$.

\(^7\)More generally, just those involved in glancing. Likewise, constant multiplicity (H4) is only
needed in our analysis for eigenvalues involved in glancing.
Similarly, it may be calculated (see [MéZ.3]) that all characteristic eigenvalues of the equations of MHD are linear or else strictly concave/convex for \( \tilde{\xi} \neq 0 \), with the exceptions of the two “slow” magnetoacoustic characteristics, which together may produce a singularity of degree at most two. Thus, again (H5) is always satisfied.

**Remark 1.5.** Assuming symmetrizability, (A1), condition (H5) like (H4) implies block structure; see the direct matrix perturbation calculations of Section 5, [Z.3], or Section 5 of this article. As was the block structure condition in the inviscid case, conditions (H4)–(H5) are used in the viscous case to make convenient certain estimates on the resolvent, and should be viewed as “first-order nondegeneracy conditions”. There is ample motivation in the example of MHD to carry out a refined analysis at the next level of degeneracy, with these conditions relaxed: in particular, the constant multiplicity requirement (H4), which presently restricts the stability analysis for MHD to the one-dimensional case. See [MéZ.3] for some work in this direction.

### 1.2. Structure and classification of profiles.

We next state some general results from [MaZ.3, Z.3] regarding structure and classification of profiles, analogous to those proved in [MP] in the strictly parabolic case. Proofs may be found in Appendix A.2; see also [MaZ.3, Z.3, Ly, LyZ.1].

Observe first, given \( \det b_{11}^2 \neq 0 \), (A2), that (H1)(i) is equivalent, by determinant identity \( \det \begin{pmatrix} dF_{11}^1 - s & dF_{12}^1 \\ b_{11}^1 & b_{22}^1 \end{pmatrix} = \det (dF_{11}^1 - s - b_{11}^1(b_{22}^1)^{-1}dF_{12}^1) \det b_{22}^1 \), to the condition

\[
\det \begin{pmatrix} dF_{11}^1 - s & dF_{12}^1 \\ b_{11}^1 & b_{22}^1 \end{pmatrix} \neq 0.
\]

Writing the traveling-wave ODE (1.3) as

\[
f^I(U) = s u^I \equiv f^I(U_-) - s u^I,
\]

\[
b_{11}^1(u^I)' + b_{22}^1(u^I)' = f^{II}(U) - f^{II}(U_-) - s(u^{II} - u_-^{II}),
\]

where \( F^1 =: \begin{pmatrix} f^I \\ f^{II} \end{pmatrix} \), we find that (1.15) is in turn the condition that (1.17) describe a nondegenerate ODE on the \( r \)-dimensional manifold described by (1.16): well-defined by the Implicit Function Theorem plus full rank of \((dF^1_{11} - s, dF^1_{12})\).

**Lemma 1.6**[Z.3, MaZ.3]. Given (H1)–(H3), the endstates \( U_\pm \) are hyperbolic rest points of the ODE determined by (1.17) on the \( r \)-dimensional manifold (1.16), i.e., the coefficients of the linearized equations about \( U_\pm \), written in local coordinates, have no center subspace. In particular, under regularity (H0), standing-wave solutions (1.2) satisfy

\[
|(d/dx_1)^k(\bar{U} - U_\pm)| \leq C e^{-\theta|x_1|}, \quad k = 0, \ldots, 4,
\]
as \( x_1 \to \pm \infty \).

Lemma 1.6 verifies the hypotheses of the gap lemma, \([GZ, ZH]^{8}\), on which we shall rely to obtain the basic ODE estimates underlying our analysis in the low-frequency/large-time regime.

Let \( i_+ \) denote the dimension of the stable subspace of \( dF_1(U_+) \), \( i_- \) denote the dimension of the unstable subspace of \( dF_1(U_-) \), and \( i := i_+ + i_- \). Indices \( i_\pm \) count the number of incoming characteristics from the right/left of the shock, while \( i \) counts the total number of incoming characteristics toward the shock. Then, the \textit{hyperbolic classification} of \( \bar{U}(\cdot) \), i.e., the classification of the associated hyperbolic shock \( (U_-, U_+) \), is:

\[
\begin{align*}
\text{Lax type} & \quad \text{if } \ i = n + 1, \\
\text{Undercompressive (u.c.)} & \quad \text{if } \ i \leq n, \\
\text{Overcompressive (o.c.)} & \quad \text{if } \ i \geq n + 2.
\end{align*}
\]

In case all characteristics are incoming on one side, i.e. \( i_+ = n \) or \( i_- = n \), a shock is called \textit{extreme}.

As in the strictly parabolic case, there is a close connection between the hyperbolic type of a shock and the nature of the corresponding connecting profile. Considering the standing-wave ODE as an \( r \)-dimensional ODE on manifold (1.16), let us denote by \( 1 \leq d_\pm \leq r \) the dimensions of stable manifold at \( U_+ \) and unstable manifold at \( U_- \), and \( d := d_+ + d_- \). Then, we have:

**Lemma 1.7** [MaZ.3, LyZ.1]. Given (H1)–(H3), there hold relations

\[
\begin{align*}
\frac{n - i_+ = r - d_+ + \dim \mathcal{U}(A_{s+}),}
\frac{n - i_- = r - d_- + \dim \mathcal{S}(A_{s-}),}
\end{align*}
\]

where \( A_s := dF_1^{11} - dF_1^{12}(b_2^{11})^{-1}b_1^{11} \), and \( \mathcal{U}(M) \) and \( \mathcal{S}(M) \) denote unstable and stable subspaces of a matrix \( M \). In particular, existence of a connecting profile implies \( n - i = r - d \).

The final assertion implies that the “viscous” and “hyperbolic” types of shock connections agree, i.e. Lax, under-, and overcompressive designations imply corresponding information about connections. The more detailed information (1.19) implies that “extreme” shocks have “extreme” connections:

**Corollary 1.8** [MaZ.3, LyZ.1]. For (right) extreme shocks, \( i_+ = n \), there holds also \( d_+ = r \), i.e. the connection is also extreme, and also \( \dim \mathcal{U}(A_s) = 0 \).

**Proof.** Immediate from (1.19), using \( d_+ \leq r \) and \( \dim \mathcal{U}(A_{s+}) \geq 0 \).

\(^{8}\text{See also the version established in [KS], independently of and simultaneously to that of [GZ].}\)
A complete description of the connection, of course, requires the further index \( \ell \) defined in (H4) as the dimension \( \ell \) of the connecting manifold between \((u_\pm, v_\pm)\) in the traveling-wave ODE. Generically, one expects that \( \ell \) should be equal to the surplus \( d - r = i - n \). In case the connection is “dimensionally transverse” in this sense, i.e.:

\[
\ell = \begin{cases} 
1 & \text{undercompressive or Lax case,} \\
i - n & \text{overcompressive case,}
\end{cases}
\tag{1.20}
\]

we call the shock “pure” type, and classify it according to its hyperbolic type; otherwise, we call it “mixed” under/overcompressive type. Throughout this article, we shall assume that (1.20) holds, so that all viscous profiles are of pure, hyperbolic type. Indeed, we shall restrict attention mainly to the classical case of pure, Lax-type profiles such as occur in the standard gas-dynamical case, confining our discussion of other cases to brief remarks. For more detailed discussion of the nonclassical, over- and undercompressive cases, we refer the reader to [Z.3] and especially [ZS].

**Remarks 1.9.** 1. Transverse Lax and overcompressive connections of (1.3) persist under change of parameters \((U_-, s)\), while transverse undercompressive connections are of codimension \( q := n + 1 - i \) (the “degree of undercompressivity”) in parameter space \((U_-, s)\).

2. For extreme indices \( d_+ = r \) or \( d_- = r \), profiles if they exist are always transverse, and likewise for indices that are “minimal” in the sense that \( d_+ = \ell \) or \( d_- = \ell \). In particular, extreme Lax or overcompressive connections (if they exist) are always transverse, hence satisfy (H3). Undercompressive shocks are never extreme.

3. The relation \( 0 \leq \dim \mathcal{U}(A_+) - \dim \mathcal{U}(A_{++}) = (r - d_+) \leq r \), may be viewed as a sort of subcharacteristic relation, by analogy with the relaxation case. Indeed, as described in [LyZ.1], it may be used to obtain the full subcharacteristic condition

\[
a_j < a_j^* < a_{j+r},
\]

as a consequence of linearized stability of constant states, where \( a_j \) denote the eigenvalues of \( A := dF_1 \) and \( a_j^* \) those of \( A_+ \); this is closely related to arguments given by Yong (see, e.g., [Yo]) in the relaxation case. The subcharacteristic relation goes the “wrong way” in the relaxation case, and so one cannot conclude a result analogous to Corollary 1.8; for further discussion, see [MaZ.1, MaZ.5].

4. In the special case \( r = 1 \), there holds \( d_+ = d_- = 1 \) whenever there is a connection, and so profiles are always of Lax type \( n - i = r - d = -1 \). This recovers the observation of Pego [P.2] in the case of one-dimensional isentropic gas dynamics that smooth (undercompressive type) phase-transitional shock profiles cannot occur under the effects of viscosity alone, even for a van der Waals-type equation of state. (They can occur, however, when dispersive, capillary pressure effects are taken into account; see, e.g., [Sl.1–5, B-G.1–2].)
1.3. Classical (inviscid) stability analysis.

The classical approach to the study of stability of shock waves, as found in the mathematical physics literature, consists mainly in the study of the related problems of structural stability, or existence and stability of profiles (1.2) as solutions of ODE (1.3), and dynamical stability, or hyperbolic stability of the corresponding ideal shock

\[ U(x,t) = \bar{U}(x_1 - st), \quad \bar{U}(x_1) := \begin{cases} U_- & \text{for } x_1 < 0, \\ U_+ & \text{for } x_1 \geq 0, \end{cases} \]

as a solution of the inviscid equations (1.4); see, for example, the excellent survey articles [BE] and [MeP]. These may be derived formally by matched asymptotic expansion, either in the vanishing-viscosity limit \( \nu \to 0 \), or, as discussed in Section 1.3, [Z.3], in the low-frequency, or large-space–time limit

\[ \bar{x} := \varepsilon x, \quad \bar{t} := \varepsilon t, \quad \varepsilon \to 0, \]

with \( \nu \) held fixed and \( \bar{t} \) varying on a bounded interval \([0, T]\). We summarize the relevant results below.

1.3.1. Structural stability. The viscous profile problem, or “inner” problem from the matched asymptotic expansion point of view, yields first of all the Rankine–Hugoniot conditions (RH)

\[ s[U] = [F^1(U)], \]

where \( [H] := H(U_+) - H(U_-) \) denotes jump across the shock in quantity \( H \), as the requirement that \( U_\pm \) must both be rest points of the traveling-wave ODE (1.3). These arise more directly in the inviscid theory, as the conditions that mass be conserved across an ideal shock discontinuity (1.21); see, e.g., [La, Sm].

In matched asymptotic expansion, solvability of the inner problem serves as the boundary (or free boundary, in this case) condition for the “outer” problem, which in this case (see Section 1.3, [Z.3]) is just the inviscid equations (1.4). By Remark 1.9.1, the Rankine–Hugoniot conditions are sufficient for existence in the vicinity of a transverse Lax- or overcompressive-type shock profile, while in the vicinity of a transverse undercompressive shock profile, existence is equivalent to (RH) plus an additional \( q \) boundary conditions, sometimes known as “kinetic conditions”, where \( q = n + 1 - i \) (in the notation of the previous section) is the degree of undercompressivity. Thus, local to an existing transverse profile, the outer, inviscid problem may in the Lax or overcompressive case be discussed without reference to the inner problem, appealing only to conservation of mass, (RH). As discussed cogently in [vN] (roundtable discussion), this remarkable fact is in sharp contrast to the situation in the case of a solid boundary layer, and is at the heart of hyperbolic shock theory.
Of course, there remains the original question of existence in the first place of any transverse smooth shock profile, which is a priori far from clear given the incomplete parabolicity of equations (1.1). Here, the genuine coupling condition (A2) plays a key role. For example, in the “completely decoupled” case that $dF_{12}^j \equiv 0$, the first equation of (1.1) is first-order hyperbolic, so supports only discontinuous traveling-wave solutions (1.21) [Sm]. On the other hand, (A2) precludes such discontinuous solutions, at least for small shock amplitude $[U] := U_+ - U_-$. For, a distributional solution (1.21) of (1.1), $\nu = 1$, may be seen to satisfy not only the Rankine–Hugoniot conditions (RH), but also, using the block structure assumption (1.8) in the symmetric, $W$ coordinates, the condition $[w^{II}] = 0$, or $[W] \in \ker B^{11}$. For $[U]$ small, however, (RH) implies that $[U]$ is approximately parallel to an eigenvector of $dF^1$, hence by (A2) cannot lie in $\ker B^{11}(U_-)$, and therefore (using smallness again) $[W]$ cannot lie in $\ker B^{11}$, a contradiction. Viewed another way, this is an example of the time-asymptotic smoothing principle introduced in [HoZi.1–2] in the specific context of compressible Navier–Stokes equations, which states that (A2) implies smoothness of time-asymptotic states as in the strictly parabolic case, despite incomplete smoothing for finite times: in particular, of stationary or traveling-wave solutions.

Indeed, Pego [P.1] has shown by a center-manifold argument generalizing that of [MP] in the strictly parabolic case that the small-amplitude existence theory for hyperbolic–parabolic systems satisfying assumption (A2) is identical to that of the strictly parabolic case, as we now describe.

First, recall the Lax structure theorem:

**Proposition 1.10** [La, Sm]. Let $F^1 \in C^2$ be hyperbolic ($\sigma(dF^1)$ real and semisimple) for $U$ in a neighborhood of some base state $U_0$, and let $a_p(U_0)$ be a simple eigenvalue, where $a_1 \leq \cdots \leq a_k$ denote the eigenvalues of $dF^1(U)$, and $r_j = r_j(U)$ the associated eigenvectors. Then, there exists a smooth function $H_p : (U_-, \theta) \to U_+$, $\theta \in \mathbb{R}$, with $H_p(U_-,0) \equiv U_-$ and $(\partial/\partial \theta)H_p(U_-,0) \equiv r_p(U_-)$, such that, for $U_+$ lying sufficiently close to $U_0$ and $[U] := U_+ - U_-$ lying sufficiently close to $[U_+ - U_-]r_p(U_0)$, the triple $(U_-, U_+, s)$ satisfies the Rankine–Hugoniot conditions (RH) if and only if $U_+ = H_p(U_-, \theta)$ for $\theta$ sufficiently small.

**Proof.** See [Sm], pp. 328–329, proofs of Theorem 17.11 and Corollary 17.12. $\square$

Note that, for $[U]$ sufficiently small and $U$ sufficiently near $U_0$, (RH) implies that $[U]$ lies approximately parallel to some $r_j(U_0)$; thus, the assumption that $[U]$ lie nearly parallel to $r_p(U_0)$ is no real restriction. For fixed $U_-$, the image of $H_p$ is known as the $(p$th) Hugoniot curve through $U_-$. For gas dynamics, under mild assumptions on the equation of state$^{10}$ the Hugoniot curves extend globally, describing the full solution set of (RH).

$^9$Without symmetric, block form, we cannot make conclusions regarding the product $B^{11}U_{E_1}$ of a discontinuous function and a possibly point measure. However, (1.8)–(1.9) together allow us to eliminate the possibility of a jump in $w^{II}$.

$^{10}$For example, existence of a global convex entropy, plus the “weak condition” of [MeP],
Definition 1.11. For $U$ lying on the $p$th Hugoniot curve through $U_-$, denote by $\sigma(U_-, U)$ the associated shock speed defined by (RH). Then, the shock triple $(U_-, U_+ = H_p(U_-, \theta_+), s = \sigma(U_-, U_+))$ is said to satisfy the strict Liu-Oleinik admissibility condition if and only if

\[(LO) \quad \sigma(U_-, U(\theta_*)) > \sigma(U_-, U_+) = s \text{ for all } \theta_*(\theta_* - \theta_+) < 0,
\]

i.e., on the segment of the Hugoniot curve between $U_-$ and $U_+$, $\sigma(U_-, U)$ takes on its minimum value at $U_+$. A general small-amplitude existence theory may now be concisely stated as follows, with the strictly parabolic theory subsumed as a special case.

**Proposition 1.12** [P.1]. Let $F^1 \in C^2$ hyperbolic and $B^{11} \in C^2$ of form (1.5)–(1.6) satisfy the genuine coupling condition (1.10) for $U$ in a neighborhood of some base state $U_0$, and let $a_p(U_0)$ be a simple eigenvalue, where $a_1 \leq \cdots \leq a_k$ denote the eigenvalues of $dF^1(U)$. Then, for a sufficiently small neighborhood $U$ of $U_0$, and for $(U_-, U_+, s)$ lying sufficiently close to $(U_0, U_0, a_p(U_0))$ and satisfying the noncharacteristic condition $a_p(U_\pm) \neq s$, there exists a traveling-wave connection (1.2) lying in $U$ if and only if the triple $(U_-, U_+, s)$ satisfies both the Rankine–Hugoniot conditions (RH) and the strict Liu–Oleinik admissibility condition (LO). Moreover, such a local connecting orbit, if it exists, is transverse, hence unique up to translation.

**Proof.** See [P.1], or the related [Fre.4].

In the case of “standard” gas dynamics, for which the equation of state admits a global convex entropy, there is a correspondingly simple global existence theory. In the contrary case of “nonstandard”, or “real” (e.g., van der Waals) gas dynamics, the global existence problem so far as we know is open.

**Proposition 1.13**[Gi, MeP]. For gas dynamics with a global convex entropy, a triple $(U_-, U_+, s)$ admits a traveling-wave connections (viscous profile) (1.2) if and only if it satisfies both (RH) and (LO). Moreover, connections if they exist are transverse.

**Proof.** See [Gi] in the “genuinely nonlinear” case $\nabla a_p r_p \neq 0$, [MeP], Appendix C in the general case. Transversality is automatic, by Remark 1.9.1.

**Remarks 1.14.** 1. In the case that eigenvalues $a_j$ retain their order relative to $a_p$, e.g., $a_p$ remains simple, along the Hugoniot curve, the Liu–Oleinik condition (LO) may be seen to be a strengthened version of the classical Lax characteristic conditions \[La\]

\[(1.23) \quad a_{p-1}(U_-) < 0 < a_p(U_-), \quad a_p(U_+) < 0 < a_{p+1}(U_+),
\]
satisfied for all known examples possessing a convex entropy (in particular, ideal gas dynamics and the larger classes of equations considered by Bethe, Weyl, Gilbarg, Wendroff, and Liu [Be, We, Gi, Wen.1–2, L.4]); see [MeP].
where \(1 \leq p \leq n\) is the principal characteristic speed associated with the shock, or, equivalently (by preservation of order),

\[(1.24) \quad a_p(U_-) > s > a_p(U_+),\]

by \(a_p(U_-) = \sigma(U_-, U_-) > s\) and symmetry with respect to \(\pm\) of (LO). In particular, the profiles described in Propositions 1.12 and 1.13 are always of classical, Lax type. Small-amplitude profiles bifurcating from a multiple eigenvalue \(a_p\), or large-amplitude profiles of general hyperbolic–parabolic systems may be of arbitrary type; see [FreS.1] and [CS], respectively.

In the genuinely nonlinear case \(\nabla a_p r_p \neq 0\), (LO) is equivalent along the Hugoniot curve \(U_+ = H_p(U_-, \theta)\) through \(U_-\) to condition (1.24), which in turn is equivalent to \(\sgn \theta = -\sgn \nabla a_p r_p(U_-) \neq 0\) [La, Sm]. Here, \(H_p\) denotes an arbitrary nondegenerate extension of the local parametrization defined in Proposition 1.10.

2. Condition (LO) yields existence and uniqueness of admissible scale-invariant solutions of (1.4) for Riemann initial data

\[(1.25) \quad U_0 = \begin{cases} U_- & \text{for } x < 0, \\ U_+ & \text{for } x \geq 0, \end{cases}\]

for small amplitudes \(|U_+ - U_-|\) and general equations [L.4], and, under mild additional assumptions (the “medium condition” of [MeP]), for large amplitudes in the case of gas dynamics with a global convex entropy [MeP]. Such Riemann solutions represent possible time-asymptotic states (after renormalization (1.22)) for the viscous equations (1.1) with “asymptotically planar” initial data in the sense that \(\lim_{x \to \pm\infty} U_0(x) = U_\pm\); see, e.g., [AMPZ.1] for further discussion.

3. For large amplitudes, the genuine coupling condition (A2) does not in general appear to preclude discontinuous traveling-wave solutions of (1.1) analogous to “subshocks” in the relaxation case [L.5, Wh], but rather must be replaced by the nonlinear version \([W] \not\in \ker \tilde{B}^{11}(U_-) = \ker \tilde{B}^{11}(U_+)\) for \((U_-, U_+, s)\) satisfying (RH). For compressible Navier–Stokes equations, this condition is satisfied globally for classical equations of state. Without loss of generality taking shock speed \(s = 0\), we find that failure corresponds to a jump in density with velocity, pressure, and temperature held fixed; see the discussion of gas-dynamical structure below Corollary 6.5, Appendix A.1. Thus, it can occur only for a nonmonotone, van der Waals-type pressure law, and corresponds to an undercompressive, phase-transitional type shock. Recall that such shocks do not possess smooth profiles under the influence of viscosity alone, Remark 1.9.4.

1.3.2. Dynamical stability. Hyperbolic stability concerns the bounded-time stability of (1.21) as a solution of inviscid equations (1.4) augmented with the free-boundary condition of solvability of the viscous profile problem, i.e., well-posedness of the “outer problem” arising through formal matched asymptotic expansion. That is, it is Hadamard-type stability, or well-posedness, that is the relevant notion in this
case, in keeping with the homogeneous nature of (1.4). In view of Remark 1.14.1, we focus our attention on the classical, Lax case, for which the free-boundary conditions are simply the Rankine–Hugoniot conditions (RH) arising through conservation of mass.

Postulating a perturbed front location $X(\tilde{x}, t)$, we may convert the problem to a more conventional fixed-boundary problem at $z := x_1 - X(\tilde{x}, t)$, following the standard approach to stability of fluid interfaces (see, e.g., [Ri, Er.1, FD, M.1–4]). This device eliminates the difficulty of measures arising in the linearized equations (and eventual nonlinear iteration) through differentiation about a moving discontinuity.

**Necessary inviscid stability condition.** Linearizing the resulting equations about (1.21), assuming without loss of generality $s = 0$, we obtain constant-coefficient problems

\[
U_t + A^1_+ U_z + \sum_{j=2}^d A^j_+ U_{x_j} = 0,
\]

where $A^j_\pm := dF^j(U_\pm)$, on $z \geq 0$, linked by $n$ transmission conditions given by the linearized Rankine–Hugoniot conditions, which involve also $t$- and $\tilde{x}$- derivatives of the (scalar) linearized front location.

Linearized inviscid stability analysis by Laplace–Fourier transform of (1.27) yields, in the case of a Lax $p$-shock (defined by (1.23), or $p := i_+$ in the notation of Section 1.2), the **weak Kreiss–Sakamoto–Lopatinski stability condition**

\[
\Delta(\vec{\xi}, \lambda) \neq 0, \quad \vec{\xi} \in \mathbb{R}^{d-1}, \quad \text{Re} \lambda > 0,
\]

where $\vec{\xi} := (\xi_2, \cdots, \xi_d)$ is the Fourier wave number in transverse spatial directions $\tilde{x} := (x_2, \cdots, x_d)$, $\lambda$ is the Laplace frequency in temporal direction $t$, and

\[
\Delta := \det (r_1^-, \cdots, r_{p-1}^-, i[F^\vec{\xi}] + \lambda[U], r_p^+, \cdots, r_n^+),
\]

with $F^\vec{\xi}(U) := \sum_{j \neq 1} \xi_j F^j(U)$, $[g] := g(U_+) - g(U_-)$, and $\{r_p^+, \cdots, r_n^+\}(\vec{\xi}, \lambda)$ denoting (analytically chosen) bases for the unstable/resp. stable subspaces of matrices

\[
A^\pm(\vec{\xi}, \lambda) := (\lambda I + dF^\vec{\xi}(U_\pm))(dF^1(U_\pm))^{-1};
\]

for details, see, e.g. [Se.2–4, ZS, Mé.5]. Condition (1.28) is obtained from the requirement that the inhomogeneous equation

\[
(r_1^-, \cdots, r_{p-1}^-, i[F^\vec{\xi}] + \lambda[U], r_p^+, \cdots, r_n^+) \alpha = \hat{f}
\]
arising through Laplace–Fourier transform of the linearized problem have a unique solution for every datum \( \hat{f} \), i.e. the columns of the matrix on the lefthand side form a basis of \( \mathbb{R}^n \).

This result was first obtained by Erpenbeck in the case of gas dynamics\(^{11}\) [Er.1–2], and in the general case by Majda [M.1–4]. Condition (1.28) is an example of a general type of stability condition introduced by Kreiss and Sakamoto in their pioneering work on hyperbolic initial-boundary value problems [Kr,Sa.1–2]; as pointed out by Majda, it plays a role analogous to that of the Lopatinski condition in elliptic theory. Zeroes of \( \Delta \) correspond to normal modes \( e^{\lambda t} e^{i\xi \cdot \vec{x}} w(x_1) \) of the linearized equations, hence (1.28) is necessary for linearized inviscid stability. Evidently, \( \Delta \) is positive homogeneous (i.e., homogeneous with respect to positive reals), degree one, as are the inviscid equations; thus, instabilities if they occur are of Hadamard type, corresponding to ill-posedness of the linearized problem.

**Sufficient inviscid stability condition.** A fundamental contribution of Majda [M.1–4], building on the earlier work of Kreiss [Kr], was to point out the importance of the uniform (or “strong”) Kreiss–Sakamoto–Lopatinski stability condition

\[
(1.32) \quad \Delta(\vec{\xi}, \lambda) \neq 0, \quad \vec{\xi} \in \mathbb{R}^{d-1}, \quad \text{Re} \lambda \geq 0, \quad (\vec{\xi}, \lambda) \neq (0,0),
\]

extending (1.28) to imaginary \( \lambda \), as a sufficient condition for nonlinear stability. Assuming (1.32), a certain “block structure” condition on matrix \( A(\vec{\xi}, \tau) \) defined in (1.30) (recall: implied by (H4)), and appropriate compatibility conditions on the perturbation at the shock (ensuring local structure of a single discontinuity, i.e., precluding the formation of shocks in other characteristic fields), Majda established local existence of perturbed ideal shock solutions

\[
(1.33) \quad \tilde{U}(x,t) =: \tilde{U}(x_1 - X(\tilde{x},t)) + U(x_1 - X(\tilde{x},t), \tilde{x},t),
\]

\( \tilde{U} \) as in (1.21), for initial perturbation sufficiently small in \( H^s \), \( s \) sufficiently large, with optimal bounds on \( \|U\|_{H^s(x,t)} \) and \( \|X\|_{H^s(\tilde{x},t)} \). Moreover, using pseudodifferential techniques, he established a corresponding result for curved shock fronts under the local version of the planar uniform stability condition, establishing both existence and stability of curved shock solutions.

These result have since been significantly sharpened and extended by Métivier and coworkers; see [Mé.5], and references therein. In particular, Métivier [Mé.1] has obtained uniform local existence and stability for vanishing shock amplitudes under assumptions (A1) and (H2) plus the additional nondegeneracy conditions that principal characteristic \( a_p(U, \xi) \) be: (i) simple and genuinely nonlinear in the normal direction to the shock, without loss of generality \( \xi = (1,0,\ldots,0) \), and (ii) strictly convex in \( \xi \) in a vicinity of \( \xi = (1,0,\ldots,0) \). In the existence result of

\(^{11}\)See also related, earlier work in [Be,D,Ko.1-2,Free.1-2,Ro,Ri].
Majda, the time of existence vanished with the shock strength, due to the singular (i.e., characteristic) nature of the small-amplitude limit.

**Remark 1.15.** Failure of (1.32) on the boundary \( \lambda = i\tau, \tau \) real, i.e., weak but non-uniform stability, corresponds to existence of a one-parameter family of (nondecaying) transverse traveling-wave solutions \( e^{i(\kappa\tau + \kappa\xi z)} w(\kappa z) \) of the linearized equations, \( \kappa \in \mathbb{R} \), moving with speed \( \sigma = \tau/|\xi| \) in direction \( \hat{\xi}/|\hat{\xi}| \), and is in general associated with loss of smoothness in the front \( X(\cdot, \cdot) \) and loss of derivatives in the iteration used to prove nonlinear stability.\(^{12}\) When \( w \) has finite energy, \( w^2_{L^2} \), such solutions are known as surface waves; however, this is not always the case. In particular, \( w \in L^2 \) only if

\[
(1.34) \quad a_{j_{\pm}}^+(\xi_1, \hat{\xi}) < \tau < a_{k_{\pm}}^+(\xi_1, \hat{\xi})
\]

for some \( j_{\pm}, k_{\pm} \) (without loss of generality \( a_{k_{\pm}}(\xi) = -a_{j_{\pm}}(-\xi) \)):

see Proposition 3.1 of [BRSZ]. For \( \xi_1 = 0 \), this implies that the traveling wave is “subsonic” in the sense that \( \sigma \) lies between the \( j_{\pm} \)th and \( k_{\pm} \)th hyperbolic characteristic speeds in direction \( \hat{\xi}/|\hat{\xi}| \) at endstates \( U_{\pm} \). It is easily seen that the set determined by (1.34) is bounded by the glancing set defined in Definition 1.3; indeed, the complement of the closure of this set is the union \( G \) of the components of \( (\xi, \tau) = (0, \pm 1) \) in the complement of the glancing set. A more fundamental division is between the larger hyperbolic domain \( R \), defined as the set of \( (\xi, \tau) \) for which the stable and unstable subspaces of \( A(\xi, i\tau) \) in (1.30) admit real bases, and its complement. (Note: \( G \subset R \).) On \( R \), the eigenvalues of \( A(\xi, i\tau) \) in (1.30) determining exponential rates of spatial decay in \( z \) are pure imaginary, and so surface waves cannot occur. The set \( R \) like \( G \) is bounded by the glancing set, at which certain eigenvalues of \( A(\xi, i\tau) \) bifurcate away from the imaginary axis.

As discussed in [BRSZ], these various cases are in fact quite different. In particular, (i) in the surface wave case, the linearized equations are “weakly well-posed” in \( L^2 \), permitting a degraded version of the energy estimates in the strongly stable case, whereas in the complementary case they are not even weakly well-posed, and (ii) zeroes \( (\xi, \tau) \) lying in the closure of \( R^c \) generically mark a boundary between regions of strong stability and strong instability in parameter space \( (U_-, U_+, s) \), whereas zeroes lying in the interior of \( R \) generically persist, marking an open set in parameter space. The explanation for the latter, at first surprising fact is that for real \( (\xi, \tau) \in R \), it may be arranged by choosing \( r_2^c \) real that \( \Delta(\xi, \cdot) \) take the imaginary axis to itself; thus, zeroes may leave the imaginary axis only by coalescing at a double (or higher multiplicity) root, either finite,

\[
(1.35) \quad \Delta(\xi, i\tau) = 0, \quad (\partial/\partial \lambda)\Delta(\xi, i\tau) = 0,
\]

or at infinity (recall, by complex symmetry, that roots escape to infinity in complex conjugate pairs). The latter possibility corresponds to one-dimensional instability

\(^{12}\)An exception is the scalar case; see [M.1–4].
\( \Delta(0, 1) = 0, \) as discussed in [Se.2, BRSZ, Z.3]. The important category of weakly stable shocks possessing a zero \( \Delta(\tilde{\xi}, i\tau) = 0 \) with \( (\tilde{\xi}, \tau) \) in \( \mathcal{R} \) was denoted in [BRSZ] as “weakly stable of real type”, and forms a third open (i.e., generic) type besides the strongly stable and strongly unstable ones.

**Evaluation in one dimension.** In one-dimension, \( \tilde{\xi} = 0 \), the basis vectors \( r_j^\pm \) may be fixed as the outgoing characteristic directions for \( A_\pm = dF^1(U_\pm) \), i.e., eigenvectors with associated eigenvalues \( a_j^\pm \geq 0 \), yielding \( \Delta(0, \lambda) = \lambda \Delta(0, 1) \). Thus, both weak and uniform stability conditions reduce to

\[
\Delta(0, 1) = \det (r_1^-, \cdots, r_{p-1}^-, [U], r_{p+1}^+, \cdots, r_n^+) \neq 0,
\]

Condition (1.36) may be regarded as a sharpened form of the Lax characteristic condition (1.23), which asserts only that the number of columns of the matrix on the left-hand side of (1.36) is correct (i.e., equal to \( n \)). Condition (1.36) can be recognized also as the condition that the associated Riemann problem \((U_-, U_+)\) be linearly well-posed (transverse), since otherwise there is a nearby family of Riemann solutions possessing the same (final) end states \( U_\pm \), but for which the data at the \( p \)-shock is modified by infinitesimal perturbations \( dU_\pm \) such that the linearized Rankine–Hugoniot relations remain satisfied:

\[
-ds[U] = F^1(U_+)dU_+ - dF^1(U_-)dU_-.
\]

That is, failure of one-dimensional inviscid stability is generically associated with the phenomenon of *wave-splitting*; see, e.g., [Er.1, ZH, Z.3], for further details.

As observed by Majda [M.1–4], in the case of a simple eigenvalue \( a_p(U_+) \), the Lax structure theorem, Theorem 1.10, ensures that *one-dimensional stability holds for sufficiently small amplitude shocks of general systems*. One-dimensional stability holds for arbitrary amplitude shocks of the equations of gas dynamics with a “standard” equation of state (e.g., possessing a global convex entropy and satisfying the “medium condition” of [MeP]), but may fail at large amplitudes for more general equations of state; see, e.g., [Er.1–2, M.1–4, MeP].

**Remark 1.16.** The set of parameters \((U_-, U_+, s)\) on which the one-dimensional dynamical stability condition (1.36) fails is of codimension one, hence generically consists of isolated points along a given Hugoniot curve \( H_p(U_-, \cdot) \). In other words, the one-dimensional dynamical stability condition *yields essentially no information* further than the Lax structure already imposed on the problem by the condition of structural stability.

---

\[^{13}\text{By positive homogeneity, } 0 \equiv \Delta_j(\tilde{\xi}_j, i\tau_j) = \Delta_j(\tilde{\xi}_j/|\tau_j|, i \text{ sgn } \tau_j) \to \Delta(0, \pm i) \text{ as } j \to \infty, \text{ where } \Delta_j \text{ are the Lopatinski conditions for a sequence of converging parameter values indexed by } j, \text{ with zeroes } (\tilde{\xi}_j, i\tau_j) \text{ such that } \sigma_j = \tau_j/|\tilde{\xi}_j| \to \infty \text{ as } j \to \infty, \text{ and } \Delta = \lim \Delta_j \text{ is the Lopatinski condition for the limiting parameter values as } j \to \infty. \text{ Hence, } \Delta \text{ vanishes at one of } (0, \pm i) \text{ (in fact both, by complex symmetry of the spectrum), yielding one-dimensional instability.} \]
Evaluation in multiple dimensions. Evaluation of the stability conditions in multi-dimensions is significantly more difficult than in the one-dimensional case, even the computation of $\Delta$ being in general nontrivial. Nonetheless, we obtain again a small-amplitude result for general systems and an arbitrary-amplitude result for gas dynamics, both with appealingly simple formulations.

Namely, under the assumptions of [Mé.1]–(A1) and (H2) plus the conditions that principal characteristic $a_p(U, \xi)$ be (i) simple and genuinely nonlinear in the normal direction to the shock, without loss of generality $\xi = (1, 0, \ldots, 0)$, and (ii) strictly convex in $\xi$ in a vicinity of $\xi = (1, 0, \ldots, 0)$—we have the simple small-amplitude result, generalizing Majda’s observation in one dimension, that sufficiently small-amplitude shocks are strongly (“uniformly”) stable. This is an indirect consequence of the detailed partial differential equation (PDE) estimates carried out by Métivier in the proof of existence and stability of small-amplitude shocks [Mé.1]. We give a direct proof in Appendix B, in the spirit of Métivier’s analysis but carried out in the much simpler linear-algebraic setting.

A detailed analysis of the large-amplitude gas-dynamical case recovering the physical stability criteria of Erpenbeck–Majda [Er.1–2, M.1–4] is presented in Appendix C. This includes in particular Majda’s theorem that for ideal gas dynamics, shocks of arbitrary amplitude are strongly (uniformly) stable. Discussion of more general equations of state may be found, e.g., in [Er.1–2, BE, MeP]. In particular, uniform stability holds for equations of state possessing a global convex entropy and satisfying the “medium condition” of [MeP], but in general may fail, depending on the geometry of the associated Hugoniot curve.

Remark 1.17. For gas dynamical shocks, direct calculation shows that weak multi-dimensional inviscid stability when it occurs is always of real type, with $R = G$. Moreover, transition from weak multi-dimensional inviscid stability of real type to strong multi-dimensional inviscid instability (failure of (1.28)) can occur only through the infinite speed limit $\sigma = \tau/|\tilde{\xi}| \to \infty$ corresponding to one-dimensional inviscid instability, $\Delta(0, 1) = 0$; see for example discussions in [Er.1] (p. 1185), [Fo]. Likewise (see Remark 1.15), the transition from weak multi-dimensional inviscid stability of real type to strong multidimensional inviscid stability can occur only through collision of $(\tilde{\xi}, \tau)$ with the glancing set $\{((\tilde{\xi}, \eta(\tilde{\xi})))\}$ defined in Definition 1.3: easily calculated for gas dynamics; see, e.g., [Z.3], example near (3.12), or Appendix C of this article. By rotational invariance and homogeneity, this transition may be detected by the single condition $\Delta(\xi_0, \eta(\xi_0)) \neq 0, \xi_0 = (1, 0, \ldots, 0)$, similarly as in the previous case. These two observations greatly simplify the determination of the various inviscid shock stability regions once $\Delta$ has been determined; see, e.g., [BRSZ] or Section 6.5 of [Z.3]. From our point

---

14Contributed by K. Jenssen and G. Lyng.
15Equivalently, $(\partial/\partial \lambda)\Delta(\tilde{\xi}, i\tau)$ and $\Delta(\tilde{\xi}, i\tau)$ do not simultaneously vanish for $\tilde{\xi}, \tau$ real (indeed, $(\partial/\partial \lambda)\Delta(\tilde{\xi}, i\tau)$ does not vanish for $(\tilde{\xi}, \tau)$ in $R$); see Appendix C and Remark 1.15.
16Precisely, together with Métivier’s theorem guaranteeing uniform stability of small-amplitude
of view, however, the main point is simply that, for gas dynamics, transition from strong inviscid stability to instability, if it occurs, occurs via passage through the open region of weak inviscid stability of real type, on which (inviscid) stability is indeterminate.

Remark 1.18. (nonclassical case) Freistühl [Fre.1–2] has extended the Majda analysis to the nonclassical, undercompressive case, obtaining in place of (1.28)–(1.29) and (1.32) the extended, \((n+q)\times(n+q)\) dimensional determinant conditions

\begin{equation}
\Delta(\tilde{\xi}, \lambda) \neq 0, \text{ for all } \tilde{\xi} \in \mathbb{R}^{d-1}, \text{ Re } \lambda > 0 \text{ (resp. Re } \lambda \geq 0 \text{ and } \lambda \neq 0),
\end{equation}

\[
\Delta(\tilde{\xi}, \lambda) := \det \left( \frac{\partial g}{\partial U_-} A_+^{-1} \frac{\partial g}{\partial U_-} A_-^{-1} \right)
\]

where

\[
g(U_-, U_+, \omega, s) = 0, \quad g \in \mathbb{R}^q
\]

encodes the \(q\) kinetic conditions needed along with (RH) to determine existence of an undercompressive connection in direction \(\omega \in S^{d-1}\) (i.e., a traveling-wave solution \(U = \bar{U}(x \cdot \omega - st)\)), and all partial derivatives are taken at the base parameters \((U_-, U_+, e_1, 0)\) corresponding to the one-dimensional stationary profile \(U = \bar{U}(x_1)\). This condition has been evaluated in [B-G.1–2] for the physically interesting case of phase boundaries, with the result of strong (i.e., uniform) stability for sufficiently small but positive viscosities.\(^{17}\) Applied to the overcompressive case, on the other hand, the analysis of Majda yields automatic instability. For, (1.31) is in this case ill-posed for all \(\tilde{\xi}, \lambda\), since the matrix on the lefthand side has fewer than \(n\) columns.

1.4. Viscous stability analysis.

The formal, matched asymptotic expansion arguments underlying the classical, inviscid stability analysis leave unclear its precise relation to the original, physical stability problem (1.1). In particular, rigorous validation of matched asymptotic expansion generally proceeds through stability estimates on the inner solution (viscous shock profile), which in the planar case is somewhat circular (recall: the inner shocks, we recover by these observations the results of the full Nyquist diagram analysis carried out in Appendix C. (In fact, these two approaches are closely related; see discussion of [Z.3], Section 6.5.)

\(^{17}\)Recall that, in the undercompressive case, higher-order derivative terms influence the inviscid problem through their effect on the connection conditions \(g\).
problem is the planar evolution problem, with additional terms comprising the formal truncation error); hence, we cannot rigorously conclude even necessity of these conditions for viscous stability. This is more than a question of mathematical nicety: in certain physically interesting cases arising in connection with nonclassical under- and overcompressive shocks, the classical stability analysis is known to give wrong results both positive and negative; see Remarks 1.18, 1.20.1, and 1.22.5, or Section 1.3, [Z.3]. Moreover, even in the classical Lax case, it is not sufficiently sharp to determine the transition from stability to instability, either in one or multiple dimensions; see Remarks 1.16 and 1.17.

These and related issues may be resolved by a direct stability analysis at the level of the viscous equations (1.1), as we now informally describe. Rigorous statements and proofs will be established throughout the rest of the article. In particular, the viscous theory not only refines (and in some cases corrects) the inviscid theory but also completes it by the addition of a third, “strong spectral stability” condition augmenting the classical conditions of structural and dynamical stability. Strong spectral stability is somewhat weaker than linearized stability— in the one-dimensional case, for example, it is more or less equivalent to stability with respect to zero-mass perturbations [ZH]— and may be recognized as the missing condition needed to validate the formal asymptotic expansion; see, for example, the related investigations [GX, GR, R, MéZ.1, GMWZ.2–4].

1.4.1. Connection with classical theory. The starting point for the viscous analysis is a linearized, Laplace–Fourier transform analysis very similar in spirit to that of the inviscid case. This yields the Evans function condition

$$D(\xi, \lambda) \neq 0, \quad \xi \in \mathbb{R}^{d-1}, \, \text{Re} \lambda > 0,$$

as a necessary condition for linearized viscous stability, where the Evans function $D$ (defined in Section 2.2, below) is a determinant analogous to the Lopatinski determinant $\Delta(\cdot, \cdot)$ of the inviscid theory. Similarly as in the inviscid case, zeroes of $D(\cdot, \cdot)$ correspond to growing modes $v = e^{\lambda t} e^{i\xi \cdot x} w(x_1)$ of the linearized equations about the background profile $\bar{U}(\cdot)$.

Definition 1.19. We define weak spectral stability (necessary but not sufficient for linearized viscous stability) as (1.39). We define strong spectral stability (by itself, neither necessary nor sufficient for linearized stability) as

$$D(\xi, \lambda) \neq 0 \quad \text{for} \quad (\xi, \lambda) \in \mathbb{R}^{d-1} \times \{ \text{Re} \lambda \geq 0 \} \setminus \{0, 0\}.$$

The fundamental relation between viscous and inviscid theories, quantifying the formal rescaling argument alluded to in (1.22), Section 1.3, is given by

Result 1 [ZS, Z.3]. (Thm 3.5) $D(\xi, \lambda) \sim \gamma \Delta(\xi, \lambda)$ as $(\xi, \lambda) \to (0, 0)$, where $\gamma$ is a constant measuring transversality of the connection $\bar{U}(\cdot)$ in (1.1), i.e., $D$ is tangent
to $\Delta$ in the low frequency limit. Here, $\Delta$ in the Lax or undercompressive case is given by (1.29) or (1.38), respectively, and in the overcompressive case by a modified determinant correcting the inviscid theory, as described in equations (3.9)–(3.10), Theorem 3.5.

That is, in the low-frequency limit ($\sim$ large-space–time, by the standard duality between spatial and frequency variables), higher-derivative terms become negligible in (1.1), and behavior is governed by its homogeneous first-order (inviscid) part (1.4). The departure from inviscid theory in the overcompressive case reflects a subtle but important distinction between low-frequency/large-space–time and vanishing-viscosity limits, as discussed in Section 1.3, [Z.3].

**1.4.2. One-dimensional stability.** The most dramatic improvement in the viscous theory over the inviscid theory comes in the one-dimensional case, for which the inviscid dynamical stability condition yields essentially no information, Remark 1.16. From Result 1, we find that vanishing of the one-dimensional Lopatinski determinant $\Delta(0,1)$ generically corresponds to crossing from the stable to the unstable complex half-plane $\text{Re} \lambda > 0$ of a root of the Evans function $D$. Indeed, we have

**Result 2** [Z.3–4]. (Props. 4.2, 4.4, and 4.5) With appropriate normalization, the condition $\gamma \Delta(0,1) > 0$ is necessary for one-dimensional linearized viscous stability.

More precisely, we have the following sharp one-dimensional stability criterion.

**Result 3** [MaZ.3–4]. (Thms. 4.8 and 4.9) Necessary and sufficient conditions for linearized and nonlinear one-dimensional viscous stability are one-dimensional structural and dynamical stability, $\gamma \Delta(0,1) \neq 0$, plus strong spectral stability, $D(0,\lambda) \neq 0$ on $\{\text{Re} \lambda \geq 0\} \setminus \{0\}$.

**Remarks 1.20.** 1. Result 2 replaces the codimension one inviscid instability condition $\gamma \Delta(0,1) = 0$ with an open condition $\gamma \Delta(0,1) > 0$ of which it is the boundary. That is, one-dimensional inviscid instability theory marks the boundary between one-dimensional viscous stability and instability. The strengthened, viscous condition $\gamma \Delta(0,1) > 0$ in particular resolves the problem of physical selection discussed in [MeP], p. 110, for which multiple Riemann solutions exist, each satisfying the classical conditions of dynamical and structural stability, but only one-- the one satisfying the one-dimensional viscous stability condition-- is selected by numerical experiment; see related discussion, Section 6.3, [Z.3]. Likewise, it yields instability of undercompressive, pulse-type (i.e., homoclinic) profiles arising in phase-transitional flow [GZ, Z.5], which, again, satisfy the classical inviscid stability criteria but numerically appear to be unstable [AMPZ.1]. Result 3 as

---

18Briefly, initial data scales as $\varepsilon$ in the low-frequency limit, but in the vanishing-viscosity limit is fixed, order one. For $\varepsilon$-order data, underdetermination of the inner problem associated with nonuniqueness of overcompressive orbits can balance with overdetermination of the outer problem; for general data, the effects occur on different scales. For related discussion outside the matched asymptotic setting, see [Fre.3], [L.2], or [FreL].
claimed completes the inviscid stability conditions with the addition of the third condition of strong spectral stability. Spectral stability has been verified for small-amplitude, Lax-type profiles in [HuZ, PZ], and for special large-amplitude profiles in [MN.1] and [Z.5]. Efficient algorithms for numerical testing of spectral stability are described, e.g., in [Br.1-2, BrZ, KL, BDG].

2. It is an interesting and physically important open question which of the three criteria of structural, dynamical, or spectral stability is the more restrictive in different regimes; in particular, it would be very interesting to know whether, or under what circumstances, spectral stability can be the determining factor in transition to instability as $|\theta|$ is gradually increased along the Hugoniot curve $U_+ = H_p(U_-, \theta)$ defined in Section 1.3.1. The latter question is closely related to the following generalized Sturm–Liouville question: Do the global symmetry assumptions (1.11)-(1.9), or perhaps the stronger condition that the system (1.1) possess a global convex entropy, as discussed in Section 1.1, preclude the existence of nonzero pure imaginary eigenvalues? For, as pointed out in [ZH,Z.3], the same calculations used to establish the signed stability condition show that eigenvalues can cross through the origin from stable to unstable complex half-plane only if $\gamma \Delta = 0$, i.e., $\Delta$ vanishes or transversality is lost. Thus, beginning with small-amplitude profiles (stable, by [HuZ,PZ]) and continuously varying the endstate $\tilde{U}_+$, we find that loss of stability, if it occurs before structural or dynamical stability fail, must correspond to a pair of complex-conjugate eigenvalues crossing the imaginary axis, presumably signaling a Poincaré–Hopf-type bifurcation\textsuperscript{19} to a family of time-periodic solutions.

Recall that such “galloping” instabilities do in fact occur in the closely related situation of detonation (see, e.g., [Ly.1,LyZ.1-2]); thus, the question of whether or not or under what circumstances they can occur for shock waves seems physically quite interesting. If they cannot occur for the class of systems we consider, then the relatively simple conditions of structural and dynamical stability—i.e., the classical conditions of inviscid stability—completely determine the transition from viscous stability to instability, a very satisfactory conclusion. If, on the other hand, they can occur, then this implies that heretofor neglected viscous effects play a far stronger role in stability than we have imagined, and this result would be philosophically still more significant.

1.4.3. Multi-dimensional stability. A deficiency of the inviscid multi-dimensional stability theory is that transition from strong inviscid stability to strong inviscid instability typically occurs through the open region of weak inviscid stability of real type, on which stability is indeterminate; see Remarks 1.15 and 1.17, or Section 1.4, [Z.3]. This indeterminacy may be remedied by a next-order correction in the low-frequency limit including second-order, viscous effects. There is also some question [BE] whether the physical transition to instability might sometimes occur before the passage to weak inviscid instability, i.e., for parameters within the

\textsuperscript{19}Degenerate, because of the well-known absence [Sat] of a spectral gap between $\lambda = 0$ and the essential spectrum of the linearized operator about the wave.
region of strong inviscid stability. This likewise can be determined, at least in prin-
ciple, by the inclusion of neglected viscous e
ffect, this time in the (global) form of
the strong spectral stability condition.

Definition 1.21. We define weak refined dynamical stability as inviscid weak
stability (1.28) augmented with the second-order condition that \( \text{Re} \beta(\bar{\xi}, i\tau) \geq 0 \)
for any real \( \bar{\xi}, \tau \) such that \( \Delta \) (hence also \( D^{i\tau}(\rho) := D(\rho\bar{\xi}, \rho i\tau) \), by Result 1) is
analytic at \( (\xi, i\tau) \), \( \Delta(\xi, i\tau) = 0 \), and \( (\partial/\partial\lambda)\Delta(\xi, i\tau) \neq 0 \), where

\[
\beta(\bar{\xi}, i\tau) := \frac{(\partial/\partial\lambda)^{\ell+1}D(\rho\bar{\xi}, \rho i\tau)}{(\partial/\partial\lambda)\Delta(\xi, i\tau)} = \frac{(\partial/\partial\lambda)^{\ell+1}D(\rho\bar{\xi}, \rho\lambda)}{(\partial/\partial\lambda)(\partial/\partial\rho)^{\ell}D(\rho\bar{\xi}, \rho\lambda)} |_{\rho=0, \lambda=i\tau}.
\]

We define strong refined dynamical stability as inviscid weak stability (1.28) aug-
mented with the conditions that \( \Delta \) be analytic at \( (\xi, i\tau) \), \( (\partial/\partial\lambda)\Delta(\xi, i\tau) \neq 0 \),
be independent for \( r^j \) defined as in (1.29) (automatic for “extreme” shocks \( p = 1 \)
or \( n \))), and \( \text{Re} \beta(\xi, i\tau) > 0 \) for any real \( \xi, \tau \) such that \( \Delta(\xi, i\tau) = 0 \). Condition
\( \Delta_\lambda(\xi, i\tau) \neq 0 \) implies that such imaginary zeroes are confined to a finite union of
smooth curves

\[
\tau = \tau_j(\bar{\xi}).
\]

Result 4 [Z.3]. (Cor. 5.2) Given one-dimensional inviscid and structural stability,
\( \gamma\Delta(0, 1) \neq 0 \), weak refined dynamical stability is a necessary condition for multi-
dimensional linearized viscous stability (indeed, for weak spectral stability as well).

Result 5 (New). (Thms. 5.4–5.5) Structural and strong refined dynamical stabil-
ity together with strong spectral stability are sufficient for linearized viscous stability
in dimensions \( d \geq 2 \) and for nonlinear viscous stability in dimensions \( d \geq 3 \) (Lax
or overcompressive case) or \( d \geq 4 \) (undercompressive case). In the strongly invis-
cid stable Lax or overcompressive case, we obtain nonlinear viscous stability in all
dimensions \( d \geq 2 \).

Remarks 1.22. 1. By the parenthetical comment in Result 4, we may substitute
in Result 5 for strong refined dynamical stability the condition of weak inviscid
stability together with the nondegeneracy conditions that for any real \( \xi, \tau \) for
which \( \Delta(\xi, i\tau) = 0 \), \( \xi, \tau \) lie off of the glancing set \( (\tau \neq \eta(\xi) \) in the notation of
Definition 1.3), \( (\partial/\partial\lambda)\Delta(\xi, i\tau) \neq 0 \), and \( \beta(\xi, i\tau) \neq 0 \). This gives a formulation
more analogous to that of Result 3 in the one-dimensional case.

\footnote{This condition is for Lax or overcompressive shocks, and must be replaced by a modified
version in the undercompressive case [Z.3].}
2. Results 4 and 5 together resolve the indeterminacy associated with transition to instability through the open region of weak inviscid stability of real type, at least in dimensions \( d \geq 3 \).\textsuperscript{21} For, \( \Delta \) is analytic at \((\tilde{\xi}, i\tau)\) for any real \((\tilde{\xi}, \tau)\) not lying in the glancing set described in Definition 1.3; see discussion below Remark 1.4. Thus, referring to Remark 1.15, we find that the transition from refined dynamical stability to instability must occur at one of three codimension-one sets: the boundary between strong inviscid stability and weak inviscid stability, marked either by one-dimensional inviscid instability \( \Delta(0,1) = 0 \) or coalescence of two roots \((\tilde{\xi}, i\tau)\) at a double zero of \( \Delta \); the boundary between weak and strong inviscid stability, at which \((\tilde{\xi}, \tau)\) associated with a root \((\tilde{\xi}, i\tau)\) of \( \Delta \) strikes the glancing set; or else the (newly defined) boundary between refined dynamical stability and instability, which \((\tilde{\xi}, \tau)\) associated with a root \((\tilde{\xi}, i\tau)\) of \( \Delta \) strikes the set \( \beta(\tilde{\xi}, i\tau) = 0 \) within the interior of \( R \). In the scalar case, \( \beta(\xi, i\tau) \) is uniformly positive and, moreover, \( \Delta \) is linear, hence globally analytic with simple roots; indeed, viscous stability holds for profiles of arbitrary strength [HoZ.3–4, Z.3]. More generally, \( \beta \) can be expressed in terms of a generalized Melnikov integral, as described in [BSZ.2, B-G.4]; evaluation (either numerical or analytical) of this integral in physically interesting cases is an important area of current investigation.

3. The condition in the definition of strong refined stability that \( \Delta \) be analytic at roots on the imaginary boundary is generically satisfied in dimension \( d = 2 \), or for rotationally invariant systems such as gas dynamics. However, it typically fails for general systems in dimensions \( d \geq 3 \). A weaker condition appropriate to this general case is to require that the curve of imaginary zeroes intersect the glancing set transversely. At the expense of further complications, all of our analysis goes through with this relaxed assumption [Z.4]. Moreover, it fails again on a codimension-one boundary.

4. Multi-dimensional spectral stability has been verified for small-amplitude Lax-type profiles in [FreS.2, PZ], completing the verification of viscous stability by Result 5 and the small-amplitude structural and dynamical stability theorems of Pego and Métivier. Classification (either numerical or analytical) of large-amplitude stability, and the associated question of whether refined dynamical stability or multi-dimensional spectral stability in practice is the determining factor, remain outstanding open problems. In particular, large-amplitude multidimensional spectral stability has so far not been verified for a single physical example. (It is automatic in the scalar case, by the maximum principle [HoZ.3–4]. It has been verified in [PZ] for a class of artificial systems constructed for this purpose.)

5. Freistühler and Zumbrun [FreZ.2] have recently established for the overcompressive version of \( \Delta \) defined in (3.9)–(3.10), Theorem 3.5, a weak version of Métivier’s theorem analogous to that presented for the Lax case in Appendix B, as-

\textsuperscript{21}Stability is more delicate in the critical dimension \( d = 2 \) or the undercompressive case, and requires a refined analysis as discussed respectively in Remark 2, end of Section 5.2.3, and Section 5.6, [Z.3].
serting uniform low-frequency stability under the assumptions of Métivier plus the additional nondegeneracy condition of invertibility of an associated one-dimensional “mass map” introduced in [FreZ.1] taking potential time-asymptotic states to corresponding initial perturbation mass. This agrees with numerical and analytical evidence given in [Fre.3, FreL, L.2, FreP.1–2, DSL], in contrast to the result of automatic inviscid instability described in Remark 1.18. The corresponding investigation of small-amplitude spectral stability remains an important open problem in both the over- and undercompressive case.

Plan of the paper. In Section 2, we recall the general Evans function and inverse Laplace transform (\(C_0\)-semigroup) machinery needed for our analysis. In Section 3, we construct the Evans function and establish the key low-frequency limit described in Result 1. In Section 4, we carry out the one-dimensional viscous theory described in Results 2 and 3, and in Section 5 the multi-dimensional viscous theory described in Results 4 and 5.

2. PRELIMINARIES.

Consider a profile \(\bar{U}\) of a hyperbolic–parabolic system (1.1), satisfying assumptions (A1)–(A3), (H0)–(H3). By the change of coordinate frame \(x_1 \to x_1 - st\) if necessary, we may arrange that \(U \equiv \bar{U}(x_1)\) be a standing-wave solution, i.e., an equilibrium of (1.1). We make this normalization throughout the rest of the paper, fixing \(s = 0\) once and for all.

Linearizing (1.1) about the stationary solution \(\bar{U}(\cdot)\) gives

\[
U_t = LU := \sum_{j,k} (B^{jk}U_{x_k})_{x_j} - \sum_j (A^jU)_{x_j},
\]

where

\[
B^{jk} := B^{jk}(\bar{U}(x_1))
\]

and

\[
A^jU := dF^j(\bar{U}(x_1))U - dB^{j1}(\bar{u}(x_1))(U, \bar{U}_{x_1})
\]

are \(C^{s-1}\) functions of \(x_1\) alone. In particular,

\[
(A^1_{11}, A^1_{12}) = (dF^1_{11}, dF^1_{12}).
\]

Taking now the Fourier transform in transverse coordinates \(\bar{x} = (x_2, \cdots, x_d)\), we obtain

\[
\hat{U}_t = L_{\xi}\hat{U} := (B^{11}\hat{U})' - (A^1\hat{U})' - i\sum_{j \neq 1} A^j \xi_j \hat{U}
\]

\[
+ i \sum_{j \neq 1} B^{j1} \xi_j \hat{U}' + i \sum_{k \neq 1} (B^{1k}\xi_k \hat{U})' - \sum_{j,k \neq 1} B^{jk} \xi_j \xi_k \hat{U},
\]

or

\[
\hat{U}_t = L_{\xi}\hat{U} := (B^{11}\hat{U})' - (A^1\hat{U})' - i\sum_{j \neq 1} A^j \xi_j \hat{U}
\]

\[
+ i \sum_{j \neq 1} B^{j1} \xi_j \hat{U}' + i \sum_{k \neq 1} (B^{1k}\xi_k \hat{U})' - \sum_{j,k \neq 1} B^{jk} \xi_j \xi_k \hat{U},
\]
where "\( t \)" denotes \( \partial/\partial x_1 \), and \( \hat{U} = \hat{U}(x_1, \widehat{\xi}, t) \) denotes the Fourier transform of \( U = U(x_1, \tilde{x}, t) \). Hereafter, where there is no danger of confusion we shall drop the hats and write \( U \) for \( \hat{U} \). Operator \( L_\xi \) reduces at \( \xi = 0 \) to the linearized operator \( L_0 \) associated with the one-dimensional stability problem.

A necessary condition for stability is that the family of linear operators \( L_\xi \) have no unstable point spectrum, i.e. the eigenvalue equations

\[
(L_\xi - \lambda)U = 0
\]

have no solution \( U \in L^2(x_1) \) for \( \bar{\xi} \in \mathbb{R}^{d-1} \), \( \text{Re} \lambda > 0 \). For, unstable \( (L^p) \) point spectrum of \( L_\xi \) corresponds to unstable \( (L^p) \) essential spectrum of the operator \( L \) for \( p < \infty \), by a standard limiting argument (see e.g. [He, Z.1]), and unstable point spectrum for \( p = \infty \). This precludes \( L^p \to L^p \) stability, by the Hille–Yosida theorem (see, e.g. [He, Fr, Pa, Z.1]). Moreover, standard spectral continuity results [Kat, He, Z.1] yield that instability, if it occurs, occurs for a band of \( \bar{\xi} \) values, from which we may deduce by inverse Fourier transform the exponential instability of (2.1) for test function initial data \( U_0 \in C_0^\infty \), with respect to any \( L^p \), \( 1 \leq p \leq \infty \). We shall see later that linearized stability of constant solutions \( U \equiv U_\pm \) (embodied in (A1)–(A3)) together with convergence at \( x_1 \to \pm \infty \) of the coefficients of \( L \), implies that all spectrum of \( L_\xi \) is point spectrum on the unstable (open) complex half-plane \( \text{Re} \lambda > 0 \); thus, we are not losing any information by restricting to point spectrum in (2.6). Of course, condition (2.6) is not sufficient for even neutral, or bounded, linearized stability of (2.1).

2.1. Spectral resolution formulae.

We begin by deriving spectral resolution, or inverse Laplace Transform formulae

\[
e^{-L_\xi t} \phi = \frac{1}{2\pi i} P.V. \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} (L_\xi - \lambda)^{-1} \phi d\lambda,
\]

and

\[
G_\xi(x_1, t; y_1) = \frac{1}{2\pi i} P.V. \int_{\eta-i\infty}^{\eta+i\infty} e^{ikt} G_{\xi, \lambda}(x_1, y_1) d\lambda,
\]

for the solution operator \( e^{-L_\xi t} \) and Green distribution \( G_\xi(x_1, t; y_1) := e^{-L_\xi t} \delta_{y_1}(x_1) \) associated with the Fourier-transformed linearized evolution equations (2.5), where \( G_{\xi, \lambda}(x_1, y_1) \) denotes the resolvent kernel \( (L_\xi - \lambda)^{-1} \delta_{y_1}(x_1) \), and \( \eta \) is any real constant greater than or equal to some \( \eta_0(\bar{\xi}) \).

\footnote{In the strictly parabolic case \( \text{Re} \sigma \sum \xi_j \xi_k B^{jk} > 0 \), or for any asymptotically constant operator \( L \) of nondegenerate type, this follows by a standard argument of Henry; see [He, Z.1]. More generally, it holds whenever the eigenvalue equation may be expressed as a nondegenerate first-order ODE [MaZ.3].}
Lemma 2.1. Operator $L_{\xi}$ satisfies

\begin{equation}
|U|_{H^1} + |B^{11}U|_{H^2} \leq C(\tilde{\xi})(|L_{\tilde{\xi}}U|_{L^2} + |U|_{L^2}),
\end{equation}

for all $U \in \mathcal{H} := \{f : f \in H^1, B^{11}f \in H^2\}$, for some well-conditioned coordinate transformation $U \to S(x_1)U$, sup$_{x_1} |S||S^{-1}| \leq C(\tilde{\xi})$, and all real $\lambda$ greater than some value $\lambda_* = \lambda_*(\tilde{\xi})$.

\textbf{Proof.} Consider the resolvent equation $(L_{\xi} - \lambda)U = f$. By the lower triangular coordinate transformation $U \to S_1U$,

\begin{equation}
S_1 := \begin{pmatrix}
I & 0 \\
-b_1^{11}(b_2^{11})^{-1} & I
\end{pmatrix},
\end{equation}

we may convert to the case that $b_1^{11} \equiv 0$, with conjugation error $EU = E_1U_x + E_0U$,

\[
E_1 = \begin{pmatrix}
0 & 0 \\
\epsilon_1 & \epsilon_2
\end{pmatrix},
\]

where $b_2^{11}$ and $A_*^{11} := A_{11} - A_{12}b_1^{11}(b_2^{11})^{-1} = A_{11}$ retain their former values, and thus their former properties of strict parabolicity and hyperbolicity with constant multiplicity, respectively. By a further, block-diagonal transformation, we may arrange also that $b_2^{11}$ be strictly positive definite and $A_{11}$ symmetric; moreover, this may be chosen so that the combined transformation is uniformly well-conditioned.

Setting $\lambda = 0$ and taking the $L^2$ inner product of $(A_*^{11}u^I_x, b_2^{11}u^{II}_{xx})$ against the resulting equation $(L_{\xi} + E)U = f$, we obtain after some rearrangement the bound

\[
C^{-1}(|u^I_x|_{L^2} + |u^{II}_{xx}|_{L^2}) \leq |A_*^{11}u^I_x|_{L^2} + |b_2^{11}u^{II}_{xx}|_{L^2} \leq C(|U|_{L^2} + |u^{II}|_{H^2}),
\]

which in the original coordinates implies (2.9)(i). Note that we have so far only used nondegeneracy of the respective leading order coefficients $A_*^{11}$ and $b_2^{11}$ in the $u^I$ and $u^{II}$ equations.

Taking the real part of the $L^2$ inner product of $U$ against the full resolvent equation $(L_{\xi} + E - \lambda)U = f$, integrating by parts, using the assumptions on $A_{11}$, $b_2^{11}$, and bounding terms of order $|U|_{L^2}|u^{II}|_{L^2}$ by Youngs inequality as $O(|U|_{L^2}^2) + \varepsilon|u^{II}_{xx}|_{L^2}^2$ with $\varepsilon$ as small as needed, we obtain after some rearrangement the estimate

\begin{equation}
\text{Re} \lambda|U|^2_{L^2} + O(|U|_{L^2}^2) + \theta|u^{II}_{xx}|_{L^2}^2 \leq |f|_{L^2}|U|_{L^2},
\end{equation}

whence we obtain the second claimed bound

\begin{equation}
|U|_{L^2} \leq |\lambda - \lambda_*|^{-1}|f|_{L^2} = |\lambda - \lambda_*|^{-1}(|L_{\tilde{\xi}} - \lambda)|U|_{L^2}
\end{equation}

for all $\lambda > \lambda_*$ on the real axis, and $\lambda_*$ sufficiently large. □
Corollary 2.2. Operator $L_\xi$ is closed and densely defined on $L^2$ with domain $\mathcal{H}$, generating a $C^0$ semigroup $e^{L_\xi t}$ satisfying $|e^{L_\xi t}|_{L^2} \leq Ce^{\omega t}$ for some real $\omega = \omega(\xi)$.

Proof. The first assertion follows in standard fashion from bound (2.9)(i); see, e.g., [Pa] or [Z.1]. The bound (2.9)(ii) applies also to the limiting, constant-coefficient operators $L_{\xi \pm}$ as $x \to \pm \infty$, whence the spectra of these operators is confined to Re $\lambda \leq \lambda_*$. Because the eigenvalue equation $(L_{\xi} - \lambda)U = 0$, after the coordinate transformation described in the proof of Lemma 2.1 above, may evidently be written as a nondegenerate ODE in $(U, u^{H'})$, we may conclude by the general theory described in Section 2.4 below that the essential spectrum of $L_{\xi}$ is also confined to the set Re $\lambda \leq \lambda_*$; see also Proposition 4.4. of [MaZ.3]. (This observation generalizes a standard result of Henry ([He], Lemma 2, pp. 138–139) in the case of a nondegenerate operator $L$.) Since (2.9) precludes point spectrum for real $\lambda > \lambda_*$, we thus find that all such $\lambda$ belong to the resolvent set $\rho(L_{\xi})$, with the resolvent bound (after the coordinate transformation described above)

\[(2.13) \quad |(L_{\xi} - \lambda)^{-1}|_{L^2} \leq |\lambda - \lambda_*|^{-1}.\]

But, this is a standard sufficient condition that a closed, densely defined operator $L_{\xi}$ generate a $C^0$ semigroup, with $|e^{L_{\xi} t}|_{L^2} \leq Ce^{\omega t}$ for all real $\omega > \lambda_*$ (see e.g. [Pa], Theorem 5.3, or [Fr,Y]). □

Corollary 2.3. For $L_{\xi}$, $e^{L_{\xi} t}$ as in Corollary 2.2, the inverse Laplace Transform (spectral resolution) formula (2.7) holds for any real $\eta$ greater than some $\eta_0$, for all $\phi \in D(L_{L_{\xi}})$, where domain $D(L_{L_{\xi}})$ is defined as in [Pa], p. 1. Likewise, (2.8) holds in the distributional sense.\(^2\)

Proof. The first claim is a general property of $C^0$ semigroups (see, e.g. [Pa], Corollary 7.5). The second follows from the first upon pairing with test functions $\phi$ (see, e.g., [MaZ.3], Section 2). □

Remarks 2.4. 1. By standard semigroup properties, the Green distribution $G_\xi(x, t; y)$ satisfies

(i) $(\partial_t - L_\xi)G_\xi(\cdot, t; y_1) = 0$ in the distributional sense, for all $t > 0$, and

(ii) $G_\xi(x_1, t; y_1) \to \delta(x - y)$ as $t \to 0$.

Here, $G_\xi$ is to be interpreted in (i) as a distribution in the joint variables $(x_1, y_1, t)$, and in (ii) as a distribution in $(x_1, y_1)$, continuously parametrized by $t$. We shall see by explicit calculation that $G_\xi$ so defined (uniquely, by uniqueness of weak solutions of (2.5) within the class of test function initial data), is a measure but not

\(^2\)Note: From the definition in [Pa], we find that the domain $D(L_{\xi})$ of $L_{\xi}$ satisfies $H^2 \subset \mathcal{H} \subset D(L_{\xi})$. Likewise, the domain $D(L_{L_{\xi}})$, consisting of those functions $V$ such that $L_{\xi}V \in D(L_{\xi})$, satisfies $H^4 \subset D(L_{L_{\xi}}).$
a function. Note that, on the resolvent set $\rho(L)$, the resolvent kernel $G_{\xi,\lambda}$ likewise has an alternative, intrinsic characterization as the unique distribution satisfying

$$
(L_{\xi} - \lambda)G_{\xi,\lambda}(x, y) = \delta(x, y)
$$

and taking $f \in L^2$ to $\langle G_{\xi,\lambda}(x, y), f(y) \rangle_y \in L^2$. This is the characterization that we shall use in constructing $G_{\xi,\lambda}$ in Section 2.4.

2. A semigroup on even a still more restricted function class than $L^2$ would have sufficed in this construction, since we are constructing objects in the very weak class of distributions. Later, by explicit computation, we will verify that $L_{\xi}$ in fact generates a $C^0$ semigroup in any $L^p$. Note also that (2.8) implies the expected, standard solution formula $e^{L_{\xi}t} f = \langle G_{\xi}(x, t; y), f(y) \rangle_y$, or, formally:

$$
(2.15)
$$

$$
e^{L_{\xi}t} f = \int G_{\xi}(x, t; y) f(y) dy,
$$

for $f$ in any underlying Banach space on which $e^{L_{\xi}t}$ is defined, in this case on $L^p$, $1 \leq p \leq \infty$.

3. As noted in [MaZ.1, MaZ.3], there is another, more concrete route to (2.7)--(2.8) and (2.15), generalizing the approach followed in [ZH] for the parabolic case. Namely, we may observe that, in each finite integral in the approximating sequence defined by the principal value integral (2.8), we may exchange orders of integration and distributional differentiation, using Fubini’s Theorem together with the fact, established in the course of our analysis, that $G_{\xi,\lambda}$ is uniformly bounded on the contours under consideration (in fact, we establish the much stronger result that $G_{\xi,\lambda}$ decays exponentially in $|x - y|$, with uniform rate), to obtain

$$
(2.16)
$$

$$
(\partial_t - L_{\xi}) \int_{\eta - iK}^{\eta + iK} e^{\lambda t} G_{\xi,\lambda}(x, y) d\lambda = \delta(x - y) \int_{\eta - iK}^{\eta + iK} e^{\lambda t} d\lambda
$$

$$
\rightarrow \delta(x - y) \otimes \delta(t)
$$

as $K \to \infty$, for all $t \geq 0$. So, all that we must verify is: (i’) the Principal Value integral (2.8) in fact converges to some distribution $G_{\xi}(x, t; y)$ as $K \to \infty$; and, (ii’) $G_{\xi}(x, t; y)$ has a limit as $t \to 0$. For, distributional limits and derivatives commute (essentially by definition), so that (i’)–(ii’) together with (2.16) imply (i)–(ii) above. Facts (i’)–(ii’) will be established by direct calculation in the course of our analysis, thus verifying formula (2.8) at the same time that we use it to obtain estimates on $G_{\xi}$. Note that we have made no reference in this argument to the semigroup machinery cited above.
2.2. Evans function framework.

We next recall the basic Evans function/asymptotic ODE tools that we will need.

2.2.1. The gap/conjugation lemma. Consider a family of first-order systems

\begin{equation}
W' = \mathbb{A}(x, \lambda)W, \tag{2.17}
\end{equation}

where \( \lambda \) varies within some domain \( \Omega \subset \mathbb{C}^k \) and \( x \) varies within \( \mathbb{R}^1 \). Equations (2.17) are to be thought of as generalized eigenvalue equations, with parameter \( \lambda \) representing a suite of frequencies. We make the basic assumption:

(h0) Coefficient \( \mathbb{A}(\cdot, \lambda) \), considered as a function from \( \Omega \) into \( L^1(\mathbb{R}) \) is analytic in \( \lambda \). Moreover, \( \mathbb{A}(\cdot, \lambda) \) approaches exponentially to limits \( \mathbb{A}_\pm(\lambda) \) as \( x \to \pm\infty \), with uniform exponential decay estimates

\begin{equation}
|\frac{\partial}{\partial x}^k(\mathbb{A} - \mathbb{A}_\pm)| \leq C_1 e^{-\theta|x|/C_2}, \quad \text{for } x \geq 0, 0 \leq k \leq K,
\end{equation}

\( C_j, \theta > 0 \), on compact subsets of \( \Omega \).

Then, there holds the following conjugation lemma of [MéZ.1], a refinement of the “gap lemma” of [GZ,KS], relating solutions of the variable-coefficient ODE (2.17) to solutions of its constant-coefficient limiting equations as

\begin{equation}
Z' = \mathbb{A}_\pm(\lambda)Z \tag{2.19}
\end{equation}
as \( x \to \pm\infty \).

**Lemma 2.5 [MéZ.1].** Under assumption (h0) alone, there exists locally to any given \( \lambda_0 \in \Omega \) a pair of linear transformations \( P_+(x, \lambda) = I + \Theta_+(x, \lambda) \) and \( P_-(x, \lambda) = I + \Theta_-(x, \lambda) \) on \( x \geq 0 \) and \( x \leq 0 \), respectively, \( \Theta_\pm \) analytic in \( \lambda \) as functions from \( \Omega \) to \( L^\infty[0, \pm\infty) \), such that:

(i) \( |P_\pm| \) and their inverses are uniformly bounded, with

\begin{equation}
|\frac{\partial}{\partial \lambda}^j\left(\frac{\partial}{\partial x}^k\Theta_\pm\right)| \leq C(j)C_1 C_2 e^{-\theta|x|/C_2} \quad \text{for } x \geq 0, 0 \leq k \leq K + 1,
\end{equation}

\( j \geq 0 \), where \( 0 < \theta < 1 \) is an arbitrary fixed parameter, and \( C > 0 \) and the size of the neighborhood of definition depend only on \( \theta, j \), the modulus of the entries of \( \mathbb{A} \) at \( \lambda_0 \), and the modulus of continuity of \( \mathbb{A} \) on some neighborhood of \( \lambda_0 \in \Omega \).

(ii) The change of coordinates \( W := P_\pm Z \) reduces (2.17) on \( x \geq 0 \) and \( x \leq 0 \), respectively, to the asymptotic constant-coefficient equations (2.19).

Equivalently, solutions of (2.17) may be conveniently factorized as

\begin{equation}
W = (I + \Theta_\pm)Z_\pm, \tag{2.21}
\end{equation}

where \( Z_\pm \) are solutions of the constant-coefficient equations (2.19), and \( \Theta_\pm \) satisfy bounds (2.20).
Proof. As described in [MéZ.1], for \( j = k = 0 \) this is a straighforward corollary of the gap lemma as stated in [Z.3], applied to the “lifted” matrix-valued ODE for the conjugating matrices \( P_{\pm} \); see also Appendix A.3, below. The \( x \)-derivative bounds \( 0 < k \leq K + 1 \) then follow from the ODE and its first \( K \) derivatives. Finally, the \( \lambda \)-derivative bounds follow from standard interior estimates for analytic functions. □

**Definition 2.6.** Following [AGJ], we define the domain of consistent splitting for problem (2.17) as the (open) set of \( \lambda \) such that the limiting matrices \( A_{+} \) and \( A_{-} \) are hyperbolic (have no center subspace), and the dimensions of their stable (resp. unstable) subspaces \( S_{+} \) and \( S_{-} \) (resp. unstable subspaces \( U_{+} \) and \( U_{-} \)) agree.

**Lemma 2.7.** On any simply connected subset of the domain of consistent splitting, there exist analytic bases \( \{V_{1}, \ldots, V_{k}\}_{+} \) and \( \{V_{k+1}, \ldots, V_{N}\}_{+} \) for the subspaces \( S_{+} \) and \( U_{+} \) defined in Definition 2.6.

**Proof.** By spectral separation of \( U_{\pm}, S_{\pm} \), the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [Kat], pp. 99–102. □

By Lemma 2.5, on the domain of consistent splitting, the subspaces

\[
S^{+} = \text{Span}\{W_{1}^{+}, \ldots, W_{k}^{+}\} := \text{Span}\{P_{+}^{+}V_{1}^{+}, \ldots, P_{+}^{+}V_{k}^{+}\}
\]

and

\[
U^{-} := \text{Span}\{W_{k+1}^{-}, \ldots, W_{N}^{-}\} := \text{Span}\{P_{-}^{-}V_{k+1}^{-}, \ldots, P_{-}^{-}V_{N}^{-}\}
\]

uniquely determine the stable manifold as \( x \to +\infty \) and the unstable manifold as \( x \to -\infty \) of (2.17), defined as the manifolds of solutions decaying as \( x \to \pm \infty \), respectively, independent of the choice of \( P_{\pm} \).

**Definition 2.8.** On any simply connected subset of the domain of consistent splitting, let \( V_{1}^{+}, \ldots, V_{k}^{+} \) and \( V_{k+1}^{-}, \ldots, V_{N}^{-} \) be analytic bases for \( S_{+} \) and \( U_{-} \), as described in Lemma 2.7. Then, the Evans function for (2.17) associated with this choice of limiting bases is defined as

\[
D(\lambda) := \det \left( W_{1}^{+}, \ldots, W_{k}^{+}, W_{k+1}^{-}, \ldots, W_{N}^{-}\right)_{|x=0, \lambda} = \det \left( P_{+}^{+}V_{1}^{+}, \ldots, P_{+}^{+}V_{k}^{+}, P_{-}^{-}V_{k+1}^{-}, \ldots, P_{-}^{-}V_{N}^{-}\right)_{|x=0, \lambda},
\]

where \( P_{\pm} \) are the transformations described in Lemma 2.5.

**Remark 2.9.** Note that \( D \) is independent of the choice of \( P_{\pm} \); for, by uniqueness of stable/unstable manifolds, the exterior products (minors) \( P_{+}^{+}V_{1}^{+} \wedge \cdots \wedge P_{+}^{+}V_{k}^{+} \) and \( P_{-}^{-}V_{k+1}^{-} \wedge \cdots \wedge P_{-}^{-}V_{N}^{-} \) are uniquely determined by their behavior as \( x \to +\infty \), \( -\infty \), respectively.
Proposition 2.10. Both the Evans function and the stable/unstable subspaces $S^+$ and $U^-$ are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for $\lambda$ within this region, equation (2.17) admits a nontrivial solution $W \in L^2(x)$ if and only if $D(\lambda) = 0$.

Proof. Analyticity follows by uniqueness, and local analyticity of $P_\pm$, $V_\pm^k$. Noting that the first $k$ columns of the matrix on the righthand side of (2.24) are a basis for the stable manifold of (2.17) at $x \to +\infty$, while the final $N - k$ columns are a basis for the unstable manifold at $x \to -\infty$, we find that its determinant vanishes if and only if these manifolds have nontrivial intersection, and the second assertion follows.

Remarks 2.11. 1. In the case that (2.17) describes an eigenvalue equation associated with an ordinary differential operator $L$, $\lambda \in \mathbb{C}^1$, Proposition 2.10 implies that eigenvalues of $L$ agree in location with zeroes of $D$. In [GJ.1–2], Gardner and Jones [GJ.1–2] have shown that they agree also in multiplicity; see also Lemma 6.1, [ZH], or Proposition 6.15 of [MaZ.3].

2. If, further, $L$ is a real-valued operator (i.e., has real coefficients), or, more generally, $A$ has complex symmetry $A(x, \bar{x}) = \bar{A}(x, \bar{x})$, where bar denotes complex conjugate, then $D$ may be chosen with the same symmetry

\begin{equation}
D(\lambda) = D(\bar{\lambda}).
\end{equation}

For, all steps in the construction of the Evans function respect complex symmetry: projection onto stable/unstable subspaces by the spectral resolution formula

\begin{equation}
P_\pm(\lambda) = \frac{1}{2\pi i} \oint_{\Gamma_\pm} (A_\pm(\lambda) - \mu)^{-1} d\mu
\end{equation}

($\Gamma_\pm$ denoting fixed contours enclosing stable/unstable eigenvalues of $A_\pm$), choice of asymptotic bases $V_\pm(\lambda)$ by solution of the analytic ODE prescribed by Kato [Kat], and finally conjugation to variable-coefficient solutions by a contraction mapping that likewise respects complex symmetry.

2.2.2. The tracking/reduction lemma. Next, consider the complementary situation of a family of equations of form

\begin{equation}
W' = A^\epsilon(x, \lambda)W,
\end{equation}

on an $(\epsilon, \lambda)$-neighborhood for which the coefficient $A^\epsilon$ does not exhibit uniform exponential decay to its asymptotic limits, but instead is slowly varying. This occurs quite generally for rescaled eigenvalue equations arising in the study of the large-frequency regime; see, e.g., [GZ,ZH,MaZ.1,Z.3].

\[\text{[24]The latter result applies specifically to ordinary differential operators of degenerate type.}\]
In this situation, it frequently occurs that not only $A^\epsilon$ but also certain of its invariant (group) eigenspaces are slowly varying with $x$, i.e., there exist matrices

\begin{equation}
L^\epsilon = \begin{pmatrix} L_1^\epsilon \\ L_2^\epsilon \end{pmatrix} (x), \quad R^\epsilon = \begin{pmatrix} R_1^\epsilon & R_2^\epsilon \end{pmatrix} (x)
\end{equation}

for which $L^\epsilon R^\epsilon(x) \equiv I$ and $|LR^\epsilon| = |L'R|$ is small relative to

\begin{equation}
M^\epsilon := L^\epsilon A^\epsilon R^\epsilon(x) = \begin{pmatrix} M_1^\epsilon & 0 \\ 0 & M_2^\epsilon \end{pmatrix} (x),
\end{equation}

where “$\equiv$” as usual denotes $\partial/\partial x$. In this case, making the change of coordinates $W^\epsilon = R^\epsilon Z$, we may reduce (2.26) to the approximately block-diagonal equation

\begin{equation}
Z^\epsilon' = M^\epsilon Z^\epsilon + \delta^\epsilon \Theta^\epsilon Z^\epsilon,
\end{equation}

where $M^\epsilon$ is as in (2.28), $\Theta^\epsilon(x)$ are uniformly bounded matrices, and $\delta^\epsilon(x) \leq \delta(\epsilon)$ is a (relatively) small scalar.

Let us assume that such a procedure has been successfully carried out, and, moreover, that the approximate flows (i.e., solution operators) $\bar{F}^\epsilon_{j \to x}$ generated by the decoupled equations

\begin{equation}
Z^\epsilon = M^\epsilon_{j} Z^\epsilon
\end{equation}

are uniformly exponentially separated to order $2\eta(\epsilon)$, in the sense that

\begin{equation}
|\bar{F}^\epsilon_{1 \to x}||(\bar{F}^\epsilon_{2 \to x})^{-1}| \leq C e^{-2\eta(\epsilon)|x-y|} \quad \text{for } x \geq y.
\end{equation}

**Remark 2.12.** A sufficient condition for and the usual means of verification of (2.31) is existence of an approximate uniform spectral gap:

\begin{equation}
\max \sigma(\text{Re} (M_1^\epsilon)) - \min \sigma(\text{Re} (M_2^\epsilon)) \leq -2\eta(\epsilon) + \alpha^\epsilon(x)
\end{equation}

for all $x$, $\eta(\epsilon) > 0$, where $\alpha^\epsilon$ is uniformly integrable, $\int |\alpha^\epsilon(x)| dx \leq C$, or, alternatively, existence of flows conjugate to $\bar{F}^\epsilon_{j \to x}$ and satisfying this condition. Here, $\text{Re} (M) := (1/2)(M^* + M)$ denotes the symmetric part of of a matrix or linear operator $M$. Note that an exact uniform spectral gap (2.32) with $\alpha^\epsilon \equiv 0$ may be achieved from (2.32) by the coordinate change $Z_1^\epsilon \to \omega Z_1^\epsilon$, where $\omega$ is a well-conditioned (scalar) exponential weight defined by $\omega' = -\alpha^\epsilon \omega$, $\omega(0) = 1$. From the exact version, we may obtain (2.31) by standard energy estimates, with $C = 1$.

Then, there holds the following reduction lemma, a refinement established in [MaZ.1] of the “tracking lemma” given in varying degrees of generality in [GZ,ZH,Z.3].
Proposition 2.13 [MaZ.1]. Consider a system \((2.29)\) satisfying the exponential separation assumption \((2.31)\), with \(\theta^c\) uniformly bounded for \(\epsilon\) sufficiently small. If, for \(0 < \epsilon < \epsilon_0\), the ratio \(\delta(\epsilon)/\eta(\epsilon)\) of formal error vs. spectral gap is sufficiently small relative to the bounds on \(\Theta\), in particular if \(\delta(\epsilon)/\eta(\epsilon) \to 0\), then, for all \(0 < \epsilon \leq \epsilon_0\), there exist (unique) linear transformations \(\Phi^c_1(x, \lambda)\) and \(\Phi^c_2(x, \lambda)\), possessing the same regularity with respect to the parameters \(\epsilon\), \(\lambda\) as do coefficients \(M^c\) and \(\delta(\epsilon)\Theta^c\), for which the graphs \(\{(Z_1, \Phi^c_2 Z_1)\}\) and \(\{(\Phi^c_1 Z_2, Z_2)\}\) are invariant under the flow of \((2.29)\), and satisfying

\[
|\Phi^c_1|, |\partial_x \Phi^c_1| \leq C \delta(\epsilon)/\eta(\epsilon) \text{ for all } x.
\]

Proof. As described in Appendix C, [MaZ.1], this may be established by a contraction mapping argument carried out for the projectivized “lifted” equations governing the flow of exterior forms; see, e.g., [Sat] for a corresponding argument in the case that \(M_1, M_2\) are scalar. We give a more direct, “matrix-valued” version of this argument in Appendix A.3, below, from which one may obtain further an explicit (but rather complicated) Neumann series for \(\Phi^c_j\) in powers of \(\delta/\eta\). □

From Proposition 2.13, we obtain reduced flows

\[
Z^c_1' = M^c_1 Z^c_1 + \delta(\Theta_{11} + \Theta_{12} \Phi^c_2) Z^c_1 = M^c_1 Z^c_1 + O(\delta^c(x)) Z^c_1
\]

and

\[
Z^c_2' = M^c_2 Z^c_2 + \delta(\Theta_{22} + \Theta_{21} \Phi^c_1) Z^c_2 = M^c_2 Z^c_2 + O(\delta^c(x)) Z^c_2
\]

on the two invariant manifolds described. Let us focus on the flow \((2.34)\), assuming without loss of generality (by a centering exponential weighting if necessary) that

\[
|F^y_{1 \rightarrow x}| \leq C e^{-\eta(\epsilon)|x-y|} \text{ for } x \leq y.
\]

Corollary 2.14 [MaZ.1]. Assuming \((2.36)\), the flow \(F^y_{1 \rightarrow x}\) of the reduced equation \((2.34)\) satisfies

\[
F^y_{1 \rightarrow x} = \tilde{F}^y_{1 \rightarrow x} + \sum_{j=1}^{J} (\delta/\eta)^j E_j(x, y, \epsilon) + O(\delta/\eta)^{J+1} e^{-\tilde{\theta}|x-y|},
\]

for \(x \leq y\), for any \(0 < \tilde{\theta} < 1\), where \(F^y_{1 \rightarrow x}\) is the flow of the associated decoupled system \((2.30)\); the iterated integrals

\[
E_{j+1} := \eta \int_y^x \tilde{F}^z_{1 \rightarrow x}(\Theta_{11} + \Theta_{12} \Phi_2)(z, \epsilon) E_j(z, y, \epsilon) dz, \quad E_0(x, y, \epsilon) := \tilde{F}^y_{1 \rightarrow x},
\]
satisfy the uniform exponential decay estimates

\[(2.39) \quad |E_j(x, y, \epsilon)| \leq C_j e^{-\delta_y|x-y|},\]

and are as smooth in \(x\) and \(y\) and as smooth in \(\epsilon\) as is \((\Theta_{11} + \Theta_{12} \Phi_2)\); and \(C_j > 0\) and \(O(\cdot)\) are uniform in \(\epsilon\), depending only on \(\bar{\theta}, M_j,\) and the constant \(C\) in (2.33). Symmetric bounds hold for the flow \(F_\epsilon^{y-x}\) of (2.35).

**Proof.** This follows by a contraction mapping argument similar to (but simpler than) that used in the proof of Proposition 2.13, where expansion (2.37) is just the \(J\)th-order Neumann expansion with remainder associated with the contraction mapping; see Appendix C of [MaZ.1], or Appendix A.3 of this paper. \(\Box\)

### 2.2.3. Formal block-diagonalization.

To complete our discussion of ODE with slowly varying coefficients, we now supply a formal block-diagonalization procedure to be used in tandem with the rigorous error bounds of Corollary 2.14. Consider (2.17) with coefficients of the form

\[(2.40) \quad A^\epsilon(x, \lambda) = A(\varepsilon x, \varepsilon \lambda/|\lambda|),\]

\(\epsilon\) small, \(\lambda/|\lambda|\) (and possibly other indexing parameters) restricted to a compact set depending on \(\epsilon\), where \(A\) has formal Taylor expansion

\[(2.41) \quad A(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k A_k(y) + O(\varepsilon^{p+1}).\]

Assume:

1. \((h0)\) \(A\) decays uniformly exponentially as \(y \to \pm \infty\) to limits \(A_{\pm}\).
2. \((h1)\) \(A_j \in C^{p+1-j}(y)\) for \(0 \leq j \leq p\), with derivatives uniformly bounded for all \(y \in \mathbb{R}\), \(\Theta(\cdot) \in C^0\) and uniformly bounded for \(y \in \mathbb{R}, \varepsilon \leq \varepsilon_0, \text{ some } \varepsilon_0 > 0\).
3. \((h2)\) \(A_0(y)\) is block-diagonalizable to form

\[(2.42) \quad D_0(y) = T_0 A_0 T_0^{-1}(y) = \text{diag}\{d_{0,1}, \ldots, d_{0,s}\}, \quad d_j \in \mathbb{C}^{n_j \times n_j},\]

with spectral separation between blocks (i.e., complex distance between their eigenvalues) uniformly bounded below by some \(\gamma > 0, \text{ for } \varepsilon \leq \varepsilon_0, y \in \mathbb{R}\).

Then, we have:

**Proposition 2.15** [MaZ.1]. *Given \((h1)-(h2),\) there exists a uniformly well-conditioned change of coordinates \(W = TW\) such that*

\[(2.43) \quad \tilde{W}' = D(\varepsilon x, \varepsilon) \tilde{W} + O(\varepsilon^{p+1}) \tilde{W}\]

uniformly for \(x \in \mathbb{R}, \varepsilon \leq \varepsilon_0,\) with

\[(2.44) \quad D(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k D_k(y) + O(\varepsilon^{p+1}).\]
and

\[ T(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k T_k(y) \]

where \( D_j, T_j \in C^{p+1-j}(y) \) with uniformly bounded derivatives for all \( 0 \leq j \leq p \), \( O(\varepsilon^{p+1}) \in C^0(y) \) uniformly bounded, and each \( D_j \) has the same block-diagonal form

\[ D_j(y) = \text{diag}\{d_{j,1}, \ldots, d_{j,s}\}, \]

\( d_{j,k} \in \mathbb{C}^{n_k \times n_k} \), as does \( D_0 \).

If there holds (h0) as well, then it is possible to choose \( T_0(\cdot) \) in such a way that, also, \( D_0 + \varepsilon D_1 \) is determined simply by the block-diagonal part of \( T_0^{-1}(A_0 + \varepsilon A_1)T_0 \), or, equivalently,

\[ T_0^{-1}(\partial/\partial y)T_0 \equiv 0. \]

Proof. See Propositions 4.4 and 4.8 of [MaZ.1], or Appendix A of this paper. Note that our slightly weakened version of (h1) is what is actually used in the proof, and not the stronger version given in [MaZ.1]. This distinction was unimportant in the relaxation case, but will be important here.

Remarks 2.16. 1. An intuitive way to see Proposition 2.15 is to express \( T \) as a product \( T_0 T_1 \cdots T_p \) and solve the resulting succession of nearby diagonalization problems; a more efficient procedure is given in [MaZ.1]. From this point of view, it is clear that one derivative in \( y \) must be given up for each power of \( \varepsilon \), since each transformation \( T_j \) introduces a diagonalization error of form \( T_j(\partial/\partial y)T_j^{-1} \); this explains also the origin of condition (2.47). The class of perturbation series (h1) is preserved under slowly varying changes of coordinates, hence quite natural for the problem at hand. In particular, the series \( D \) obtained by diagonalization is in the same class, and so the procedure is convenient for iteration. If desired, \( W^{3, \infty}(y) \) may be substituted for \( C^2(y) \) everywhere.

2. We call attention to the important statement in Proposition 2.15 that any additional regularity with respect to \( \varepsilon \) in the error term \( O(\varepsilon^{p+1}) \) in (2.41) translates to the same regularity \( C^q(\varepsilon \to C^0(y)) \) for the error terms \( O(\varepsilon^{p+1}) \) in (2.44)-(2.45). In particular, if the remainder term is \( C^\omega(\varepsilon \to C^0(y)) \), as often happens (for example, for both relaxation and real viscous profiles), then the error term has regularity \( C^\omega(\varepsilon \to C^0(y)) \). This observation is used in conjunction with Proposition 2.13 and Corollary 2.14 above; specifically, it gives regularity with respect to \( \varepsilon \) in error terms \( \Theta_{jk} \), and the resulting graphs \( \Phi_j \), and thus in terms \( \mathcal{F}^{u-y} \), \( \mathcal{E}_j \), and \( O(\delta/\eta)^{j+1} e^{-\theta|x-y|} \) of expansion (2.37). Likewise, we point out, in the typical case that (2.40) is obtained by rescaling, that analyticity in \( \lambda \) in the original (unrescaled) coordinates, if it holds, may also preserved by appropriate choice of \( T_0 \), even though
we have expressed $A$ in terms of the nonanalytic parameter $\lambda/|\lambda|$, and this property also is inherited in the fixed-point constructions of Proposition 2.13 and Corollary 2.14; for details, see Remarks 4.11–4.12, [MaZ.1].

2.2.4. Splitting of a block-Jordan block. Augmenting the above discussion, we briefly consider the complementary case of an expansion (2.41) for which (h0)–(h2) hold, but (h3) is replaced by

(h3')(i) $A_0$ is a standard block-Jordan block of order $s$,

\begin{equation}
A_0(y) = J_{s,q} + \mu(y)I_{sq}, \quad J_{s,q} := \begin{pmatrix}
0 & I_q & \cdots & 0 \\
0 & 0 & I_q & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & I_q \\
0 & \cdots & \cdots & 0
\end{pmatrix};
\end{equation}

and (ii) for each $y$, the spectrum of the lower lefthand block $M := (A_1)_{s1}$ of $A_1$ lies in a compact set on which $z \to z^{1/s}$ is analytic; in particular, $\det(M) \neq 0$.

Such an expansion typically arises as a single block $d_j(y, \epsilon) := \sum_{k=0}^{p} \epsilon^k d_{k,j}(y)$ of the matrix $D(y, \epsilon)$ obtained by the block-diagonalization procedure just described. Indeed, the situation may be expected to occur in general for operators of degenerate (mixed) type, indicating the simultaneous presence of multiple scales. (Clearly, still more degenerate situations may occur, and could be treated on a case-by-case basis.)

Remark 2.17. We have assumed that $A_0$ has already been put in form (2.48) because (h3')(ii) is not preserved under general coordinate changes, due to dynamical effects $T_1 \partial y T_0$. On the other hand a calculation similar to that of Proposition 4.8, [MaZ.1] shows that $(T_0^{-1}(\partial y)T_0)_{s1} = 0$ for transformations preserving the block form (2.48) (namely, block upper-triangular matrices with diagonal blocks equal to some common invertible $\alpha$), and so (h3')(ii) is independent of the manner in which (2.48) is achieved from (2.41).

Performing the “balancing” transformation $A \to B^{-1}AB$,

\begin{equation}
B := \text{diag}\{1, \epsilon^{1/s}, \ldots, \epsilon^{(s-1)/s}\},
\end{equation}

we may convert this to an expansion

\begin{equation}
A(y, \epsilon) = \mu I + \sum_{k=1}^{p} \epsilon^{k/s}A_k(y) + O(\epsilon^{(p+1)/s})
\end{equation}

in powers of $\epsilon^{1/s}$, $A_j \in C^{p+1-j}$, $O(\epsilon^{(p+1)/s}) \in C^0$, where

\begin{equation}
A_1 = \begin{pmatrix}
0 & I_q & \cdots & 0 \\
0 & 0 & I_q & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & I_q \\
M & \cdots & \cdots & 0
\end{pmatrix}.
\end{equation}
and $M$ satisfies the spectral criterion $(h3')(ii)$.

Then, the spectrum $(\sigma M)^{1/s}$ of $A_1$ separates into $s$ groups of $q$, corresponding to the $s$ different analytic representatives of $z^{1/s}$, and so $A_1$ is $(q \times q)$ block-diagonalizable, reducing to the previous case with $\epsilon$ replaced by $\epsilon^{1/s}$ and $A_0$ shifted to $A_1$. This corresponds to the identification of a new, slow scale $\epsilon^{1/s}$ in the problem. Note that differentiation still brings down a full factor of $\epsilon << \epsilon^{1/s}$, so that the first step in the diagonalization process can still be carried out, despite the shift in the series. Alternatively, we could remove the $\mu I$ term by an exponential weighting, and rescale $y$ to remove the initial factor $\epsilon^{1/s}$, thus converting to the standard situation described in Section 2.2.3.

2.2.5. Approximation of stable/unstable manifolds. Proposition 2.15 and Corollary 2.14 together give a recipe for the estimation of the stable/unstable manifolds of ODE with slowly varying coefficients, given (h0)--(h2): namely, expand formally to order $\delta = \epsilon^p$ such that $\delta/\eta \to 0$, then apply Corollary 2.14 to obtain a rigorous approximation in terms of the resulting block-decoupled system. In the case relevant to traveling waves that all $A_j$ (not just $A_0$) converge exponentially in $y$ to constants $A_j^\pm$ as $y \to \pm\infty$, we have, provided that (i) $d_{0,j}$ are scalar multiples of the identity; and (ii) there holds the necessary condition of neutral separation to zeroth order, $\Re \sigma(d_{0,j}) \leq 0 \leq \Re \sigma(d_{0,k})$, $j \leq S < k$, for all $y$; that this procedure is possible if and only if the corresponding eigenspaces of the original matrix $A^\epsilon$, or equivalently of $\epsilon^p D_p$, have a uniform spectral gap of order $\epsilon^p$ at the limits $x = \pm\infty$, i.e., there holds the easily checkable condition

\begin{equation}
\text{(2.52)}
\Re \sigma(d_{p,j}^\pm) \leq -\theta < 0 < \theta \leq \Re \sigma(d_{p,k}^\pm), \quad j \leq S < k,
\end{equation}

for some uniform $\theta > 0$. For, this condition is clearly necessary by Lemma 2.5. On the other hand, suppose that the condition holds. The diagonalizing construction described in Proposition 2.15 preserves the property of exponential decay to constant states. Thus, making a change of coordinates such that

\[
\Re d_{p,j}^\pm \leq -\theta/2 < 0 < \theta/2 \leq \Re d_{p,k}^\pm, \quad j \leq S < k,
\]

we obtain by zeroth order neutrality (preserved, by assumption (i)) plus exponential decay, that (2.32) holds with integrable error $\alpha(x) = O(\epsilon e^{-\theta e|x|})$, verifying (2.31).

We remark that $d_{0,j}$ scalar is equivalent to constant multiplicity of the associated eigenvalue of $A_0$. For neutrally zeroth-order stable (resp. unstable) blocks, this is a standard structural assumption, corresponding in the critical regime $\lambda/|\lambda|$ pure imaginary to local well-posedness of the underlying evolution equations. For strictly zeroth-order stable (resp. unstable) blocks, diagonalizability is not necessary either for our construction or for local well-posedness.

Likewise, uniform spectral gap at $\pm\infty$ for a set of problems $\epsilon$, $(\lambda/|\lambda|)(\epsilon)$ is roughly equivalent to asymptotic direction $\lim(\lambda/|\lambda|)$ pointing into the domain of consistent splitting as $|\lambda| \to \infty$, or equivalently $\epsilon \to 0$. Thus, the requirement of exponential
separation is in practice no restriction for applications to stability of traveling waves, the domain of feasibility lying in general asymptotic to the domain of consistent splitting. The only real requirement is sufficient regularity in the coefficients of the linearized equations to carry out the formal diagonalization procedure to the necessary order $p$. Whether or not our regularity requirement is sharp is an interesting technical question, to which we do not know the answer. We point out only that the regularity required in the sectorial case is just $C^{0+\alpha}$, as is correct; see [ZH], or Remark 5.4, [MaZ.3].

2.3. Hyperbolic–parabolic smoothing.

We now recall some important ideas of Kawashima et al concerning the smoothing effects of hyperbolic–parabolic coupling. The following results assert that hyperbolic effects can compensate for degenerate viscosity $B$, as revealed by the existence of a compensating matrix $K$.

**Lemma 2.18 ([KSh]).** Assuming $A^0$, $A$, $B$ symmetric, $A^0 > 0$, and $B \geq 0$, the genuine coupling condition

$$(GC) \quad \text{No eigenvector of } A \text{ lies in } \ker B$$

is equivalent to either of:

$$(K1) \quad \text{There exists a smooth skew-symmetric matrix function } K(A^0,A,B) \text{ such that}$$

$$
(2.53) \quad \text{Re } (K(A^0)^{-1}A + B)(U) > 0.
$$

$$(K2) \quad \text{For some } \theta > 0, \text{ there holds}$$

$$
(2.54) \quad \text{Re } \sigma(-i\xi(A^0)^{-1}A - |\xi|^2(A^0)^{-1}B) \leq -\theta|\xi|^2/(1 + |\xi|^2),
$$

for all $\xi \in \mathbb{R}$.

**Proof.** These and other useful equivalent formulations are established in [KSh]; see also [MaZ.4, Z.4].

**Corollary 2.19.** Under (A1)–(A3), there holds the uniform dissipativity condition

$$
(2.55) \quad \text{Re } \sigma \left( \sum_j i\xi_j A^j - \sum_{j,k} \xi_j \xi_k B^{jk} \right)_{\pm} \leq -\theta|\xi|^2/(1 + |\xi|^2), \quad \theta > 0.
$$

Moreover, there exist smooth skew-symmetric “compensating matrices” $K_{\pm}(\xi)$, homogeneous degree one in $\xi$, such that

$$
(2.56) \quad \text{Re } \left( \sum_{j,k} \xi_j \xi_k \tilde{B}^{jk} - K(\xi)(\tilde{A}^0)^{-1} \sum_k \xi_k \tilde{A}^k \right)_{\pm} \geq \theta > 0
$$
for all \( \xi \in \mathbb{R}^d \setminus \{0\} \).

**Proof.** By the block-diagonal structure of \( \tilde{B}^{jk} \), (GC) holds also for \( A_j^{\pm} \) and \( \hat{B}^{jk} := (\hat{A}^0)^{-1} \text{Re} \, \hat{B}^{jk} \), since

\[
\ker \sum_{j,k} \xi_j \xi_k \hat{B}^{jk} = \ker \sum_{j,k} \xi_j \xi_k \text{Re} \, \hat{B}^{jk} = \ker \sum \xi_j \xi_k B^{jk}.
\]

Applying Lemma 2.18 to \( \tilde{A}_0 := \tilde{A}_0^{\pm} \), \( \hat{A} := (\hat{A}^0)^{-1} \sum_k \xi_k \hat{A}^k \), \( \hat{B}^{jk} := (\hat{A}^0)^{-1} \sum_{j,k} \xi_j \xi_k \text{Re} \, \hat{B}^{jk} \), we thus obtain (2.56) and

\[
(2.57) \quad \text{Re} \, \sigma[(\hat{A}^0)^{-1}(-\sum_j i \xi_j \hat{A}^j - \sum_{j,k} \xi_j \xi_k \text{Re} \, \hat{B}^{jk})]_{\pm} \leq -\theta_1 |\xi|^2/(1 + |\xi|^2),
\]

\( \theta_1 > 0 \), from which we readily obtain

\[
(2.58) \quad (-\sum_j i \xi_j \hat{A}^j - \sum_{j,k} \xi_j \xi_k \hat{B}^{jk})_{\pm} \leq -\theta_2 |\xi|^2/(1 + |\xi|^2).
\]

Observing that \( M > \theta_1 \iff (\hat{A}^0)^{-1/2} M (\hat{A}^0)^{-1/2} > \theta \) and \( \sigma(\hat{A}^0)^{-1/2} M (\hat{A}^0)^{-1/2} > \theta \iff \sigma(\hat{A}^0)^{-1} M > \theta \), together with \( S > \theta \iff \sigma S > \theta \) for \( S \) symmetric, we obtain (2.55) from (2.58). Because all terms other than \( K \) in the lefthand side of (2.56) are homogeneous, it is evident that we may choose \( K(\cdot) \) homogeneous as well (restrict to the unit sphere, then take homogeneous extension). \( \Box \)

**Remark 2.20** [GMWZ.4]. In the special case \( A^0 = I \), \( B = \text{block-diag} \{0, b\} \), and \( \text{Re} \, b > 0 \), (GC) is equivalent to the condition that no eigenvector of \( A_{11} \) lie in the kernel of \( A_{21} \). If, also, \( A_{11} \) is a scalar multiple of the identity, therefore, \( \text{Re} \, A_{12} A_{21} > 0 \). Thus, on any compact set of such \( A, B \) satisfying (GC), \( K(I, A, B) \) may be taken as a linear function

\[
(2.59) \quad K(A) = \theta \begin{pmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{pmatrix}
\]

of \( A \) alone, for \( \theta > 0 \) sufficiently small. This implies, in particular, for systems (1.1) that can be simultaneously symmetrized, and for which each \( A_j^k \) is a scalar multiple of the identity, there is a linear, or “differential”, choice \( K(\sum_j \xi_j A^j) = \sum_j \xi_j K^j \) such that \( \sum_{jk}(K^j A^k + B^{jk})\xi_j \xi_k \geq 0 \) on any compact set of \( U \). In particular, this holds for the standard Navier–Stokes equations of gas dynamics and MHD, as pointed out in [Kaw]; more generally, it holds for all of the variants described in Appendix A.1: that is, for all examples considered in this article.
2.4. Construction of the resolvent kernel.

We conclude these preliminaries by deriving explicit representation formulae for the resolvent kernel \( G_{\xi}(x, y) \) using a variant of the classical construction (see e.g. [CH], pp. 371–376) of the Green distribution of an ordinary differential operator in terms of decaying solutions of the homogeneous eigenvalue equation \((L_{\xi} - \lambda)U = 0\), or

\[
\begin{split}
L_0 U &= (B^{11} U')' - (A^1 U)' - i \sum_{j \neq 1} A^j \xi_j U + i \sum_{j \neq 1} B^{j 1} \xi_j U' \\
&+ i \sum_{k \neq 1} (B^{1 k} \xi_k U)' - \sum_{j, k \neq 1} B^{j k} \xi_j \xi_k U - \lambda U = 0,
\end{split}
\]

matched across the singularity \( x = y \) by appropriate jump conditions, in the process obtaining standard decay estimates on the resolvent kernel (see (2.85), Proposition 2.23, below) suitable for analysis of intermediate frequencies \( \lambda \). Here, \( A^j \) and \( B^{j k} \) are as defined as in (2.1)–(2.4), \( U \in \mathbb{R}^n \), and ‘ as usual denotes \( d/dx_1 \). Our treatment follows the general approach introduced in [MaZ.3] to treat the case of an ordinary differential operator, such as \( L \) above, that is of degenerate type, i.e., the coefficient of the highest-order derivative is singular, but for which the eigenvalue equation can nonetheless be written as a nondegenerate first-order ODE (2.17) in an appropriate reduced phase space: in this case,

\[
W' = A(x_1, \xi, \lambda)W
\]

with \( W = (U, b_1^{11} u' + b_2^{11} u''') \in \mathbb{C}^{n+r} \). For related analyses in the nondegenerate case, see, e.g., [AGJ, K.1–2, ZH, MaZ.1].

2.4.1. Domain of consistent splitting. Define

\[
\Lambda^{\tilde{\xi}} := \cap_{j=1}^n \Lambda_{\tilde{\xi}}^{\pm}(\tilde{\xi})
\]

where \( \Lambda_{\tilde{\xi}}^{\pm}(\tilde{\xi}) \) denote the open sets bounded on the left by the algebraic curves \( \lambda_{\tilde{\xi}}^{\pm}(\xi_1, \tilde{\xi}) \) determined by the eigenvalues of the symbols \(-\xi^2 B_{\pm} - i \xi A_{\pm}\) of the limiting constant-coefficient operators

\[
L_{\pm} w := B_{\pm} w''' - A_{\pm} w'
\]

as \( \xi_1 \) is varied along the real axis, with \( \tilde{\xi} \) held fixed. The curves \( \lambda_{\tilde{\xi}}^{\pm}(\cdot, \tilde{\xi}) \) comprise the essential spectrum of operators \( L_{\xi_{\pm}} \).
The existence of a center manifold thus corresponds with existence of solutions

\( (2.67) \)

characteristic equation

\( \lambda \) properties do not hold together on the boundary of

\( (2.68) \)

set

\( \{ \lambda : \Re \lambda > -\eta(|\Im \lambda|^2 + |\bar{\xi}|^2)/(1 + |\Im \lambda|^2 + |\bar{\xi}|^2) \} \), \( \eta > 0 \).

**Proof.** We do not need to explicitly calculate the matrix \( \Lambda \in \mathbb{C}^{(n+r) \times (n+r)} \) that results, but only to point out that its existence follows by invertibility of \( A^1 \) and \( b^1 \), (H1), once we rewrite \( (2.60) \) as the lower triangular system of equations

\( (2.65) \)

\[ u' = (A^1)^{-1}(-A^1_{12}(b^1_{11})^{-1}z - (A^1_{11} + \lambda)u - A^1_{12}v^1) - i \sum_{j \neq 1} (A^j_{11}u + A^j_{12}v^1), \]

\[ v' = (b^1_{21})^{-1}z - (b^1_{21})^{-1}b^1_{11}u', \]

\[ z' = (A^1_{21} - A^1_{22}(b^1_{21})^{-1}b^1_{11})u^1 + A^1_{21}(b^1_{21})^{-1}z + A^1_{21}u^1 + (A^1_{22} + \lambda)v^1 + \ldots, \]

with \( (u, v, z) := (u', u^{I1}, b^1_{11}u' + b^1_{21}v') \). (We will need to carry out explicit computations only for our treatment in Section 4.3.2 of one-dimensional high-frequency bounds.) Note, as follows from the second equation, the important fact that every solution of \( (2.65) \) indeed corresponds to a solution \( U = (u, v) \) of \( (2.60) \), with

\( z = b^1_{11}u' + b^1_{21}v' \).

The final assertion, bound \( (2.64) \), follows easily from the bound \( (2.55) \) on the dispersion curves \( \lambda^\pm_j(\xi) \). To establish the second assertion, we must show that, on the set \( \Lambda \), the limiting eigenvalue equations

\( (2.66) \)

\[(L_{\xi^\pm} - \lambda)w = 0 \]

have no center manifold and the stable manifold at \(+\infty\) and the unstable manifold at \(-\infty\) have dimensions summing to the full dimension \( n \); moreover, that these properties do not hold together on the boundary of \( \Lambda^\xi \).

The fundamental modes of \( (2.66) \) are of form \( e^{\mu z}V \), where \( \mu, V \) satisfy the characteristic equation

\[ \left[ \begin{array}{c} \mu^2 B^1_{11} + \mu(-A^1_{11} + i \sum_{j \neq 1} B^1_{11} \xi_j + i \sum_{k \neq 1} B^1_{jk} \xi_k) \\ -(i \sum_{j \neq 1} A^j_{11} \xi_j + \sum_{j \neq 1} B^j_{1k} \xi_j \xi_k + \lambda I) \end{array} \right] V = 0. \]  

The existence of a center manifold thus corresponds with existence of solutions \( \mu = i\xi_1, V \) of \( (2.67) \), \( \xi_1 \) real, i.e., solutions of the dispersion relation

\[ (- \sum_{j, k} B^1_{1k} \xi_j \xi_k - i \sum_{j} A^1_{1j} \xi_j - \lambda I)V = 0. \]
But, $\lambda \in \sigma(-B^\xi - iA^\xi_\pm)$ implies, by definition (2.62), that $\lambda$ lies outside of $\Lambda^\xi$, establishing nonexistence of a center manifold. Moreover, it is clear from the same argument that a center manifold does exist on the boundary of $\Lambda$, since this corresponds to existence of pure imaginary eigenmodes.

Finally, nonexistence of a center manifold, together with connectivity of $\Lambda$, implies that the dimensions of stable/unstable manifolds at $+\infty/-\infty$ are constant on $\Lambda$. Taking $\lambda \to +\infty$ along the real axis, with $\xi \equiv 0$, we find that these dimensions sum to the full dimension $n + r$ as claimed. For, Fourier expansion about $\xi_1 = \infty$ of the one-dimensional $(\tilde{\xi} = 0)$ dispersion relation (see Appendix A.4, below) yields $n - r$ “hyperbolic” modes

$$\lambda_j = -i\xi_1 a^*_j + \ldots, \quad j = 1, \ldots, n - r,$$

where $a^*_j$ denote the eigenvalues of $A^1_\pm$, and $r$ “parabolic” modes

$$\lambda_{n-r+j} = -b_j \xi_1^2 + \ldots, \quad j = 1, \ldots, r,$$

where $b_j$ denote the eigenvalues of $b^{11}_\pm$; here, we have suppressed the $\pm$ indices for readability. Inverting these relationships to solve for $\mu := i\xi_1$, we find, for $\lambda \to \infty$, that there are $n - r$ hyperbolic roots $\mu_j \sim -\lambda/a^*_j$, and $2r$ parabolic roots $\mu^\pm_{n-r+j} \sim \pm \sqrt{\lambda/b_j}$. By assumption $(\tilde{H}1)(iii)$, the former yield a fixed number $k/(n - r - k)$ of stable/unstable roots, independent of $x_1$, and thus of $\pm$. Likewise, $(\tilde{H}1)(i)$ implies that the latter yields $r$ stable, $r$ unstable roots. Combining, we find the desired consistent splitting, with $(k + r)/(n - k)$ stable/unstable roots at both $\pm \infty$. □

A standard fact, for asymptotically constant-coefficient ordinary differential operators (see, e.g. [He], Lemma 2, pp. 138–139) is that components of the domain of consistent splitting agree with components of the complement of the essential spectrum of the variable-coefficient operator $L$. Thus, $\Lambda$ is a maximal domain in the essential spectrum complement, consisting of the component containing real, plus infinity. Moreover, provided that the coefficients of $L$ approach their limits at integrable rate, it can be shown (see, e.g. [AGJ,GZ,ZH,MaZ.1]) that each connected component consists either entirely of eigenvalues, or else entirely of normal points, defined as resolvent points or isolated eigenvalues of finite multiplicity. The latter fact will be seen directly in the course of our construction.

### 2.4.2. Basic construction.

We now carry out the resolvent construction at points $\lambda$ in $\Lambda$, more generally, at any point in the domain of consistent splitting, by the method of [MaZ.3]. Our starting point is a duality relation between solutions $U$ of the eigenvalue equation and solutions $\tilde{U}$ of the adjoint eigenvalue equation, which may be obtained by the observation that $E$ defined by

$$(d/dx_1)E := \langle \tilde{U}, (L_\xi - \lambda)U \rangle - \langle (L_\xi^* - \bar{\lambda})\tilde{U}, U \rangle = \langle \tilde{U}, L_\xi U \rangle - \langle L_\xi^* \tilde{U}, U \rangle$$
is a conserved quantity whenever \( U \) and \( \tilde{U} \) are solutions of the eigenvalue and adjoint eigenvalue equations, respectively.

Since \( \langle \hat{U}, L_{\xi} \hat{U} \rangle - \langle L_{\xi} \hat{U}, U \rangle \) is a perfect derivative involving derivatives of \( U, \hat{U} \) of order strictly less than the maximum order of \( L_{\xi} \), the quantity \( E \) may always be expressed as a quadratic form in phase space, in this case

\[
(2.69) \quad E(x_1) = \langle \hat{U}, B^{11} U' \rangle + \langle \hat{U}, (-A^1 + iB^{1\xi} + iB^{1\xi_1})U \rangle + \langle U', -B^{11} U \rangle,
\]

or

\[
(2.70) \quad E(x_1) = \left\langle \begin{pmatrix} \hat{U} \\ U' \end{pmatrix}, S^{\xi} \begin{pmatrix} U \\ U' \end{pmatrix} \right\rangle(x_1),
\]

where

\[
(2.71) \quad S^{\xi} := \begin{pmatrix} -A^1 + iB^{1\xi} + iB^{1\xi_1} B^{11} \\ -B^{11} \end{pmatrix}.
\]

In general, we obtain a cross-banded matrix \( S \), with elements of the \( j \)th band consisting of alternating \( \pm \) multiples of the \( j \)th coefficient of the operator \( L_{\xi} \), the anti-diagonal being filled with multiples of the principal coefficient, and the bands below the anti-diagonal being filled with zero blocks. Thus, for nondegenerate operators, it is immediate that \( S \) is invertible, and we obtain the characterization [ZH, MaZ.1] of solutions of the eigenvalue equation as those functions \( U \) with the property that \( E(x_1) \) is preserved for any solutions \( \tilde{U} \) of the adjoint equation.

For degenerate operators, \( S \) is of course not invertible, and we must work instead in a reduced phase space. In the present case, we may rewrite (2.69) as

\[
(2.72) \quad E(x_1) = \left\langle \begin{pmatrix} \hat{U} \\ Z \end{pmatrix}, S^{\xi} \begin{pmatrix} U \\ Z \end{pmatrix} \right\rangle,
\]

where

\[
(2.73) \quad S^{\xi} := \begin{pmatrix} -A^1 + iB^{1\xi} + iB^{1\xi_1} & 0 \\ -(b^{11}_2)^{-1}b^{11}_I & -I_r \end{pmatrix}
\]

and

\[
(2.74) \quad Z := (b^{11}_1, b^{11}_2)U', \quad \tilde{Z} := (0, b^{11}_2)\hat{U}'.
\]

Applying an elementary column operation (subtracting from the first column the second column times \( (b^{11}_2)^{-1}b^{11}_I \)), we find that

\[
(2.75) \quad \det S^{\xi} \equiv \det A^1_* \neq 0,
\]

by (H1)(i). Thus, we may conclude, similarly as in the nondegenerate case:
Lemma 2.22. \( W = (U, Z) \) satisfies eigenvalue equation (2.60) if and only if

\[
\begin{align*}
\tilde{W}^* \tilde{S} W &\equiv \text{constant}
\end{align*}
\]

for all \( \tilde{W} = (\tilde{U}, \tilde{Z}) \) satisfying the adjoint eigenvalue equation, and vice versa.

For future reference, we note the representation

\[
\begin{align*}
(S^\xi)^{-1} &= \begin{pmatrix}
-(A_1^*)^{-1} & 0 & (A_1^*)^{-1}A(\xi)_{12} \\
(b_2^{11})^{-1}b_1^{11}(A_1^*)^{-1} & 0 & -(b_2^{11})^{-1}b_1^{11}(A_1^*)^{-1}A(\xi)_{12} - I \\
-\tilde{A}(\xi)(A_1^*)^{-1} & I & -\tilde{A}(\xi)_{22} + \tilde{A}(\xi)(A_1^*)^{-1}A_{12}
\end{pmatrix}
\end{align*}
\]

\[
A_1^* := A_{11}^* - A_{12}^* (b_2^{11})^{-1}b_1^{11},
\]

\[
\tilde{A}(\xi) := \alpha_{21}(\xi) - \alpha_{22}(\xi)(b_2^{11})^{-1}b_1^{11},
\]

\[
\alpha(\xi) := A^1 - iB^1\xi - iB^1\xi,
\]

which may be obtained by the above-mentioned column operation, followed by straightforward row reduction.

Let

\[
\phi_j^+ := P_+ v_j^+, \quad j = 1, \ldots, k,
\]

and

\[
\phi_j^- = j = k + 1, \ldots, n + r
\]

denote the locally analytic bases of the stable manifold at \(+\infty\) and the unstable manifold at \(-\infty\) of solutions of the variable-coefficient equation (2.60) that are found in Section 3.1, and set

\[
\Phi^+ := (\phi_1^+, \ldots, \phi_k^+), \quad \Phi^- := (\phi_{k+1}^-, \ldots, \phi_{n+r}^-),
\]

and

\[
\Phi := (\Phi^+, \Phi^-).
\]

Define the solution operator from \( y_1 \) to \( x_1 \) of (2.60), denoted by \( F^{y_1 \rightarrow x_1} \), as

\[
F^{y_1 \rightarrow x_1} = \Phi(x_1; \lambda) \Phi^{-1}(y_1; \lambda)
\]

and the projections \( \Pi_{y_1}^\pm \) on the stable manifolds at \( \pm\infty \) as

\[
\Pi_{y_1}^+ = (\Phi^+(y_1; \lambda) \ 0) \Phi^{-1}(y_1; \lambda) \quad \text{and} \quad \Pi_{y_1}^- = (0 \ \Phi^-(y_1; \lambda)) \Phi^{-1}(y_1; \lambda).
\]

Then, we have the following universal result (see, e.g. [ZH,MaZ.1] in the nondegenerate case, or [MaZ.3] for the one-dimensional hyperbolic–parabolic case).
Proposition 2.23. With respect to any $L^p$, $1 \leq p \leq \infty$, the domain of consistent splitting consists entirely of normal points of $L_\xi^*$, i.e., resolvent points, or isolated eigenvalues of constant multiplicity. On this domain, the resolvent kernel $G_{\xi, \lambda}$ is meromorphic, with representation

\[ G_{\xi, \lambda}(x_1, y_1) = \begin{cases} (I_n, 0)F_{y_1-x_1} \Pi^+_y (\tilde{S}_\xi)^{-1}(y_1)(I_n, 0)^{tr} & x_1 > y_1, \\ -(I_n, 0)F_{y_1-x_1} \Pi^-_y (\tilde{S}_\xi)^{-1}(y_1)(I_n, 0)^{tr} & x_1 < y_1, \end{cases} \]

(\tilde{S}_\xi)^{-1} as described in (2.77). Moreover, on any compact subset $K$ of $\rho(L_\xi) \cap \Lambda$ ($\rho(L_\xi)$ denoting resolvent set), there holds the uniform decay estimate

\[ |G_{\xi, \lambda}(x_1, y_1)| \leq Ce^{-\eta|x_1-y_1|}, \]

where $C > 0$ and $\eta > 0$ depend only on $K$, $L_\xi$.

Proof. As discussed further in [Z.1,MaZ.1], to show that $\lambda$ is in the resolvent set of $L_\xi$, with respect to any $L^p$, $1 \leq p \leq \infty$, it is enough to: (i) construct a resolvent kernel $G_{\xi, \lambda}(x_1, y_1)$ satisfying

\[ (L_\xi - \lambda)G_{\xi, \lambda} = \delta_{y_1}(x_1) \]

and obeying a uniform decay estimate

\[ |G_{\xi, \lambda}(x_1, y_1)| \leq Ce^{-\eta|x_1-y_1|}; \]

and, (ii) show that there are no $L^p$ solutions of $(L_\xi - \lambda)W = 0$, i.e., $\lambda$ is not an eigenvalue of $L_\xi$, or, equivalently, a zero of the Evans function $D(\cdot)$ (For $p < \infty$, these are necessary as well as sufficient, [Z.1]). For, then, the Hausdorff–Young inequality yields that the distributional solution formula $(L_\xi - \lambda)^{-1}f := \int G_\lambda(x_1, y_1)f(y_1)dy$ yields a bounded right inverse of $(L_\xi - \lambda)$ taking $L^p$ to $L^p$, while nonexistence of eigenvalues implies that this is also a left inverse.

On the domain of consistent splitting, we shall show that (ii) implies (i) (they are in fact equivalent [Z.1,MaZ.1]), which immediately implies the first assertion by the properties of analytic functions and the correlation between eigenvalues and zeroes of $D$, Remark 2.11. It remains, then, to establish existence of a solution of (2.86) satisfying the bound (2.87), under the assumption that $D(\lambda) \neq 0$.

Note, by Lemma 2.5, that the stable manifold $\Phi^+$ decays $x_1 \geq 0$ with uniform rate $Ce^{\eta|x_1|}$, $\eta > 0$. Moreover, if $D \neq 0$, then $\Phi^+$ does not decay at $-\infty$, whereupon we may conclude from Lemma 2.5 that it grows exponentially as $x_1 \to -\infty$, so that it in fact decays uniformly exponentially in $x_1$ on the whole line. Likewise, $\Phi^-$ decays exponentially as $x_1$ decreases, uniformly on the whole line. Thus, we find that (2.84) satisfies (2.85) as claimed. Clearly, it also satisfies (2.86) away from the
singular point $x_1 = y_1$. Finally, we note that, by the construction of $\tilde{S}^{\xi}$, we have the general fact that
\[
\int_{y_1-}^{y_1+} L_\xi G_{\xi,\lambda} = (I_n, 0)\tilde{S}^{\xi}.
\]
Substituting from (2.84), we obtain
\[
\left(\begin{array}{c}
[G_{\xi,\lambda}] \\
[(b_1^{11}, b_2^{11})]G'_{\xi,\lambda}
\end{array}\right) = (I_n, 0)\tilde{S}^{\xi} \mathcal{F}_{y_1-} y_1 (\tilde{S}^{\xi})^{-1} (I_n, 0)^{tr} = I_n,
\]
validating the jump condition across $x_1 = y_1$, and completing the proof. \qed

Proposition 2.23 includes a satisfactory intermediate-frequency bound (2.85) on the resolvent kernel. More careful analyses will be required in the large- and small-frequency limits.

Remark 2.24. Similarly as in the strictly parabolic case [ZH], formula (2.84) extends to the full phase-variable representation
\begin{equation}
(2.88)
\begin{array}{c}
G_{\xi,\lambda}(x_1, y_1) \\
(b_1^{11}, b_2^{11})(\partial/\partial x_1)G_{\xi,\lambda}(x_1, y_1)
\end{array}
\begin{array}{c}
(\partial/\partial y_1)G_{\xi,\lambda}(x_1, y_1)(0, (b_2^{11})^t) \\
(b_1^{11}, b_2^{11})(\partial/\partial x_1)(\partial/\partial y_1)G_{\xi,\lambda}(x_1, y_1)(0, (b_2^{11})^t)
\end{array}
\begin{cases}
\mathcal{F}_{y_1-} x_1 \Pi_{y_1-} (\tilde{S}^{\xi})^{-1}(y_1) & x_1 > y_1, \\
-\mathcal{F}_{y_1-} x_1 \Pi_{y_1-} (\tilde{S}^{\xi})^{-1}(y_1) & x_1 < y_1.
\end{cases}
\end{equation}
This formula (though not the specific phase variable) is also universal, and follows from (2.84), which states that the upper lefthand corner is correct, together with the fact that the columns of the righthand side satisfy the forward eigenvalue equation (written as a first-order system) with respect to $x_1$, by inspection, while the rows satisfy the adjoint eigenvalue equation (again, written as a first-order system) with respect to $y_1$, by duality relation (2.76), a simple consequence of which is that the columns and rows do respect the phase-variable formulation and so we may deduce from correctness of the upper lefthand corner that all entries are correct. Formula (2.88) is important in the development of the effective spectral theory given in Section 4.3.3, below; see discussion, proof of Proposition 4.38.

2.4.3. Generalized spectral decomposition. For the treatment of low-frequency behavior, we develop a modified representation of the resolvent kernel consisting of a scattering decomposition in solutions of the forward and adjoint eigenvalue equations.

From (2.76), it follows that if there are $k$ independent solutions $\phi_1^+, \ldots, \phi_k^+$ of \((L_\xi - \lambda I)W = 0\) decaying at $+\infty$, and $n - k$ independent solutions $\phi_{k+1}^-, \ldots, \phi_n^-$ of \((L_\xi^* - \lambda I)\tilde{W} = 0\) decaying at $-\infty$, then there exist $n - k$ independent solutions $\hat{\psi}_k^+, \ldots, \hat{\psi}_n^+$ of \((L_\xi^* - \lambda I)\tilde{W} = 0\) decaying at $+\infty$, and $k$ independent solutions $\hat{\psi}_1^-, \ldots, \hat{\psi}_k^-$ decaying at $-\infty$. More precisely, setting
\begin{equation}
(2.89)
\Psi^+(x_1; \lambda) = (\psi_{k+1}^+(x_1; \lambda) \ldots \psi_n^+(x_1; \lambda)) \in \mathbb{R}^{n \times (n-k)},
\end{equation}
\[
\Psi^- (x_1; \lambda) = (\psi^-_1 (x_1; \lambda) \quad \cdots \quad \psi^-_k (x_1; \lambda)) \in \mathbb{R}^{n \times k},
\]
and
\[
\Psi (x_1; \lambda) = (\Psi^- (x_1; \lambda) \quad \Psi^+ (x_1; \lambda)) \in \mathbb{R}^{n \times n},
\]
where \(\psi^\pm_j\) are exponentially growing solutions obtained through Lemma 2.5, we may define dual exponentially decaying and growing solutions \(\tilde{\psi}^\pm_j\) and \(\tilde{\phi}^\pm_j\) via
\[
(\tilde{\Psi} \quad \tilde{\Phi})^\pm \mathcal{S}^\xi (\Psi \quad \Phi)^\pm = I.
\]

Then, we have:

**Proposition 2.25.** The resolvent kernel may alternatively be expressed as
\[
G^\xi_{\lambda} (x_1, y_1) = \begin{cases}
(I_n, 0)\Phi^+(x_1; \lambda)M^+(\lambda)\tilde{\Psi}^- (y_1; \lambda)(I_n, 0)^{tr} & x_1 > y_1, \\
-(I_n, 0)\Phi^- (x_1; \lambda)M^- (\lambda)\tilde{\Psi}^+ (y_1; \lambda)(I_n, 0)^{tr} & x_1 < y_1,
\end{cases}
\]
where
\[
M(\lambda) := \text{diag}(M^+(\lambda), M^- (\lambda)) = \Phi^{-1} (z; \lambda) (\mathcal{S}^\xi)^{-1} (z) \tilde{\Psi}^{-1} (z; \lambda),
\]
\[
\tilde{\Psi} := (\tilde{\Psi}^- \quad \tilde{\Psi}^+). \text{ (Note: the righthand side of (2.94) is independent with respect to } z, \text{ as a consequence of Lemma 2.22.)}
\]

**Proof.** Immediate, by rearrangement of (2.86). \(\Box\)

**Remarks.** 1. Representation (2.84) reflects the classical duality principle (see, e.g. [ZH], Lemma 4.2) that the transposition \(G^\xi_{\lambda} (y_1, x_1)\) of the Green distribution \(G^\xi_{\lambda} (x_1, y_1)\) associated with operator \((L^\xi - \lambda)\) should be the Green distribution for the adjoint operator \((L^* \xi - \tilde{\lambda})\).

2. As before, it is possible to represent the matrix \(G^\xi_{\lambda} \) by means of intrinsic objects such as solution operators and projections on stable manifolds
\[
G^\xi_{\lambda} (x_1, y_1) = \begin{cases}
(I_n, 0)\mathcal{F}^{z \rightarrow x_1} \Pi^+_z (\mathcal{S}^\xi)^{-1} (z) \tilde{\Pi}^-_z \tilde{\mathcal{F}}^{z \rightarrow y_1} (I_n, 0)^{tr} & x_1 > y_1, \\
-(I_n, 0)\mathcal{F}^{z \rightarrow x_1} \Pi^-_z (\mathcal{S}^\xi)^{-1} (z) \tilde{\Pi}^+_z \tilde{\mathcal{F}}^{z \rightarrow y_1} (I_n, 0)^{tr} & x_1 < y_1,
\end{cases}
\]
\[
\mathcal{F}^{z \rightarrow y_1} := \tilde{\Psi}^{-1} (z; \lambda) \tilde{\Psi} (y_1; \lambda), \quad \tilde{\Pi}^+_z := \tilde{\Psi}^{-1} \begin{pmatrix} 0 \\ \tilde{\Psi}^+ \end{pmatrix} (z; \lambda), \quad \tilde{\Pi}^-_z := \tilde{\Psi}^{-1} \begin{pmatrix} \tilde{\Psi}^- \\ 0 \end{pmatrix} (z; \lambda).
\]
The exponential decay asserted in Proposition 2.23 is somewhat more straightforward to see in this dual formulation, by judiciously choosing \(z\) in (2.95) and using the exponential decay of forward and adjoint flows on \(x_1 \geq 0\) alone.

From Proposition 2.25, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant–coefficient case.
Corollary 2.26 [Z.3]. On $\Lambda \cap \rho(L_{\xi})$, there hold

$$G_{\xi,\lambda}(x_1, y_1) = \sum_{j,k} M_{jk}^+(\lambda) \phi_j^+(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^*$$

for $y_1 \leq 0 \leq x_1$,

$$G_{\xi,\lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^+(\lambda) \phi_j^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^* - \sum_k \psi_k^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^*$$

for $y_1 \leq x_1 \leq 0$, and

$$G_{\xi,\lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^-(\lambda) \phi_j^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^* + \sum_k \phi_k^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^*$$

for $x_1 \leq y_1 \leq 0$, with

$$M^+ = (-I, 0) (\Phi^+ \Phi^-)^{-1} \Psi^-$$

and

$$d^\pm = (0, I) (\Phi^+ \Phi^-)^{-1} \Psi^-.$$

Symmetric representations hold for $y_1 \geq 0$.

**Proof.** The matrix $M^+$ in (2.94) may be expanded using duality relation (2.76) as

$$M^+ = (-I, 0) (\Phi^+ \Phi^-)^{-1} A^{-1} (\tilde{\Psi}^- \tilde{\Phi}^-)^{-1} \left[ I \begin{array}{c} 0 \end{array} \right]_z$$

$$= (-I, 0) (\Phi^+ \Phi^-)^{-1} (\Psi^- \Phi^-) \left[ I \begin{array}{c} 0 \end{array} \right]_z$$

$$= (-I, 0) (\Phi^+ \Phi^-)^{-1} \Psi^-_z,$$

yielding (2.99) for $x_1 \geq y_1$, in particular for $y_1 \leq 0 \leq x_1$.

Next, expressing $\phi_j^\pm(x_1; \lambda)$ as a linear combination of basis elements at $-\infty$, we obtain the preliminary representation

$$G_{\lambda}(x_1, y_1) = \sum_{j,k} d_{jk}^+(\lambda) \phi_j^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^* + \sum_{j,k} e_{jk}^+(\lambda) \phi_j^-(x_1; \lambda) \tilde{\psi}_k^-(y_1; \lambda)^*,$$

valid for $y_1 \leq x_1 \leq 0$. Duality, (2.76), with (2.84), and the fact that $\Pi_+ = I - \Pi_-$, gives

$$- \begin{pmatrix} d^+ \\ e^+ \end{pmatrix} = (\tilde{\Phi}^- \tilde{\Psi}^-)^* A \Pi_+ \Psi^-_z$$

$$= - (\Phi^- \Psi^-)^{-1} [I - (0, I) (\Phi^+ \Phi^-)^{-1}] \Psi^-$$

$$= \begin{pmatrix} 0 \\ -I_k \end{pmatrix} - \begin{pmatrix} 0 & I_{n+r-k} \\ 0 & 0 \end{pmatrix} (\Phi^+ \Phi^-)^{-1} \Psi^-.$$
yielding (2.97) and (2.100) for \( y_1 \leq x_1 \leq 0 \). Relations (2.98) and (2.100) follow for \( x_1 \leq y_1 \leq 0 \) in similar, but more straightforward fashion from (2.76) and (2.84).

**Remark 2.27.** In the constant-coefficient case, with a choice of common bases \( \Psi^\pm = \Phi^\pm \) at \( \pm \infty \), (2.96)–(2.100) reduce to the simple formula

\[
G_{\xi,\lambda}(x_1,y_1) = \begin{cases} 
-\sum_{j=k+1}^{n+r} \phi_j^+ (x_1;\lambda) \tilde{\phi}_j^{+*} (y_1;\lambda) & x_1 > y_1, \\
\sum_{j=1}^{k} \phi_j^- (x_1;\lambda) \tilde{\phi}_j^{-*} (y_1;\lambda) & x_1 < y_1,
\end{cases}
\]

where, generically, \( \phi_j^\pm \), \( \tilde{\phi}_j^\pm \) may be taken as pure exponentials

\[
\phi_j^\pm (x_1) \tilde{\phi}_j^{\pm*} (y_1) = e^{\mu_j^\pm (\lambda)(x_1-y_1)} V_j^\pm (\lambda) V_j^{\pm*} (\lambda).
\]

Note that, moving individual modes \( \phi_j^\pm \tilde{\phi}_j^\pm \) in the spectral resolution formula (2.8), using Cauchy’s Theorem, to contours \( \mu_j (\lambda) \equiv i\xi \) lying along corresponding dispersion curves \( \lambda = \lambda_j (\xi) \), we obtain the standard decomposition of \( e^{Lt} \) into eigen-modes of continuous spectrum: in this (constant-coefficient) case, just the usual representation obtained by Fourier transform solution.

## 3. The Evans Function, and Its Low-Frequency Limit.

We next explore the key link between the Evans function \( D \) and the Lopatinski determinant \( \Delta \) in the limit as frequency goes to zero, establishing Result 1 of the introduction.

### 3.1. The Evans function.

We first recall the following important result, established by Kreiss in the strictly hyperbolic case (see [Kr], Lemma 3.2, and discussion just above; see also [CP], Theorem 3.5, p. 431, and [ZS], Lemma 3.5) and by Métivier [Mé.4] in the constant multiplicity case. The result holds, more generally, for systems satisfying the block structure condition of Kreiss–Majda [Kr,M.1–4].

**Lemma 3.1.** Let there hold (A1) and (H1)–(H2), and (H4), or, more generally, \( \sigma (dF^\xi (U_\pm)) \) real, semisimple, and of constant multiplicity for \( \xi \in \mathbb{R}^d \setminus \{0\} \) and \( \det dF^1 \neq 0 \). Then, the vectors \( \{ r_1^-, \cdots, r_{n-1}^- \}, \{ r_{i+1}^+, \cdots, r_n^+ \} \) (defined as in (1.30), Section 1.3), and thus \( \Delta (\xi,\lambda) \) may be chosen to be homogeneous (degree zero for \( r_j^\pm \), degree one for \( \Delta \)), analytic on \( \tilde{\xi} \in \mathbb{R}^{d-1} \), \( \text{Re} \lambda > 0 \) and continuous at the boundary \( \tilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\} \), \( \text{Re} \lambda = 0 \).

**Proof.** See Exercises 4.23–4.24 and Remark 4.25, Section 4.5.2 of [Z.3] for a proof in the general case (three alternative proofs, based respectively on [Kr], [CP], and [ZS]). In the case of main interest, when (H5) holds as well, we will see this
later in the course of the explicit computations of Section 5.3; see Remark 5.20.

\[\square\]

**Remark 3.2.** Typically, \(\Delta\) has a conical singularity at \((\tilde{\xi}, \lambda) = (0, 0)\), in the sense that the level set through the origin is a cone and not a plane. That is, it is degree one homogeneous but not linear, with a gradient discontinuity at the origin. This reflects the fact that \(r_j^\pm\) are degree zero homogeneous but are not constant unless all \(dF^j\) commute.

Following the construction of Section 2.2, we may define an Evans function

\[
D(\tilde{\xi}, \lambda) := \det \left( W_1^+, \ldots, W_k^+, W_{k+1}^-, \ldots, W_{n+r}^- \right)_{x=0, \lambda}
= \det \left( P_+ V_1^+, \ldots, P_+ V_k^+, P_- V_{k+1}^-, \ldots, P_- V_{n+r}^- \right)_{x=0, \lambda},
\]

(3.1)

associated with the linearized operator \(L_{\tilde{\xi}}\) about the viscous shock profile, on its domain of consistent splitting, in particular on the set

\[
\Lambda := \{(\tilde{\xi}, \lambda) : \lambda \in \Lambda_{\tilde{\xi}}\}
= \{(\tilde{\xi}, \lambda) : \text{Re} \lambda > -\eta(|\text{Im} \lambda|^2 + |\tilde{\xi}|^2)/(1 + |\text{Im} \lambda|^2 + |\tilde{\xi}|^2)\},
\]

\(\Lambda_{\tilde{\xi}}\) as defined in (2.62), where \(P_\pm\) are the transformations described in Lemma 2.5, \(W_j^\pm\) denote solutions of variable-coefficient eigenvalue problem \((L_{\tilde{\xi}} - \lambda)U = 0\) written as a first-order system in phase-variables \(W = (U, z_2)\), \(z_2 = b_1^I u^I + b_2^I u^{II}\), and \(V_j^\pm\) solutions of the associated limiting, constant-coefficient systems as \(x_1 \to \pm \infty\). By construction, \(D\) is analytic on \(\Lambda\).

For comparison with the inviscid case, it is convenient to introduce polar coordinates

\[
(\tilde{\xi}, \lambda) =: (\rho \tilde{\xi}_0, \rho \lambda_0),
\]

\(\rho \in \mathbb{R}_+, \tilde{\xi}_0, \lambda_0 \in \mathbb{R}^{d-1} \times \{\text{Re} \lambda \geq 0\} \setminus \{(0,0)\}\), and consider \(W_j^\pm, V_j^\pm\) as functions of \((\rho, \tilde{\xi}_0, \lambda_0)\).

**Lemma 3.3 [MéZ.2].** Under assumptions (A1)–(A3) and (H0)–(H3), the functions \(V_j^\pm\) may be chosen within groups of \(r\) “fast modes” bounded away from the center subspace of coefficient \(A_\pm\), analytic in \((\rho, \tilde{\xi}_0, \lambda_0)\) for \(\rho \geq 0\), \(\tilde{\xi}_0 \in \mathbb{R}^{d-1}\), \(\text{Re} \lambda_0 \geq 0\), and \(n\) “slow modes” approaching the center subspace as \(\rho \to 0\), analytic in \((\rho, \tilde{\xi}_0, \lambda_0)\) for \(\rho > 0\), \(\tilde{\xi}_0 \in \mathbb{R}^{d-1}\), \(\text{Re} \lambda_0 \geq 0\) and continuous at the boundary \(\rho = 0\), with limits

\[
V_j^\pm(0, \tilde{\xi}_0, \lambda_0) = \left( (A_{\pm}^1)^{-1} r_j^\pm(\tilde{\xi}_0, \lambda_0) \right),
\]

(3.3)

\(r_j^\pm\) defined as in Lemma 3.1.
Proof. We here carry out the case $\text{Re}\,\lambda_0 > 0$. In the case of interest, that (H5) also holds, we shall later obtain the case $\text{Re}\,\lambda = 0$ as a straightforward consequence of the more detailed computations in Section 5.3; see Remark 5.20. The general case (without (H5)) follows by the analysis of [McZ.2].

Substituting $v = e^{\mu x}v$ into the limiting eigenvalue equations $(L_\xi - \lambda)v = 0$ written in polar coordinates, we obtain the polar characteristic equation,

\[
\begin{bmatrix}
\mu^2 B_{11}^{11} + \mu(-A_{11}^{1} + i\rho \sum_{j \neq 1} B_{1j}^{1} \xi_j + i\rho \sum_{k \neq 1} B_{1k}^{1} \xi_k) \\
-(i\rho \sum_{j \neq 1} A_{1j}^{j} \xi_j + \rho^2 \sum_{j,k \neq 1} B_{1j}^{jk} \xi_j \xi_k + \rho \lambda I)
\end{bmatrix} v = 0,
\]

(3.4)

where for notational convenience we have dropped subscripts from the fixed parameters $\xi_0, \lambda_0$. At $\rho = 0$, this simplifies to

\[
\left(\mu^2 B_{11}^{11} - \mu A_{11}^{1}\right) v = 0,
\]

which, by the analysis in Appendix A.2 of the linearized traveling-wave ordinary differential equation $(\mu B_{11}^{11} - A_{11}^{1}) v = 0$, has $n$ roots $\mu = 0$, and $r$ roots $\text{Re}\,\mu \neq 0$. The latter, “fast” roots correspond to stable and unstable subspaces, which extend analytically as claimed by their spectral separation from other modes; thus, we need only focus on the bifurcation as $\rho$ varies near zero of the $n$-dimensional center manifold associated with “slow” roots $\mu = 0$.

Positing a first-order Taylor expansion

\[
\begin{align*}
\mu &= 0 + \mu^1 \rho + o(\rho), \\
v &= v^0 + \nu^1 \rho + o(\rho),
\end{align*}
\]

(3.5)

and matching terms of order $\rho$ in (3.4), we obtain:

\[
(-\mu^1 A_{11}^{1} - i \sum_{j \neq 1} A_{1j}^{j} \xi_j - \lambda I)v^0 = 0,
\]

(3.6)

or $-\mu^1$ is an eigenvalue of $(A_{11}^{1})^{-1}(\lambda + iA_{\xi})$ with associated eigenvector $v^0$.

For $\text{Re}\,\lambda > 0$, $(A_{11}^{1})^{-1}(\lambda + iA_{\xi})$ has no center subspace. For, substituting $\mu^1 = i\xi_1$ in (3.6), we obtain $\lambda \in \sigma(iA_{\xi})$, pure imaginary, a contradiction. Thus, the stable/unstable spectrum splits to first order, and we obtain the desired analytic extension by standard matrix perturbation theory, though not in fact the analyticity of individual eigenvalues $\mu$.  $\square$
Corollary 3.4. Denoting $W = (U, b_1^{11} u^1 + b_2^{11} u^{11})$ as above, we may arrange at $\rho = 0$ that all $W_\pm$ satisfy the linearized traveling-wave ODE $(B^{11} U')' - (A^1 U)' = 0$, with constant of integration

\begin{equation}
B^{11} U_j^\pm - A^1 U_j^\pm \equiv \begin{cases} 0, & \text{for fast modes,} \\ r_j^\pm, & \text{for slow modes,} \end{cases}
\end{equation}

$r_j^\pm$ as above, with fast modes analytic at the $\rho = 0$ boundary and independent of $(\xi, \lambda_0)$ for $\rho = 0$, and slow modes continuous at $\rho = 0$.

Proof. Immediate \hfill \square

3.2. The low-frequency limit.

We are now ready to establish the main result of this section, validating Result 1 of the introduction. This may be recognized as a generalization of the basic Evans function calculation pioneered by Evans [E.4], relating behavior near the origin to geometry of the phase space of the traveling wave ODE and thus giving an explicit link between PDE and ODE dynamics. The corresponding one-dimensional result was established in [GZ]; for related calculations, see, e.g., [J, AGJ, PW].

Theorem 3.5 [ZS, Z.3]. Under assumptions (A1)–(A3) and (H0)–(H4), there holds

\begin{equation}
D(\xi, \lambda) = \gamma \Delta(\xi, \lambda) + O(|\xi| + |\lambda|)^{\ell+1},
\end{equation}

where $\Delta$ is given in the Lax case by the inviscid stability function described in (1.29), in the undercompressive case by the analogous undercompressive inviscid stability function (1.38) with $g$ appropriately chosen,\textsuperscript{25} and in the overcompressive case by the special, low-frequency stability function

\begin{equation}
\Delta(\tilde{\xi}, \lambda) := \det (r_1^-, \cdots, r_{n-i-1}^-, m_\delta^1, \cdots, m_\delta^i, r_{i+1}^+, \cdots, r_n^+),
\end{equation}

where

\begin{equation}
m(\tilde{\xi}, \lambda, \delta) := \lambda \int_{-\infty}^{\infty} (\tilde{U}^\delta(x) - \bar{U}(x))dx \\
+ i \int_{-\infty}^{\infty} (F^{\tilde{\xi}} (\tilde{U}^\delta(x)) - F^{\tilde{\xi}} (\bar{U}(x)))dx.
\end{equation}

In each case the factor $\gamma$ is a constant measuring transversality of the intersection of unstable/resp. stable manifolds of $U_-/U_+$, in the phase space of the traveling-wave ODE ($\gamma \neq 0 \iff$ transversality), while the constant $\ell$ as usual denotes the dimension

\textsuperscript{25}Namely, as a Melnikov separation function associated with the undercompressive connection; see [ZS] for details.
of the manifold \( \{\bar{U}^\delta\} \), \( \delta \in \mathcal{U} \subset \mathbb{R}^\ell \) of connections between \( U\pm \) (see (H3), Section 1.1, and discussion, Section 1.2). In the Lax or undercompressive case, \( \ell = 1 \) and \( \{\bar{U}^\delta\} = \{\bar{U}(-\delta)\} \) is simply the manifold of translates of \( \bar{U} \).

That is, \( D(\cdot, \cdot) \) is tangent to \( \Delta(\cdot, \cdot) \) at \( (0, 0) \); equivalently, \( \Delta(\cdot, \cdot) \) describes the low-frequency behavior of \( D(\cdot, \cdot) \). Note that \( \Delta(\cdot, \cdot) \) is evidently homogeneous degree \( \ell \) in each case (recall, \( \ell = 1 \) for Lax, undercompressive cases), hence \( O(|\xi| + |\lambda|)^{\ell+1} \) is indeed a higher order term.

**Remark 3.6.** In the one-dimensional setting, \( m(0, 1, \delta) \) has an interpretation as “mass-map”, see [FreZ.1]; likewise, \( \Delta(0, 1) \) (\( \Delta \) as in (3.10)) arises naturally in determining shock shift/distribution of mass resulting from a given perturbation mass.

**Proof.** We will carry out the proof in the Lax case only. The proofs for the under- and overcompressive cases are quite similar, and may be found in [ZS]. We are free to make any analytic choice of bases, and any nonsingular choice of coordinates, since these affect the Evans function only up to a nonvanishing analytic multiplier which does not affect the result. Choose bases \( W_j^\pm \) as in Lemma 3.4, \( W_j = (U, z_j^0) \), \( z_2 = b_1^1 u^I + b_2^1 u^II \). Noting that \( L_0 \bar{U}^0 = 0 \), by translation invariance, we have that \( \bar{U}^0 \) lies in both \( \text{Span}\{U^+_1, \ldots, U^+_K\} \) and \( \text{Span}\{U^-_{K+1}, \ldots, U^-_{n+r}\} \) for \( \rho = 0 \), hence without loss of generality

\[
U^+_1 = U^-_{n+r} = \bar{U}',
\]

independent of \( \xi, \lambda \). (Here, as usual, \( \rho \) denotes \( \partial/\partial x_1 \).

More generally, we order the bases so that \( W^+_1, \ldots, W^+_K \) and \( W^-_{n+r-k-1}, \ldots, W^-_{n+r} \) are fast modes (decaying for \( \rho = 0 \)) and \( W^+_{K+1}, \ldots, W^+_K \) and \( W^-_{K+1}, \ldots, W^-_{n+r-k-2} \) are slow modes (asymptotically constant for \( \rho = 0 \)), fast modes analytic and slow modes continuous at \( \rho = 0 \) (Corollary 3.4).

Using the fact that \( U^+_1 \) and \( U^-_{n+r} \) are analytic, we may express

\[
U^+_1(\rho) = U^+_1(0) + U^+_{1,\rho}\rho + o(\rho),
\]

\[
U^-_{n+r}(\rho) = U^-_{n+r}(0) + U^-_{n+r,\rho}\rho + o(\rho).
\]

Writing out the eigenvalue equation in polar coordinates,

\[
(B^{11}U')' = (A^1U)' - i\rho \sum_{j \neq 1} B^{j1} \xi_j U'
\]

\[
- i\rho \sum_{k \neq 1} B^{1k} \xi_k U' + i\rho \sum_{j \neq 1} A^j \xi_j U + \rho \lambda U
\]

\[
- \rho^2 \sum_{j, k \neq 1} B^{jk} \xi_j \xi_k U,
\]

(3.12)
we find that $Y^+ := U^+_{1,\rho}(0)$ and $Y^- := U^-_{n+r,\rho}(0)$ satisfy the variational equations

\[
(B^{11} Y')' = (A^1 Y)' - i \sum_{j \neq 1} B^{1j} \xi_j \bar{U}'' \\
- i \left( \sum_{k \neq 1} B^{1k} \xi_k \bar{U}' \right)' + i \sum_{j \neq 1} A^j \xi_j \bar{U}' + \lambda \bar{U}',
\]

(3.13)

with boundary conditions $Y^+(+\infty) = Y^-(+\infty) = 0$. Integrating from $+\infty$, $-\infty$ respectively, we obtain therefore

\[
B^{11} Y^\pm - A^1 Y^\pm = i F^\xi(\bar{U}) - i B^{11} \xi(\bar{U}) \bar{U}' \\
- i B^{11} (\bar{U}) \bar{U}' + \lambda \bar{U} - \left[ i F^\xi(U_\pm) + \lambda U_\pm \right],
\]

(3.14)

hence $\tilde{Y} := (Y^- - Y^+)$ satisfies

\[
B^{11} \tilde{Y}' - A^1 \tilde{Y} = i[F^\xi] + \lambda[u].
\]

(3.15)

By hypothesis (H1), \( \begin{pmatrix} A_{11}^1 & A_{12}^1 \\ b_{11}^1 & b_{22}^1 \end{pmatrix} \) is invertible, hence

\[
(U, z_2) \rightarrow (z_2, -z_1, z_2 - (A_{21}^1 + b_{11}^1', A_{22}^1 + b_{22}^1')U) \\
= (z_2, B^{11} U' - A^1 U)
\]

(3.16)

is a nonsingular coordinate change, where \( \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} := \begin{pmatrix} A_{11}^1 & A_{12}^1 \\ b_{11}^1 & b_{22}^1 \end{pmatrix} U. \)

Fixing $\tilde{\xi}_0$, $\lambda_0$, and using $W^+_1(0) = W^-_{n+r}(0)$, we have

\[
D(\rho) = \det \left( W^+_1(0) + \rho W^+_l(0) + o(\rho), \cdots, W^+_K(0) + o(1), \right. \\
\left. W^-_{K+1}(0) + o(1), \cdots, W^-_{n+r}(0) + \rho W^-_{n+r,\rho}(0) + o(\rho) \right)
\]

(3.17)

\[
= \det \left( W^+_1(0) + \rho W^+_l(0) + o(\rho), \cdots, W^+_K(0) + o(1), \right. \\
\left. W^-_{K+1}(0) + o(1), \cdots, \rho \tilde{Y}(0) + o(\rho) \right)
\]

\[
= \det \left( W^+_1(0), \cdots, W^+_K(0), W^-_{K+1}(0), \cdots, \rho \tilde{Y}(0) \right) + o(\rho)
\]

Applying now coordinate change (3.16) and using (3.7) and (3.15), we obtain

(3.18)
D(\rho) = C \det \left( \begin{array}{cc}
\text{fast} & \text{slow} \\
\{ z_{2,1}^+, \ldots, z_{2,k}^+ \} & \{ \ast, \ldots, \ast, \ast, \ldots, \ast \} \\
0, \ldots, 0, & \{ r_{i+1}^+, \ldots, r_n^+, r_1^-, \ldots, r_{n-i}^- \}
\end{array} \right) \\
\text{fast} & \text{slow} \\
\{ \bar{z}_{2,n+r-k-1}, \ldots, \bar{z}_{2,n+r-1} \} & \{ \ast \} \\
0, \ldots, 0, & \{ i[F^\xi(U)] + \lambda[U] \}
\right)_{x_1 = 0} + o(\rho)

= \gamma \Delta(\bar{\xi}, \lambda) + o(\rho)

as claimed, where

\gamma := C \det \left( \begin{array}{c}
z_{2,1}^+, \ldots, z_{2,k}^+, \bar{z}_{2,n+r-k-1}, \ldots, \bar{z}_{2,n+r-1} \end{array} \right)_{x_1 = 0}.

Noting that \{z_{2,1}^+, \ldots, z_{2,k}^+\} and \{\bar{z}_{2,n+r-k-1}, \ldots, \bar{z}_{2,n+r}\} span the tangent manifolds at \bar{U}(\cdot) of the stable/unstable manifolds of traveling wave ODE (1.17) at \bar{U}_+/\bar{U}_-, respectively, with \bar{z}_{2,1}^+ = \bar{z}_{2,n+r} = (b_1^{11}, b_2^{11}) \bar{U}' in common, we see that \gamma indeed measures transversality of their intersection; moreover, \gamma is constant, by Corollary 3.4.

4. ONE-DIMENSIONAL STABILITY.

We now focus attention on the one-dimensional case, establishing results 2 and 3 of the introduction. For further discussion/applications, see, e.g., [GS, BSZ.1, ZH, MaZ.4] and [Z.3], Section 6.

4.1. Necessary conditions: the stability index.

In the one-dimensional case \bar{\xi} \equiv 0, we obtain by the construction of Section 3 an Evans function \bar{D}(\lambda) depending only on the temporal frequency, associated with the one-dimensional linearized operator \bar{L} := \bar{L}_0. Since \bar{L} is real-valued, we obtain by Remark 2.11.2 that \bar{D}(\cdot) may be chosen with complex symmetry

\bar{D}(\bar{\lambda}) = \bar{D}(\lambda),

where bar denotes complex conjugate, so that, in particular, \bar{D}(\lambda) is real-valued for \lambda real. Further, the energy estimates of Lemma 2.1 show that (since there is no spectrum there) \bar{D} is nonzero for \lambda real and sufficiently large, so that

\begin{equation}
\text{sgn } \bar{D}(+\infty) := \lim_{\lambda \to \text{real } +\infty} \text{sgn } \bar{D}(\lambda)
\end{equation}
is well-defined. Finally, Theorem 3.5 gives \((d/d\lambda)^\ell D(0) = \ell! \gamma \Delta(0, 1)\).

**Definition 4.1.** Combining the above observations, we define the stability index

\[(4.2) \quad \Gamma := \text{sgn} \,(d/d\lambda)^\ell D(0)D(+\infty) = \text{sgn} \,\gamma \Delta(0,1)D(+\infty),\]

where \(\gamma, \Delta\) as defined in Section 1.4 denote transversality and inviscid stability coefficients, and \(\ell\) as defined in Section 1.2 the dimension of the manifold \(\{\tilde{U}^\delta\}\) of connections between \(U_\pm\). In the Lax or undercompressive case, \(\ell = 1\).

**Proposition 4.2.** The number of unstable eigenvalues \(\lambda \in \{\lambda : \text{Re}\,\lambda > 0\}\) has even parity if \(\Gamma > 0\) and odd parity if \(\Gamma < 0\). In particular, \(\Gamma > 0\) is necessary for one-dimensional linearized viscous stability with respect to \(L^1 \cap L^\infty\) or even test function \((C^\infty_0)\) initial data.

**Proof.** By complex symmetry (2.25), nonreal eigenvalues occur in conjugate pairs, hence do not affect parity. On the other hand, the number of real roots clearly has the parity claimed. This establishes the first assertion, from which it follows that \(\Gamma \geq 0\) is necessary for stability with respect to \(L_1 \cap L^\infty\) data (recall that zeroes of \(D\) on \(\{\text{Re}\,\lambda > 0\}\) correspond to exponentially decaying eigenmodes).

The assertion that \(\gamma \Delta(0,1) = 0\) implies linearized instability, and instability with respect to test function initial data follow from the more detailed calculations of Section 4.3.4.

**Proposition 4.2** gives a weak version of Result 2 of the introduction, the deficiency being that the normalizing factor \(\text{sgn} \,D(+\infty)\) is a priori unknown. As described in Section 6, [Z.3], the basic definition (4.2) may nonetheless yield useful conclusions as a measure of spectral flow/change in stability under homotopy of model or shock parameters. In particular, in the Lax case for which stability of small-amplitude shocks is known, one may conclude that, moving along the Hugoniot curve \(H_p(U_-, \theta)\) in direction of increasing \(|\theta|\), instability occurs when \(\gamma \Delta(0,1)\) first changes sign.

On the other hand, formula (4.2) is of little use as a measure of absolute stability, e.g., for nonclassical shocks, or Lax-type shocks occurring on disconnected components of the Hugoniot curve. It is useful therefore, as described with increasing levels of generality in [GS, BSZ.1, Z.3], to give an alternative, “absolute” version of the stability index in which \(\text{sgn} \,D(+\infty)\) is explicitly evaluated.

**Lemma 4.3 [Z.3].** Let matrix \(A\) be symmetric, invertible, and matrix \(B\) positive semidefinite, \(\text{Re} \,(B) \geq 0\). Then, the cones \(S(A^{-1}B) \oplus (N(A) \cap \ker B)\) and \(U(A)\) are transverse, where \(S(M), U(M)\) refer to stable/unstable subspaces of \(M\) and \(N(M)\) to the cone \(\{v : \text{Re} \,(v, Mv) \leq 0\}\).

**Proof.** Suppose to the contrary that \(x_0 \neq 0\) lies both in the cone \(S(A^{-1}B) \oplus (N(A) \cap \ker B)\) and in \(U(A)\), i.e.,

\[x_0 = x_1 + x_2\]
where \(x_1 \in \mathcal{S}(A^{-1}B)\), \(x_2 \in (N(A) \cap \ker B)\), and \(x_0 \in U(A)\). Define \(x(t)\) by the ordinary differential equation \(x' = A^{-1}Bx\), \(x(0) = x_0\). Then \(x(t)\) to \(x_2\) as \(t \to +\infty\) and thus \(\lim_{t \to +\infty} \langle x(t), Ax(t) \rangle \leq 0\). On the other hand,

\[
\langle x, Ax \rangle' = 2 \langle A^{-1}Bx, Ax \rangle = 2 \langle Bx, x \rangle \geq 0
\]

by assumption, hence \(\langle x_0, Ax_0 \rangle \leq 0\), contradicting the assumption that \(x_0\) belongs to \(U(A)\). \(\square\)

**Proposition 4.4** [Z.3]. Given (A1)-(A3) and (H0)-(H3), we have

\[
\text{sgn } D(\lambda) = \text{sgn } \det (\tilde{S}^+, \tilde{U}^+) \det (\pi Z^+, \varepsilon \tilde{S}^+) \det (\varepsilon \tilde{U}^-, \pi Z^-)_{|\lambda=0} \neq 0,
\]

for sufficiently large, real \(\lambda\), where \(\pi\) denotes projection of \(Z = (z_1, z_2, z'_2)\) onto \((z_1, z_2)\) with

\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} :=
\begin{bmatrix}
  A_{11} & A_{22} \\
  b_1 & b_2
\end{bmatrix} U,
\]

\(\varepsilon\) denotes extension of \((z_1)\) to \((z_1, 0)\), and \(\mathbb{S}(x_1)\), \(U(x_1)\) are bases of the stable/unstable subspaces of \(A^1_4 = (A^1_{11} - A_{12}(b^1_2)^{-1}b^1_1)\) (note: \((n - r)\) dimensional).

**Proof.** Working in standard coordinates \(W = (u^I, u^{II}, z'_2)\), we show equivalently that

\[
\text{sgn } D(\lambda) = \text{sgn } \det (\tilde{S}^+, \tilde{U}^+) \det (\pi W^+, \varepsilon \tilde{S}^+) \det (\varepsilon \tilde{U}^-, \pi W^-)_{|\lambda=0} \neq 0,
\]

where \(\pi\) denotes projection of \(W = (u^I, u^{II}, z'_2)\) onto \((u^I, u^{II})\) components, \(z_2 := b^1_1 u^I + b^1_2 u^{II}\), \(\varepsilon u^I := (u^I, -(b^1_2)^{-1}b^1_1 u^{II})\) denotes extension, and \(\mathbb{S}(x_1), U(x_1)\) are bases of the stable/unstable subspaces of \(A^1_4\).

Though it is not immediately obvious, we may after a suitable coordinate change arrange that \(A^1_{1\pm}\) be symmetric, \(B^{11}_{1\pm} = \begin{pmatrix} 0 & 0 \\ 0 & b^{11}_1 \end{pmatrix}_\pm\), and \(\text{Re } (\hat{b}^{11}_{1\pm}) > 0\). For, as pointed out in [KSh], we may without loss of generality take \(\tilde{A}^0_{1\pm}\) block-diagonal in (1.11), by multiplying coefficients on the left by \(T^T\) and on the right by \(T\) a suitably chosen upper block-triangular matrix; for further discussion, see the proof of Proposition 6.3, Appendix A.1. Multiplying on the left and right by \((\tilde{A}^0_{1\pm})^{-1/2}\) then reduces the equation to the desired form, in coordinate \(V = (\tilde{A}^0_{1\pm})^{-1/2}W\).

It is sufficient to show that quantity (4.5) does not vanish in the class (A1), (A3). For, since \(D(\lambda)\) does not vanish either, for real \(\lambda\) sufficiently large, we can then establish the result by homotopy of the symmetric matrix \(A^1_{1\pm}\) to an invertible real diagonal matrix (straightforward, using the unitary decomposition \(A^1 = UD U^*, U^* U = I\), and the fact that the unitary group is arcwise connected\(^{26}\)) and of \(B^{11}_{1\pm}\)

\(^{26}\)See, e.g., [Se.5], Chapter 7.
to \( \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \) (e.g., by linear interpolation of the positive definite \( b_{IIr}^{1+} \) to \( I_r \)), in which case it can be seen by explicit computation. (Note that the endpoint of this homotopy is on the boundary of but not in the Kawashima class \( (A2) \), since eigenvectors of \( A_1 \) are in the kernel of \( B_r \).)

As to behavior at \( \lambda = 0 \), a bifurcation analysis as in Section 3 of the limiting constant-coefficient equations at \( \pm \infty \) shows that the projections \( \Pi \) of slow modes of \( W_+ \) may be chosen as the unstable eigenvectors \( r^+_j \) of \( A_1^+ \), corresponding to outgoing characteristic modes, and the projections of fast modes as the stable (i.e. \( \text{Re } \mu < 0 \)) solutions of

\[
(A^1 - \mu B^{11})_+ \begin{pmatrix} u^I \\ u^{II} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

or without loss of generality

\[
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \mu b_2 \end{pmatrix}_+ \begin{pmatrix} u^I \\ u^{II} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

and thus of form

\[
\begin{pmatrix} -(A_{11})^{-1} A_{12} u^{II} \\ u^{II} \end{pmatrix}_+ = 0,
\]

where

\[
(b_2^{-1}(A_{22} - A_{21}(A_{11})^{-1} A_{12}) - \mu I)_+ u^{II} = 0.
\]

Likewise, using \( b_{11}^{1+} = 0 \), we find from the definitions of \( S, \varepsilon \) that stable solutions \( S^+ \) are in the stable subspace of \( A_{11}^+ \), with \( \varepsilon S^+ = (\hat{S}^+)_0 \), hence vectors \( \varepsilon S^+ \) lie in the intersection of the stable subspace of \( A^+_1 \) and the kernel of \( B_{11}^+ \). Our claim is that these three subspaces are independent, spanning \( \mathbb{C}^n \). Rewording this assumption, we are claiming that the stable subspace of \( (A^1)^{-1} B_{11}^+ \), the center subspace \( \ker B_{11}^+ \) intersected with the stable subspace of \( A^+_1 \), and the unstable subspace of \( A^+_1 \) are mutually independent. (Note: that dimensions are correct follows by consistent splitting, a consequence of \( (H1)(i) \)). But, this follows by Lemma 4.3, with \( A := A^+_1 \) and \( B := B_{11}^+ \), hence \( \det (\pi \mathbb{C}^+, \varepsilon \hat{S}^+) \neq 0 \). A symmetric argument shows that \( \det (\varepsilon \hat{U}^-, \pi \mathbb{C}^-)_{\lambda = 0} \neq 0 \), completing the result. \( \square \)

Formula (4.5) gives in principle an explicit evaluation of \( \Gamma \), involving only information that is linear-algebraic or concerning the dynamics of the traveling-wave ODE; however, it is still quite complicated. In the extreme Lax shock case, the formula simplifies considerably.
Proposition 4.5. In the case of an extreme right \((n\text{-shock})\) Lax profile,

\[
\Gamma = \text{sgn} \gamma^2 \det (r_1^-, \ldots, r_{n-1}^-, [U]) \det (r_1^-, \ldots, r_{n-1}^-, \bar{U}'/\bar{U}'(-\infty)) > 0
\]

is necessary for one-dimensional stability. A symmetric formula holds for extreme left \((1\text{-shock})\) Lax profiles.

**Proof.** From Corollary 1.8, we may deduce that \(\gamma\) for an extreme right \((\text{i.e., } n\text{-shock})\) Lax profile consists of a Wronskian involving only modes from the \(+\infty\) side, and is therefore explicitly evaluable; in particular, working in \((z_1, z_2, z_2')\) coordinates, we obtain \(\gamma\) as a determinant of \(z_2\) components only. Moreover, the expression (4.5) simplifies greatly. For, in \((z_1, z_2)\) coordinates, we have that \(U = \emptyset\), \(S\) is full dimension \(n - r\), and \(\epsilon S\) consists of vectors of the simple form \((z, 0)\). This means that \(\det (S, U)\) simplifies to just \(\det S\), while \(\det (Z^-, \epsilon S^+)\) simplifies to the product of \(\det S^+\) and \(\gamma\). Therefore, this term, similarly as in the strictly parabolic case, combines with like term \(\gamma\) in the computation of the stability index. Finally, \(\det (\epsilon U^0, \Pi Z^-)\) simplifies to \(\det (r_1^-, \ldots, r_{n-1}^-, \bar{U}')\), completing the result. \(\square\)

**Remark 4.6.** Condition (4.10), our strongest version of Result 2, is identical with that of the strictly parabolic case. The only very weak information required from the connection problem is the orientation of \(\bar{u}'\) as \(x \to -\infty\), i.e., the direction in which the profile leaves along the one-dimensional unstable manifold. In the case \(r = 1\), for example for isentropic gas dynamics, the traveling-wave ODE is scalar, and so the orientation of \(\bar{U}'\) is determined by the direction of the connection. In this case, (4.10) gives a full evaluation.

### 4.2 Sufficient conditions.

By (H3), there exists an \(\ell\)-parameter family of stationary solutions \(\{\bar{U}^\delta\}\) of (1.1) near \(\bar{U}\), \(\ell \geq 1\). Indeed, by Lemma 1.6, these lie arbitrarily close to \(\bar{U}\) in \(L^1 \cap C^4(x_1)\), precluding one-dimensional asymptotic stability of \(\bar{U}\) for any reasonable class of initial perturbation (say, test function initial data). The appropriate notion of stability of shock profiles in one dimension is, rather, “orbital” stability, or convergence to the manifold of stationary solutions \(\{\bar{U}^\delta\}\).

**Definition 4.7.** We define nonlinear orbital stability as convergence of \(\bar{U}(\cdot, t)\) as \(t \to \infty\) to \(\bar{U}^\delta(t)\), where \(\delta(\cdot)\) is an appropriately chosen function, for any solution \(\bar{U}\) of (1.1) with initial data sufficiently close in some norm to the original profile \(\bar{U}\). Likewise, we define linearized orbital stability as convergence of \(U(\cdot, t)\) to \((\partial \bar{U}^\delta/\partial \delta) \cdot \delta(t)\) for any solution \(U\) of the linearized equations (2.1) with initial data bounded in some chosen norm.

Note that differentiation with respect to \(\delta_j\) of the standing-wave ODE \(F^1(\bar{U}^\delta)_{x_1} = (B^1(\bar{U}^\delta)U^\delta_{x_1})_{x_1}\) yields \(L_0 \partial \bar{U}^\delta/\partial \delta_j = 0\), so that \((\partial \bar{U}^\delta/\partial \delta_j)\) represent stationary solutions of (2.1), with \(\text{Span}\{\partial \bar{U}^\delta/\partial \delta_j\}\) lying tangent to the stationary manifold \(\{\bar{U}^\delta\}\). In the Lax or undercompressive case, \(\ell = 1\), and the stationary manifold
is just the set of translates $\tilde{U}^\delta(x_1) := \tilde{U}(x_1 - \delta)$ of $\tilde{U}$, with the tangent stationary manifold $\text{Span} \{ \tilde{U}_{x_1} \}$ corresponding to the usual translational zero eigenvalue arising in the study of stability of traveling waves.

Result 3 of the introduction is subsumed in the following two results, to be established throughout the rest of the section.

**Theorem 4.8** [MaZ.4]. (Linearized stability) Let $\tilde{U}$ be a shock profile (1.2) of (1.1), under assumptions (A1)–(A3) and (H0)–(H3). Then, $\tilde{U}$ is $L^1 \cap L^p \to L^p$ linearly orbitally stable in dimension $d = 1$, for all $p > 1$, if and only if it satisfies the conditions of structural stability (transversality), dynamical stability (inviscid stability), and strong spectral stability. More precisely, for initial data $U_0 \in L^1 \cap L^p$, $p \geq 1$, the solution $U(x,t)$ of equations (1.1) linearized about $\tilde{U}$ satisfies

$$|U(\cdot,t) + \delta(t)\tilde{U}'(\cdot)|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}|U_0|_{L^1 \cap L^p},$$

(4.11)

**Theorem 4.9.** (Nonlinear stability) Let $\tilde{U}$ be a Lax-type profile (1.2) of a general real viscosity model (1.1), satisfying (A1)–(A3), (H0)–(H3), and the necessary conditions of structural, dynamical, and strong spectral stability. Then, $\tilde{U}$ is $L^1 \cap H^3 \to L^p \cap H^3$ nonlinearly orbitally stable in dimension $d = 1$, for all $p \geq 2$. More precisely, the solution $\tilde{U}(x,t)$ of (1.1) with initial data $U_0$ satisfies

$$|\tilde{U}(x,t) - \tilde{U}(x - \delta(t))|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}|U_0|_{L^1 \cap H^3},$$

$$|\tilde{U}(x,t) - \tilde{U}(x - \delta(t))|_{H^3} \leq C(1 + t)^{-\frac{1}{2}}|U_0|_{L^1 \cap H^3}$$

(4.12)

for initial perturbations $U_0 := \tilde{U}_0 - \tilde{U}$ that are sufficiently small in $L^1 \cap H^3$, for all $p \geq 2$, for some $\delta(t)$ satisfying $\delta(0) = 0$,

$$|\delta(t)| \leq C(1 + t)^{-\frac{1}{2}}|U_0|_{L^1 \cap H^3},$$

(4.13)

and

$$|\delta(t)| \leq C|U_0|_{L^1 \cap H^3}.$$

(4.14)

**Remarks.** 1. For slightly more regular initial perturbation $U_0 \in L^1 \cap H^4$, we may obtain by the same argument, using higher-derivative Green distribution bounds (described, e.g., in [MaZ.2, MaZ.4]), the derivative estimate

$$|\tilde{U}(x,t) - \tilde{U}(x - \delta(t))|_{W^{1,p}} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}|U_0|_{L^1 \cap H^4}$$

generalizing (4.12), which yields in turn the extension of (4.12) to low norms $L^p$, $1 \leq p \leq 2$, by a simple bootstrap argument (described in the Remark just below the proof of Theorem 4.9).

2. Theorem 4.8 and the extension of 4.9 just described contain in particular the results of $L^1 \to L^1$ linear and $L^1 \cap H^4 \to L^1 \cap H^4$ nonlinear global bounded stability, indicating that $L^1$ and $L^1 \cap H^4$, respectively, are natural norms for the linearized and nonlinear viscous shock perturbation problem in one dimension.
4.2.1. Linearized estimates.

Theorem 4.8 (and to some extent Theorem 4.9) is obtained as a consequence of detailed, pointwise bounds on the Green distribution $G(x, t; y)$ of the linearized evolution equations (2.1), which we now describe. For readability, we defer the rather technical proof to Section 4.3. In the remainder of this section, for ease of notation, we drop superscripts on $A^1$, $F^1$ and $B^{11}$, and subscripts on $x_1$ and $y_1$, denoting them simply as $A$, $F$, $B$, $x$, and $y$. We also make the simplifying assumption

(P0) Either $A_{\pm}$ are strictly hyperbolic, or else $A_{\pm}$ and $B_{\pm}$ are simultaneously symmetrizable.

As described in Remark 4.13 below, the bounds needed for our stability analysis persist in the general case, but the full description of the Green’s function does not.

Let $a_j^{\pm}$, $j = 1, \ldots, (n-r)$ denote the eigenvalues of $A(\pm\infty)$, and $l_j^{\pm}$ and $r_j^{\pm}$ associated left and right eigenvectors, respectively, normalized so that $l_j^{\pm} t_j r_k^{\pm} = \delta_k^j$. In case $A_{\pm}$ is strictly hyperbolic, these are uniquely defined up to a scalar multiplier.

Otherwise, we require further that $l_j^{\pm}$, $r_j^{\pm}$ be left and right eigenvectors also of $P_j^{\pm} B_{\pm} P_j^{\pm}$, $P_j^{\pm} := R_j^{\pm} L_j^{\pm t}$, where $L_j^{\pm}$ and $R_j^{\pm}$ denote $m_j^{\pm} \times m_j^{\pm}$ left and right eigenblocks associated with the $m_j^{\pm}$-fold eigenvalue $a_j^{\pm}$, normalized so that $L_j^{\pm} R_j^{\pm} = I_{m_j^{\pm}}$. (Note: The matrix $P_j^{\pm} B_{\pm} P_j^{\pm} \sim L_j^{\pm t} B_{\pm} R_j^{\pm}$ is necessarily diagonalizable, by simultaneous symmetrizability of $A_{\pm}$, $B_{\pm}$).

Eigenvalues $a_j^{\pm}$, and eigenvectors $l_j^{\pm}, r_j^{\pm}$ correspond to large-time convection rates and modes of propagation of the degenerate model (1.1). Likewise, let $a_j^* (x)$, $j = 1, \ldots, (n-r)$ denote the eigenvalues of

\begin{equation}
A_* := A_{11} - A_{12} B_{22}^{-1} B_{21},
\end{equation}

$A := dF(\bar{U}(x))$, and $l_j^* (x)$, $r_j^* (x) \in \mathbb{R}^{n-r}$ associated left and right eigenvectors, normalized so that $l_j^* t_j r_j^* \equiv \delta_j^j$. More generally, for an $m_j^*$-fold eigenvalue, we choose $(n-r) \times m_j^*$ blocks $L_j^*$ and $R_j^*$ of eigenvectors satisfying the dynamical normalization

\begin{equation}
L_j^* \partial_x R_j^* \equiv 0,
\end{equation}

along with the usual static normalization $L_j^* R_j^* \equiv \delta_j^j I_{m_j^*}$; as shown in Lemma 4.9, [MaZ.1], this may always be achieved with bounded $L_j^*$, $R_j^*$. Associated with $L_j^*$, $R_j^*$, define extended, $n \times m_j^*$ blocks

\begin{equation}
\mathcal{L}_j^* := \begin{pmatrix} L_j^* & 0 \\ 0 & R_j^* \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -B_{22}^{-1} B_{21} R_j^* \end{pmatrix}.
\end{equation}

Eigenvalues $a_j^*$ and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (1.1).
Define time-asymptotic, effective diffusion coefficients
\[
\beta_j^\pm := (t_j^R B_j)_{\pm}, \quad j = 1, \ldots, n,
\]
and local, \(m_j \times m_j\) dissipation coefficients
\[
\eta_j^\pm(x) := -L_j^x D_j R_j^\pm(x), \quad j = 1, \ldots, J \leq n - r,
\]
where
\[
D_j(x) := A_{12}B_{22}^{-1}\left[ A_{21} - A_{22}B_{22}^{-1}B_{21} + A_{*}B_{22}^{-1}B_{21} + B_{22}\partial_x(B_{22}^{-1}B_{21}) \right]
\]
is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

At \(x = \pm \infty\), these reduce to the corresponding quantities identified by Zeng [Ze.1,LZe] in her study by Fourier transform techniques of decay to constant solutions \((\bar{u}, \bar{v}) \equiv (u_\pm, v_\pm)\) of hyperbolic–parabolic systems, i.e., of limiting equations
\[
U_t = L_\pm U := -A_\pm U_x + B_\pm U_{xx}.
\]
These arise naturally through Taylor expansion of the one-dimensional (frozen-coefficient) Fourier symbol \((-i\xi A - \xi^2 B)_\pm\), as described in Appendix A.4; in particular, as a consequence of dissipativity, (A2), we obtain (see, e.g., [Kaw, LZe, MaZ.3], or Lemma 2.18)
\[
\beta_j^\pm > 0, \quad \text{Re} \sigma(\eta_j^\pm) > 0 \quad \text{for all } j.
\]
However, note that the dynamical dissipation coefficient \(D_j(x)\) does not agree with its static counterpart, possessing an additional term \(B_{22}\partial_x(B_{22}^{-1}B_{21})\), and so we cannot conclude that (4.21) holds everywhere along the profile, but only at the endpoints. This is an important difference in the variable-coefficient case; see Remarks 1.11-1.12 of [MaZ.3] for further discussion.

**Proposition 4.10 [MaZ.3].** In spatial dimension \(d = 1\), under assumptions (A1)–(A3), (H0)–(H3), and (P0), for Lax-type shock profiles satisfying the conditions of transversality, hyperbolic stability, and strong spectral stability, the Green distribution \(G(x, t; y)\) associated with the linearized evolution equations (2.1) may be decomposed as
\[
G(x, t; y) = H + E + S + R,
\]
where, for \(y \leq 0\):
\[
H(x, t; y) := \sum_{j=1}^{J} a_j^{x-1}(x)a_j^x(y)R_j^*(x)\xi_j^*(y,t)\delta_{x-a_j^t}(-y)\mathcal{L}_j^{xt}(y)
\]
\[
= \sum_{j=1}^{J} R_j^*(x)O(e^{-\eta_j t})\delta_{x-a_j^t}(-y)\mathcal{L}_j^{xt}(y),
\]
\[ E(x, t; y) := \sum_{a_k > 0} \left[ c_{k,-}^0 \right] U'(x) t_k^{-t} \left( \text{erf} \left( \frac{y + a_k^t t}{\sqrt{4 \beta_k t}} \right) - \text{erf} \left( \frac{y - a_k^t t}{\sqrt{4 \beta_k t}} \right) \right), \]

and

\[ S(x, t; y) := \chi(t \geq 1) \sum_{a_k < 0} r_k^{-1} t_k^{-l} (4 \pi \beta_k t)^{-1/2} e^{-(x-y-a_k^t) t^2 / 4 \beta_k t} + \chi(t \geq 1) \sum_{a_k > 0} r_k^{-1} t_k^{-l} (4 \pi \beta_k t)^{-1/2} e^{-(x-y-a_k^t) t^2 / 4 \beta_k t} \left( \frac{e^{-x}}{e^x + e^{-x}} \right), \]

\[ + \chi(t \geq 1) \sum_{a_k^+ > 0, a_k^- < 0} [e_{k,-}^j] r_k^{-1} t_k^{-l} (4 \pi \beta_k^{-1} t)^{-1/2} e^{-(x-z_j) t^2 / 4 \beta_k^{-1} t} \left( \frac{e^{-x}}{e^x + e^{-x}} \right), \]

\[ + \chi(t \geq 1) \sum_{a_k^+ > 0, a_k^- > 0} [e_{k,-}^j] r_k^{-1} t_k^{-l} (4 \pi \beta_k^{+1} t)^{-1/2} e^{-(x-z_j) t^2 / 4 \beta_k^{+1} t} \left( \frac{e^x}{e^x + e^{-x}} \right) \]

denote hyperbolic, excited, and scattering terms, respectively, \( \eta_0 > 0 \), and \( R \) denotes a faster decaying residual (described in Proposition 4.39 below). Symmetric bounds hold for \( y \geq 0 \).

Here, the averaged convection rates \( \bar{a}_j^+ = \bar{a}_j^+ (x, t) \) in (4.23) denote the time-averages over \( [0, t] \) of \( a_j^+ (x) \) along backward characteristic paths \( z_j^+ = z_j^+ (x, t) \) defined by

\[ \frac{dz_j^+}{dt} = a_j^+ (z_j^+), \quad z_j^+ (t) = x, \]

and the dissipation matrix \( \bar{\zeta}_j^+ = \bar{\zeta}_j^+ (x, t) \in \mathbb{R}^{m_j^+ \times m_j^+} \) is defined by the dissipative flow

\[ \frac{d\bar{\zeta}_j^+}{dt} = -\eta_j^+ (z_j^+) \bar{\zeta}_j^+, \quad \bar{\zeta}_j^+ (0) = I_{m_j}. \]

Similarly, in (4.25),

\[ \bar{z}_{jk}^\pm (y, t) := a_j^\pm \left( t - \left| \frac{y}{a_k^\pm} \right| \right) \]

and

\[ \bar{\beta}_{jk}^\pm (x, t; y) := \frac{|x^\pm|}{|a_j^\pm t|} \beta_j^\pm + \left| \frac{y}{a_k^\pm t} \right| \left( \frac{a_j^\pm}{a_k^\pm} \right)^2 \beta_k^-, \]

represent, respectively, approximate scattered characteristic paths and the time-averaged diffusion rates along those paths. In all equations, \( a_j, a_j^{+\pm}, l_j, L_j^{+\pm} \),
are as defined just above, and scattering coefficients \( [c^i_{j,k} -], i = -, 0, + \), are uniquely determined by

\[
\sum_{a_j^- < 0} [c^i_{j,k} -] r_j^r + \sum_{a_j^+ > 0} [c^i_{j,k} +] r_j^r + [c^0_{j,k} -] (U(+\infty) - U(-\infty)) = r_k^-
\]

for each \( k = 1, \ldots, n \), and satisfying

\[
\sum_{a_k^- > 0} [c^0_{j,k} -] l_k^- = \sum_{a_k^+ < 0} [c^0_{j,k} +] l_k^+ = \pi,
\]

where the constant vector \( \pi \) is the left zero effective eigenfunction of \( L \) associated with the right eigenfunction \( \bar{U}' \). Similar decompositions hold in the over- and undercompressive case.

Proposition 4.10, the variable-coefficient generalization of the constant-coefficient results of [Ze.1, LZe], was established in [MaZ.3] by Laplace transform (i.e., semigroup) techniques generalizing the Fourier transform approach of [Ze.1–2, LZe]; for discussion/geometric interpretation, see [Z.6, MaZ.1–2]. In our stability analysis, we will use only a small part of the detailed information given in the proposition, namely \( L^p \to L^q \) estimates on the time-decaying portion \( H + S + R \) of the Green distribution \( G \) (see Lemma 4.1, below). However, the stationary portion \( E \) of the Green distribution must be estimated accurately for an efficient stability analysis.

**Remark 4.11**[MaZ.2]. The function \( H \) described in Proposition 4.10 satisfies

\[
H(x, t; y)\Pi_2 \equiv 0
\]

and

\[
\Pi_2(A^0(x))^{-1}H(x, t; y) \equiv 0,
\]

where \( \Pi_2 := \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \) denotes projection onto final \( r \) coordinates, or equivalently \( L^* j \Pi_2 \equiv 0 \) and \( \Pi_2(A^0(x))^{-1}R^* j \equiv 0, 1 \leq j \leq J \).

These may be seen by the intrinsic property of \( L^* j \) and \( R^* j \) (readily obtained from our formulae) that they lie, respectively, in the left and right kernel of \( B \). This gives the first relation immediately, by the structure of \( B \) given in (A3). Likewise, we find that

\[
0 \equiv BR^* j := \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} (A^0)^{-1}R^* j
\]

\[
= \begin{pmatrix} b\Pi_2(A^0)^{-1}R^* j \\ 0 \end{pmatrix},
\]
yielding the second result by invertibility of \( b \) (see condition (A3)). This quantifies the observation that, in \( W = (w^I, w^II)^t \) coordinates, data in the \( w^II \) coordinate is smoothed under the evolution of (2.1), whereas data in the \( w^I \) coordinate is not; likewise, the “parabolic” variable \( w^II \) experiences smoothing while the “hyperbolic” coordinate \( w^I \) does not. (Recall that \( W = (A^0)^{-1}U \).

**Remark 4.12.** Following [Ho], we may recover the hyperbolic evolution equations (4.23), (4.26)–(4.27) by direct calculation, considering the evolution of a jump discontinuity along curve \( x = z(t) \) in the original equation \( U_t + (AU)_x = (BU)_x \); see also the related calculation of Section 1.3, [MaZ.1] in the relaxation case. For simplicity, write \( U = (u, v) \). Introducing the parabolic variable \( \tilde{v} := b_1u + b_2v \), and denoting jump across \( z \) of a variable \( f \) by \([f]\), we may write (2.1) as the pair of equations

\[
(4.35) \quad u_t + (A_*u)_x + (A_{12}b_2^{-1}\tilde{v})_x = 0
\]

and

\[
(4.36) \quad v_t + \left( (A_{21} - A_{22}b_2^{-1}b_1 + b_1, x - b_{2, x}b_2^{-1}b_1)u \right)_x + \left( (A_{22}b_2^{-1} + b_{2, x}b_2^{-1})\tilde{v} \right)_x - (\tilde{v})_x = 0.
\]

Applying the Rankine–Hugoniot conditions, we obtain relations

\[
(A_* - \tilde{z})[u] = 0, \quad [\tilde{v}] = 0, \quad \text{and}
\]

\[
(4.37) \quad [\tilde{v}_x] = \left( A_{21} - A_{22}b_2^{-1}b_1 + \tilde{z}b_2^{-1}b_1 + b_2\partial_x(b_2^{-1}b_1) \right)[u] = \left( A_{21} - A_{22}b_2^{-1}b_1 + b_2^{-1}b_1A_* + b_2\partial_x(b_2^{-1}b_1) \right)[u] =: d_*[u],
\]

yielding \( \tilde{z}_j = a_j(z_j(t)) \), \([u] = R_j^*(z_j(t))\zeta_j(t)\), \([\tilde{v}] = 0\), and \([\tilde{v}_x] = d_*R_j^*(z_j(t))\zeta_j(t)\), \(\zeta_j \in \mathbb{R}\). Substituting these Ansätze into (4.35) yields

\[
[u_t] + a^*_j[u_x] + a^*_{j,x}[u] + A_{12}b_2^{-1}[\tilde{v}_x] + (A_{12}b_2^{-1})_x[\tilde{v}] = 0,
\]

and thus

\[
[u] + a^*_{j,x}[u] + D_*[u] = 0,
\]

where all coefficients are evaluated at \( z_j(t) \) and \( D_* = A_{12}b_2^{-1}d_* \) as defined in (4.19). Taking inner product with \( L_j^*(z_j(t)) \) and rearranging using

\[
L_j^tR_j^* = -L_j^tR_j^* = -a_j^*(\partial_xL_j^{*t})R_j^*,
\]

we therefore obtain the characteristic evolution equations

\[
(4.38) \quad \hat{\zeta}_j = (a_j^*(\partial_xL_j^{*t})R_j^* - \partial_xa_j - \eta_j)\zeta_j,
\]
\[ \eta_j = L_j^s D_x R_j^s \] as defined in (4.18). Under the normalization \((\partial_x L_j^s) R_j^s = 0\) (note: automatic in the gas-dynamical case considered in [Ho], for which \(L_j^s\) and \(R_j^s\) are scalar), this yields in the original coordinate \(U\) precisely the solution operator \(a_j^{-1}(z_j) a_j^s(y) R_j^s(z_j) \zeta_j(y, t) L_j^s(y)\) described in (4.23).

**Remark 4.13.** In the case that \(A_\pm\) is nonstrictly hyperbolic and \(B_\pm\) are not simultaneously symmetrizable, the simple description of scattering terms breaks down, and much more complicated behavior appears to occur. However, the excited term \(E\) remains much the same, with scalar diffusion replaced by a multi-mode heat kernel and associated multi-mode error function. Likewise, a review of the pointwise Green’s function arguments of Section 4.3.3 shows that they still yield sharp modulus bounds on the scattering term even though we do not know the leading-order behavior, and so all bounds relevant to the ultimate stability argument go through.

### 4.2.2 Linearized stability.

As described in [MaZ.2–3], linearized orbital stability follows immediately from the pointwise bounds of Proposition 4.10. We reproduce the argument here, both for completeness and to motivate the nonlinear argument to follow in Section 4.2.4. Similarly as in [Z.6, MaZ.1–3], define the linear instantaneous projection:

\[
\varphi(x, t) := \int_{-\infty}^{+\infty} E(x, t; y) U_0(y) \, dy
\]

where \(U_0\) denotes the initial data for (2.1), and \(\bar{U} = \bar{U}(x)\) as usual. The amplitude \(\delta\) may be expressed, alternatively, as

\[
\delta(t) = -\int_{-\infty}^{+\infty} e(y, t) U_0(y) \, dy,
\]

where

\[
E(x, t; y) =: \bar{U}'(x) e(y, t),
\]

i.e.,

\[
e(y, t) := \sum_{a_k^- > 0} [C_0^-] l_k^- \left( \text{erf} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^- t}} \right) - \text{erf} \left( \frac{y - a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right)
\]

for \(y \leq 0\), and symmetrically for \(y \geq 0\).
Then, the solution $U$ of (2.1) satisfies
\begin{equation}
U(x, t) - \varphi(x, t) = \int_{-\infty}^{+\infty} (H + \tilde{G})(x, t; 0)U_0(y) \, dy,
\end{equation}
where
\begin{equation}
\tilde{G} := S + R
\end{equation}
is the regular part and $H$ the singular part of the time-decaying portion of the Green distribution $G$.

**Lemma 4.14 [MaZ.2].** In spatial dimension $d = 1$, under assumptions (A1)–(A3), (H0)–(H3), for Lax-type shock profiles satisfying the conditions of transversality, hyperbolic stability, and strong spectral stability, $\tilde{G}$ and $H$ satisfy
\begin{equation}
|\int_{-\infty}^{+\infty} \tilde{G} \cdot f(y) \, dy|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1/q - 1/p)}|f|_{L^q},
\end{equation}
\begin{equation}
|\int_{-\infty}^{+\infty} \tilde{G} \cdot f(y) \, dy|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1/q - 1/p) - 1/2}|f|_{L^q} + Ce^{-\eta t}|f|_{L^p},
\end{equation}
and
\begin{equation}
|\int_{-\infty}^{+\infty} H \cdot f(y) \, dy|_{L^p} \leq Ce^{-\eta t}|f|_{L^p},
\end{equation}
\begin{equation}
|\int_{-\infty}^{+\infty} H \cdot f(y) \, dy|_{W^{1,p}} \leq Ce^{-\eta t}|f|_{W^{1,p}},
\end{equation}
for all $t \geq 0$, some $C$, $\eta > 0$, for any $1 \leq q \leq p$ (equivalently, $1 \leq r \leq p$) and $f \in L^q \cap W^{1,p}$, where $1/r + 1/q = 1 + 1/p$.

**Proof.** Assuming $(P0)$, bounds (4.44)–(4.45) follow by the Hausdorff-Young inequality together with bounds (4.25) and the more stringent bounds on $R$ and its derivatives given in Proposition 4.39 below; see [MaZ.2–3] for further details. Bound (4.46) follows by direct computation and the fact that particle paths $z_j^*(x, t)$ satisfy uniform bounds
\[ 1/C \leq |(\partial/\partial x)z_j| < C, \]
for all $x$, $t$, by the fact that characteristic speeds $a_j(z)$ converge exponentially as $z \to \pm \infty$ to constant states. Finally, (4.47) follows from the relations
\[ \delta_{x-a_j^* t}(-y) = a_j^*(y)^{-1}\delta \left( t - \int_y^x \frac{dz}{a_j^*(z)} \right) \]
and
\[
\left( \frac{\partial}{\partial x} \right) \delta \left( t - \int_y^x \frac{dz}{a_j^*(z)} \right) = a_j^*(y)a_j^*(x)^{-1} \left( \frac{\partial}{\partial y} \right) \delta \left( t - \int_y^x \frac{dz}{a_j^*(z)} \right),
\]
where \( \bar{a}_j^* = \bar{a}_j^*(x,t) \) is defined as the average of \( a_j^*(z) \) along the backward characteristic \( z^*_j(x,t) \) defined by (4.26), which are special cases of the more general relations
\[
\delta h_{(x,t)}(y) = f_y(x,y,t)\delta(f(x,y,t))
\]
and
\[
(\partial/\partial x)\delta(f(x,y,t)) = (f_x/f_y)(\partial/\partial y)\delta(f(x,y,t)),
\]
where \( h(x,t) \) is defined by \( f(x,h(x,t),t) \equiv 0 \). Noting that the same modulus bounds, though not the pointwise description, persist also when (P0) fails, Remark 4.13, we obtain the general case as well. \( \Box \)

**Proof of Theorem 4.8.** It is equivalent to show that, for initial data \( U_0 \in L^1 \cap L^p \), the solution \( U(x,t) \) of (2.1) satisfies
\[
|U(\cdot,t) - \varphi(\cdot,t)|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}(|U_0|_{L^1} + |U_0|_{L^p}).
\]
But, this follows immediately from (4.42) and bounds (4.44) and (4.46), with \( q = p \). The estimate (4.11) gives sufficiency of structural, dynamical, and strong spectral stability for viscous linearized orbital stability. Necessity will be established by a separate, and somewhat simpler computation in Section 4.3.5; see Proposition 4.43 and Corollary 4.44. \( \Box \)

### 4.2.3 Auxiliary energy estimate.

Along with Proposition 4.10, the proof of Theorem 4.9 relies on the following auxiliary energy estimate, generalizing estimates of Kawashima [Kaw] in the constant–coefficient case. Following [MaZ.2], define nonlinear perturbation
\[
U(x,t) := \bar{U}(x + \delta(t),t) - \bar{U}(x),
\]
where the “shock location” \( \delta \) is to be determined later; for definiteness, fix \( \delta(0) = 0 \). Evidently, decay of \( U \) is equivalent to nonlinear orbital stability as described in (4.12).

**Proposition 4.15**[MaZ.4]. Under the hypotheses of Theorem 4.9 let \( U_0 \in H^3 \), and suppose that, for \( 0 \leq t \leq T \), both the supremum of \( |\delta| \) and the \( W^{2,\infty} \) norm of the solution \( U = (u^I, u^{II})^t \) of (1.1), (4.49) remain bounded by a sufficiently small constant \( \varsigma > 0 \). Then, for all \( 0 \leq t \leq T \),
\[
|U(t)|_{H^3}^2 \leq C|U(0)|_{H^3}^2 e^{-\varsigma t} + C \int_0^t e^{-\varsigma_2(t-\tau)}(|U|_{L^2}^2 + |\delta|^2(\tau)) d\tau.
\]
Remarks 4.16. 1. Estimate (4.50) asserts that the $H^3$ norm is controlled essentially by the $L^2$ norm, that is, it reveals strong hyperbolic-parabolic smoothing of the initial data. Weaker versions of this estimate may be found in [Kaw, MaZ.1–2, MaZ.4] (used there, as here, to control higher derivatives in the nonlinear iteration yielding time-asymptotic decay).

2. As described in [MaZ.2], $H^s$ energy estimates may for small-amplitude profiles be obtained essentially as a perturbation of the constant-coefficient case, using the fact that coefficients are in this case slowly varying; in particular, hypothesis (H1) is not needed in this case. In the large-amplitude case, there is a key new ingredient in the analysis beyond that of the constant-coefficient case, connected with the “upwind” hypothesis (H1)(ii) requiring that hyperbolic convection rates be of uniform sign relative to the shock speed $s$. This allows us to control terms of form $R \mathcal{O}(|\bar{U}_x|)|\nu|^2$ that were formerly controlled by the small-amplitude assumption $|\bar{U}_x| = \mathcal{O}(\varepsilon), \varepsilon := |\bar{U}_- - \bar{U}_+| << 1$, instead by a “Goodman-type” weighted norm estimate in the spirit of [Go.1–2], thus closing the argument.

Proof of Proposition 4.15. A straightforward calculation shows that $|U|_{H^r} \sim |W|_{H^r},$

\begin{equation}
W = \tilde{W} - \bar{W} := W(\bar{U}) - W(\tilde{U}),
\end{equation}

for $0 \leq r \leq 3$ provided $|U|_{W^{2,\infty}}$ remains bounded, hence it is sufficient to prove a corresponding bound in the special variable $W$. We first carry out a complete proof in the more straightforward case that the equations may be globally symmetrized to exact form (1.7), i.e., with conditions (A1)–(A3) replaced by the following global versions, indicating afterward by a few brief remarks the changes needed to carry out the proof in the general case.

\begin{enumerate}
\item[(A1')] $\tilde{A}^i, \tilde{A}^{ik} := \tilde{A}_{11}^i, \tilde{A}_0^i > 0.$
\item[(A2')] No eigenvector of $\sum \xi_j dF^j(U)$ lies in the kernel of $\sum \xi_j \xi_k B^{jk}(U)$, for all nonzero $\xi \in \mathbb{R}^d$. (Equivalently, no eigenvector of $\sum \xi_j \tilde{A}^i(\tilde{A}_0^i)^{-1}(W)$ lies in the kernel of $\sum \xi_j \xi_k \tilde{B}^{jk}(W)$.)
\item[(A3')] $\tilde{B}^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}$, with $\text{Re} \sum \xi_j \xi_k \tilde{b}^{jk}(W) \geq \theta |\xi|^2$ for some $\theta > 0$, for all $W$ and all $\xi \in \mathbb{R}^d$, and $\bar{G} \equiv 0.$
\end{enumerate}

Substituting (4.49) into (1.11), we obtain the quasilinear perturbation equation

\begin{equation}
(\tilde{A}_0^0 \tilde{W}_t - A_0^0 \bar{W}_t) + (\tilde{A} \tilde{W}_x - A \bar{W}_x) - (\tilde{B} \tilde{W}_x - \bar{B} \bar{W}_x)_x = \delta(t) \tilde{A}_0^0 \tilde{W}_x,
\end{equation}

where

\begin{equation}
\tilde{A}_0^0 := A_0^0(\tilde{W}), A_0^0 := A_0^0(\bar{W}); \quad \tilde{A} := A(\tilde{W}), \bar{A} := A(\bar{W}); \quad \tilde{B} := B(\tilde{W}), \bar{B} := B(\bar{W}),
\end{equation}
using the quadratic Leibnitz relation
\begin{equation}
4.54 \quad A_2 W_2 - A_1 W_1 = A_2(W_2 - W_1) + (A_2 - A_1) W_1,
\end{equation}
and recalling the block structure assumption (A3), we obtain the alternative perturbation equation:
\begin{equation}
4.55 \quad \tilde{A}^0 W_t + \tilde{A} W_x - (\tilde{B} W_x)_x = M_1 \tilde{W}_x + (M_2 \tilde{W}_x)_x + \delta(t) \tilde{A}^0 W_x + \delta(t) \tilde{A}^0 \tilde{W}_x,
\end{equation}
where
\begin{equation}
4.56 \quad M_1 = M_1(W, \tilde{W}) := \tilde{A} - \hat{A} = \left( \int_0^1 dA(\tilde{W} + \theta W) d\theta \right) W,
\end{equation}
and
\begin{equation}
4.57 \quad M_2 = M_2(W, \tilde{W}) := \tilde{B} - \hat{B} = \left( \begin{array}{cc}
0 & 0 \\
0 & (\int_0^1 db(\tilde{W} + \theta W) d\theta) W
\end{array} \right).
\end{equation}

We now carry out a series of successively higher order energy estimates of the type formalized by Kawashima [Kaw]. The origin of this approach goes back to [Kan,MNi] in the context of gas dynamics; see, e.g., [HoZ.1] for further discussion/references. The novelty here is that the stationary background state \( \tilde{W} \) is not constant in \( x \), nor even slowly varying as in [MaZ.2]. We overcome this difficulty by introducing the additional ingredient of a Goodman-type weighted norm in order to control hyperbolic modes, assumed for this purpose (see hypothesis (H1)(ii)) to be uniformly transverse to the background shock profile.

Let \( \tilde{K} \) denote the skew-symmetric matrix described in Lemma 2.18 associated with \( \tilde{A}^0, \tilde{A}, \tilde{B} \), satisfying
\[ \tilde{K}(\tilde{A}^0)^{-1} \tilde{A} + \tilde{B} > 0. \]

Then, regarding \( \tilde{A}^0, \tilde{K}, \) we have the bounds
\begin{equation}
4.58 \quad \tilde{A}^0_x = dA^0(\tilde{W}) \tilde{W}_x, \quad \tilde{K}_x = dK(\tilde{W}) \tilde{W}_x, \quad \tilde{A}_x = dA(\tilde{W}) \tilde{W}_x, \quad \tilde{B}_x = dB(\tilde{W}) \tilde{W}_x,
\end{equation}
\[ \tilde{A}^0_t = dA^0(\tilde{W}) \tilde{W}_t, \quad \tilde{K}_t = dK(\tilde{W}) \tilde{W}_t, \quad \tilde{A}_t = dA(\tilde{W}) \tilde{W}_t, \quad \tilde{B}_x = dB(\tilde{W}) \tilde{W}_t. \]

and (from the defining equations):
\begin{equation}
4.59 \quad |\tilde{W}_x| = |W_x + \tilde{W}_x| \leq |W_x| + |\tilde{W}_x|
\end{equation}
and
\begin{equation}
4.60 \quad |\tilde{W}_t| \leq C(|\tilde{W}_x| + |\tilde{W}_{xx}| + |\delta||\tilde{W}_x|) \leq C(|W_x| + |\tilde{W}_x| + |\tilde{W}_{xx}| + |\delta||W_x| + |\delta||\tilde{W}_x|) \leq C(|W_x| + |\tilde{W}_x| + |\tilde{W}_{xx}|).\]
Thus, in particular
\begin{equation}
|\dot{\delta}|, |\tilde{A}_0^0|, |\tilde{A}_{xx}|, |\tilde{K}_x|, |\tilde{K}_{xx}|, |\tilde{A}_x|, |\tilde{A}_{xx}|, |\tilde{B}_x|, |\tilde{B}_{xx}|, |\tilde{A}_t|, |\tilde{K}_t|, |\tilde{A}_t|, |\tilde{B}_t| \\
\leq C(\zeta + |\tilde{U}_x|).
\end{equation}

Finally, we introduce the weighted norms and inner product
\begin{equation}
|f|_\alpha := |\alpha^{1/2} f|_{L^2}, \quad |f|_{H^s_\alpha} := \sum_{r=0}^s |\partial_x^r f|_\alpha, \quad \langle f, g \rangle_\alpha := \langle \alpha f, g \rangle_{L^2},
\end{equation}
\noindent $\alpha(x)$ scalar, uniformly positive, and uniformly bounded. For the remainder of this section, we shall for notational convenience omit the subscript $\alpha$, referring always to $\alpha$-norms or -inner products unless otherwise specified. For later reference, we note the commutator relation
\begin{equation}
\langle f, g_x \rangle = -\langle f_x + (\alpha_x/\alpha) f, g \rangle,
\end{equation}
and the related identities
\begin{equation}
\langle f, S f_x \rangle = -(1/2) \langle f, (S_x + (\alpha_x/\alpha) S) f \rangle,
\end{equation}
\begin{equation}
\langle f, (S f)_x \rangle = (1/2) \langle f, (S_x - (\alpha_x/\alpha) S) f \rangle,
\end{equation}
valid for symmetric operators $S$.

By (H1)(ii), we have that $\bar{A}_{11}(\bar{A}_{11}^0)^{-1}$ has real spectrum of uniform sign, without loss of generality negative, so that the similar matrix
\begin{equation}
(\bar{A}_{11}^0)^{-1/2} \bar{A}_{11}(\bar{A}_{11}^0)^{-1/2} = (\bar{A}_{11}^0)^{-1/2} \bar{A}_{11}(\bar{A}_{11}^0)^{-1}(\bar{A}_{11}^0)^{1/2}
\end{equation}
has real, negative spectrum as well. (Recall, $\bar{A}_{11}^0$ is symmetric negative definite as a principal minor of the symmetric negative definite matrix $\bar{A}^0$.) It follows that $\bar{A}_{11}$ is uniformly symmetric negative definite, i.e.,
\begin{equation}
\bar{A}_{11} \leq -\theta < 0.
\end{equation}

Defining $\alpha$, following Goodman [Go.2], by the ODE
\begin{equation}
\alpha_x = C_*|\tilde{U}_x|\alpha, \quad \alpha(0) = 1,
\end{equation}
where $C_* > 0$ is a large constant to be chosen later, we have by (4.66)
\begin{equation}
(\alpha_x/\alpha) \bar{A}_{11} \leq -C_*\theta|\tilde{U}_x|.$
Young’s inequality, the energy estimate rearrangement, integration by parts using (4.63)–(4.64), and several applications of systems [Fri]. Taking the zeroth- and first-order “Friedrichs-type” estimate for symmetrizable hyperbolic systems [Fri] and expanding $\tilde{A} = A + \mathcal{O}(\zeta)$, $\tilde{A}^x = \mathcal{O}(|\tilde{U}^x| + \zeta)$, we obtain by (4.68) the key property
\begin{equation}
\langle W, \tilde{A}W_x \rangle = (1/2) \langle u, (\alpha_x/\alpha)\tilde{A}11u \rangle + \mathcal{O}\left( |\tilde{U}_x||W|, |W|, (\alpha_x/\alpha)|W|, |\zeta W| + |w^I| \right)
\end{equation}
(4.70)
\begin{equation}
\leq -(C_\ast /3)|w^I|, |\tilde{W}_x||w^I| + C\zeta|w^I|^2 + C(C_\ast)|w^I|^2,
\end{equation}
by which we shall control transverse modes, provided $C_\ast$ is chosen sufficiently large, or, more generally,
\begin{equation}
\langle \partial^k_x W, \tilde{A} \partial^k_x W_x \rangle \leq -(C_\ast \theta /3)|\partial^k_x u|, |\tilde{U}_x||\partial^k_x u| + C\zeta|\partial^k_x u|^2 + C(C_\ast)|\partial^k_x u|^2.
\end{equation}
Here and below, $C(C_\ast)$ denotes a suitably large constant depending on $C_\ast$, while $C$ denotes a fixed constant independent of $C_\ast$; likewise, $\mathcal{O}(\cdot)$ indicates a bound independent of $C_\ast$.

**Zeroth order “Friedrichs-type” estimate.** We first perform a standard, zeroth- and first-order “Friedrichs-type” estimate for symmetrizable hyperbolic systems [Fri]. Taking the $\alpha$-inner product of $W$ against (4.55), we obtain after rearrangement, integration by parts using (4.63)–(4.64), and several applications of Young’s inequality, the energy estimate
\begin{equation}
\frac{1}{2} \langle W, \tilde{A}^0 W_t \rangle = \langle W, \tilde{A}^0 W_t \rangle + \frac{1}{2} \langle W, \tilde{A}^0_t W \rangle
\end{equation}
\begin{equation}
= -\langle W, \tilde{A}W_x \rangle + \langle W, (\tilde{B}W)_x \rangle + \langle W, M_1 \tilde{U} \rangle + \langle W, (M_2 \tilde{W})_x \rangle
\end{equation}
\begin{equation}
+ \hat{\delta}(t) \langle W, \tilde{A}^0 W_x \rangle + \hat{\delta}(t) \langle W, \tilde{A}^0 \tilde{U}_x \rangle + \frac{1}{2} \langle W, \tilde{A}^0_t W \rangle
\end{equation}
\begin{equation}
= \frac{1}{2} \langle W, (\tilde{A}_x + (\alpha_x/\alpha)\tilde{A})W \rangle - \langle W_x - (\alpha_x/\alpha)W, \tilde{B}W \rangle + \langle W, M_1 \tilde{U} \rangle
\end{equation}
\begin{equation}
- \langle W_x + (\alpha_x/\alpha)W, M_2 \tilde{U}_x \rangle - \frac{1}{2} \hat{\delta}(t) \langle W, (\tilde{A}_x^0 + (\alpha_x/\alpha)\tilde{A}^0)W \rangle
\end{equation}
\begin{equation}
+ \hat{\delta}(t) \langle W, \tilde{A}^0 \tilde{U}_x \rangle + \frac{1}{2} \langle W, \tilde{A}^0_t W \rangle
\end{equation}
\begin{equation}
\leq -\langle W_x, \tilde{B}W \rangle + C(C_\ast) \int \alpha \left( |W_x| + |\tilde{U}_x| \right) |W|^2
\end{equation}
\begin{equation}
+ |w^I_x||W||(|W_x| + |\tilde{U}_x|) + |\hat{\delta}|W||\tilde{U}_x| \right)
\end{equation}
\begin{equation}
\leq -\theta|v_x|^2 + C(C_\ast) \left( |W|^2_x + |\hat{\delta}|^2 \right).
Here, we have freely used the bounds (2.18), as well as (4.61). We have also used in a crucial way the block-diagonal form of $M_2$ in estimating $|\langle W_x, M_2 \bar{U}_x \rangle| \leq C \int |v_x||W||\bar{U}_x|$ in the first inequality.

**First order “Friedrichs-type” estimate.** For first and higher derivative estimates, it is crucial to make use of the favorable terms (4.71) afforded by the introduction of $\alpha$-weighted norms. Differentiating (4.55) with respect to $x$, taking the $\alpha$-inner product of $W_x$ against the resulting equation, and substituting the result into the first term on the righthand side of (4.73)

\[
\frac{1}{2} \langle W_x, \tilde{A}^0 W_x \rangle_t = \langle W_x, (\tilde{A}^0 W_t)_x \rangle - \langle W_x, (\tilde{A}_x^0 W_t)_x \rangle + \frac{1}{2} \langle W_x, \tilde{A}^0_t W_x \rangle,
\]

we obtain after various simplifications and integrations by parts:

(4.74)  
\[
\frac{1}{2} \langle W_x, \tilde{A}^0 W_x \rangle_t = -\langle W_x, (\tilde{A} W_x)_x \rangle + \langle W_x, (\tilde{B} W_x)_{xx} \rangle + \langle W_x, (M_1 \bar{W}_x)_x \rangle \\
+ \langle W_x, (M_2 \bar{W}_x)_{xx} \rangle + \tilde{\delta}(t) \langle W_x, (\tilde{A}^0 W_x)_x \rangle \\
+ \tilde{\delta}(t) \langle W_x, (\tilde{A}^0 W_x)_x \rangle - \langle W_x, (\tilde{A}^0 W_t)_x \rangle + \frac{1}{2} \langle W_x, \tilde{A}^0 W_x \rangle \\
= -\langle W_x, \tilde{A} W_{xx} \rangle - \langle W_x, \tilde{A}_x W_x \rangle - \langle W_{xx} + (\alpha_x/\alpha) W_x, \tilde{B} W_{xx} + \tilde{B}_x W_x \rangle \\
+ \langle W_x, (M_1 \bar{U}_x)_x \rangle - \langle W_{xx} + (\alpha_x/\alpha) W_x, (M_2 \bar{U}_x)_x \rangle \\
+ \frac{1}{2} \tilde{\delta}(t) \langle W_x, (\tilde{A}^0 - (\alpha_x/\alpha) \tilde{A}^0) W_x \rangle + \tilde{\delta}(t) \langle W_x, \tilde{A}^0 \bar{W}_x \rangle \\
+ \tilde{\delta}(t) \langle W_x, \tilde{A}^0 \bar{W}_{xx} \rangle - \langle W_x, \tilde{A}^0_t W_t \rangle + \frac{1}{2} \langle W_x, \tilde{A}^0_t W_x \rangle.
\]

Estimating the first term on the righthand side of (4.74) using (4.71), $k = 1$, and substituting $(\tilde{A}^0)^{-1}$ times (4.55) into the second to last term on the righthand side of (4.74), we obtain by (4.61) plus various applications of Young’s inequality the
next-order energy estimate:

\[ \frac{1}{2} \langle W_x, \tilde{A}^0 W_x \rangle_t \leq -\langle W_x, \tilde{A} W_{xx} \rangle - \langle W_{xx}, \tilde{B} W_{xx} \rangle \]

\[ + C(C_*) \int (|W|^2 + |\delta|^2)(|\tilde{U}_{xx}| + |\tilde{U}_x|) \]

\[ + C(C_*)(|W|^2_{L^2} + |\tilde{U}|^2_{L^2}) \]

\[ + C(C_*)(|w_{xx}^I| + \zeta |W_x|, (|W| + |W_x|)|\tilde{U}_x| + |w_{xx}^I|) \]

\[ + C(C_*)\int (|W_x| + |w_{xx}^I|), |\tilde{U}_x||(|W| + |W_x|)) \]

\[ \leq -\theta |w_{xx}^I|^2 - (C_* \theta/3)\langle |\tilde{U}_x||w_x^I|, |w_x^I| \rangle + C\zeta |w_x^I|^2 + C(C_*)|w_x^I|^2 \]

\[ + C(C_*)\left( (|W|^2_{L^2} + |\delta|^2) \right) \]

\[ + C(C_*)\left( |v_x| + \zeta |W_x|, (|W| + |W_x|)|\tilde{U}_x| + |w_{xx}^I| \right) \]

\[ + C(C_*)\langle (|W_x| + |w_{xx}^I|), |\tilde{U}_x||(|W| + |W_x|) \rangle \]

\[ \leq -\left( \frac{\theta}{2} \right) |w_{xx}^I|^2 - (C_* \theta/4)\langle |\tilde{W}_x||w_x^I|, |w_x^I| \rangle + C(C_*)\zeta |w_x^I|^2 + C(C_*)|w_x^I|^2 \]

\[ + C(C_*)\left( (|W|^2_{L^2} + |\delta|^2) \right), \]

provided \( C_* \) is sufficiently large and \( \zeta \) sufficiently small.

**First order “Kawashima-type” estimate.** Next, we perform a “Kawashima-

\[ \text{type” derivative estimate. Taking the } \alpha \text{-inner product of } W_x \text{ against } \tilde{K}(\tilde{A}^0)^{-1} \text{ times (4.55), and noting that (integrating by parts, and using skew-symmetry of } \tilde{K}) \]

\[ \frac{1}{2} \langle W_x, \tilde{K} W \rangle_t = \frac{1}{2} \langle W_x, \tilde{K} W \rangle + \frac{1}{2} \langle W_{xt}, \tilde{K} W \rangle + \frac{1}{2} \langle W_x, \tilde{K}_t W \rangle \]

\[ = \frac{1}{2} \langle W_x, \tilde{K} W \rangle - \frac{1}{2} \langle W_{t}, \tilde{K} W_x \rangle \]

\[ - \frac{1}{2} \langle W_{t}, (\alpha_x/\alpha) W \rangle + \frac{1}{2} \langle W_x, \tilde{K}_t W \rangle \]

\[ = \langle W_x, \tilde{K} W \rangle + \frac{1}{2} \langle W, (\tilde{K}_x + (\alpha_x/\alpha)) W_t \rangle + \frac{1}{2} \langle W_x, \tilde{K}_t W \rangle, \]

we obtain by calculations similar to the above the auxiliary energy estimate:

\[ \frac{1}{2} \langle W_x, \tilde{K} W \rangle_t \leq -\langle W_x, \tilde{K}(\tilde{A}^0)^{-1} \tilde{A} W_x \rangle \]

\[ + C(C_*)|w_x^I|^2 + C(|\tilde{U}_x| + \zeta + \tilde{\zeta})|w_x^I|, |w_x^I| \]

\[ + C\tilde{\zeta}^{-1}|w_{xx}^I|^2 + C(C_*)|W|^2_{L^2} + |\delta(t)|^2, \]

where \( \tilde{\zeta} > 0 \) is an arbitrary constant arising through Young’s inequality. (Here, we have estimated term \( \langle A U_{xx}, (\alpha_x/\alpha) U \rangle \) arising in the middle term of the righthand side of (4.76) using (4.64) by \( C(C_*) \int |\tilde{U}_x||U|^2 \leq C(C_*)|U|^2_{L^\infty}. \)
Combined, weighted $H^1$ estimate. Choosing $\zeta << \bar{\zeta} << 1$, adding (4.77) to the sum of (4.72) and (4.75) times a suitably large positive constant $M(C_*, \bar{\zeta}) >> \bar{\zeta}^{-1}$, and recalling (2.53), we obtain, finally, the combined first-order estimate

$$
\frac{1}{2} \left( M(C_*, \bar{\zeta}) \langle W, \bar{A}^0 W \rangle + \langle W_x, \bar{K} W \rangle + M(C_*, \bar{\zeta}) \langle W_x, \bar{A}^0 W_x \rangle \right)_t \\
\leq -\theta (|W_x|^2 + |w_{xx}^I|^2) + C(C_*) \left( |W|_{L^2}^2 + |\delta|^2 \right),
$$

(4.78)

$\theta > 0$, for any $\bar{\zeta}, \zeta(\bar{\zeta}, C_*)$ sufficiently small, and $C_*, C(C_*)$ sufficiently large.

Higher order estimates. Performing the same procedure on the twice- and thrice-differentiated versions of equation (4.55), we obtain, likewise, Friedrichs estimates

$$
\frac{1}{2} \langle \partial_x^q W, \bar{A}^0 \partial_x^q W \rangle_t \leq -\langle (\bar{\zeta}/2) |\partial_x^{q+1} w_{II}|^2 - (C_*/4) \langle |\bar{W}_x| |\partial_x^q w_I|, |\partial_x^q w_I| \rangle \\
+ C(C_*) (\zeta |\partial_x^q w_I|^2 + |\partial_x^q w_{II}|^2 + |W_x|_{H^2}^2 + |W|_{L^2}^2 + |\delta(t)|^2),
$$

(4.79)

and Kawashima estimates

$$
\frac{1}{2} \langle \partial_x^q W, \bar{K} \partial_x^q W \rangle_t \leq -\langle \partial_x^q W, \bar{K} (\bar{A}^0)^{-1} \bar{A} \partial_x^q W \rangle \\
+ C(C_*) |\partial_x^q w_{II}|^2 + C((|\bar{W}_x| + \bar{\zeta} + \zeta) |\partial_x^q u|, |\partial_x^q u|) \\
+ C \bar{\zeta}^{-1} |\partial_x^{q+1} w_{II}|^2 + C(C_*) (|W_x|_{H^2}^2 + |W|_{L^2}^2 + |\delta(t)|^2),
$$

(4.80)

for $q = 2, 3$, provided $\bar{\zeta}, \zeta(\bar{\zeta}, C_*)$ are sufficiently small, and $C_*, C(C_*)$ are sufficiently large. The calculations are similar to those carried out already; see also the closely related calculations of Appendix A, [MaZ.2].

Final estimate. Adding $M(C_*, \bar{\zeta})^2$ times (4.78), $M(C_*, \bar{\zeta})$ times (4.79), and (4.80), with $q = 2$, where $M$ is chosen still larger if necessary, we obtain

$$
\frac{1}{2} \left( M(C_*, \bar{\zeta})^2 \langle W, \bar{A}^0 W \rangle + M(C_*, \bar{\zeta})^2 \langle W_x, \bar{K} W \rangle + M(C_*, \bar{\zeta})^2 \langle W_x, \bar{A}^0 W_x \rangle \\
+ \langle \partial_x^2 W, \bar{K} \partial_x W \rangle + M(C_*, \bar{\zeta}) \langle \partial_x^2 W, \bar{A}^0 \partial_x^2 W \rangle \right)_t \\
\leq -\theta (|W_x|_{H^1}^2 + |w_{xx}^I|_{H^2}^2) + C(C_*) \left( |W|_{L^2}^2 + |\delta|^2 \right).
$$

(4.81)

Adding now $M(C_*, \bar{\zeta})^2$ times (4.81), $M(C_*, \bar{\zeta})$ times (4.79), and (4.80), with
\( q = 3 \), we obtain the final higher-order estimate
\[
\frac{1}{2} \left( M(C_*, \tilde{\zeta})^5 \langle W, \tilde{A}^0 W \rangle + M(C_*, \tilde{\zeta})^4 \langle W_x, \tilde{K} W \rangle + M(C_*, \tilde{\zeta})^5 \langle W_x, \tilde{A}^0 W_x \rangle \\
+ M(C_*, \tilde{\zeta})^2 \langle \partial_x^2 W, \tilde{K} \partial_x W \rangle + M(C_*, \tilde{\zeta})^3 \langle \partial_x^3 W, \tilde{A}^0 \partial_x^2 \rangle \\
+ \langle \partial_x^3 W, \tilde{K} \partial_x^2 W \rangle + M(C_*, \tilde{\zeta}) \langle \partial_x^2 W, \tilde{A}^0 \partial_x^2 \rangle \right) t \\
\leq -\theta (|W_x|^2_{H^3_\alpha} + |w_x^{II}|^3_{H^2_\alpha}) + C(C_*) \left( |W|^2_{L^2} + |\dot{\delta}|^2 \right).
\]

Denoting
\[
E(W) := \frac{1}{2} \left( M(C_*, \tilde{\zeta})^5 \langle W, \tilde{A}^0 W \rangle + M(C_*, \tilde{\zeta})^4 \langle W_x, \tilde{K} W \rangle + M(C_*, \tilde{\zeta})^5 \langle W_x, \tilde{A}^0 W_x \rangle \\
+ M(C_*, \tilde{\zeta})^2 \langle \partial_x^2 W, \tilde{K} \partial_x W \rangle + M(C_*, \tilde{\zeta})^3 \langle \partial_x^3 W, \tilde{A}^0 \partial_x^2 \rangle \\
+ \langle \partial_x^3 W, \tilde{K} \partial_x^2 W \rangle + M(C_*, \tilde{\zeta}) \langle \partial_x^2 W, \tilde{A}^0 \partial_x^2 \rangle \right),
\]
we have by Young’s inequality that \( E^{1/2} \) is equivalent to norms \( H^3 \) and \( H^3_\alpha \), hence (4.78) yields
\[
E_t \leq -\theta_2 E + C(C_*) \left( |W|^2_{L^2} + |\dot{\delta}|^2 \right).
\]

Multiplying by integrating factor \( e^{-\theta_2 t} \), and integrating from 0 to \( t \), we thus obtain
\[
E(t) \leq e^{-\theta_2 t} E(0) + C(C_*) \int_0^t e^{-\theta_2 s} \left( |W|^2_{L^2} + |\dot{\delta}|^2 \right) (s) \, ds.
\]

Multiplying by \( e^{-\theta_2 t} \), and using again that \( E^{1/2} \) is equivalent to \( H^3 \), we obtain the result.

The general case. It remains only to discuss the general case that hypotheses (A1)–(A3) hold as stated and not everywhere along the profile, with \( \tilde{G} \) possibly nonzero. These generalizations requires only a few simple observations. The first is that we may express matrix \( \tilde{A} \) in (4.55) as
\[
(4.83) \quad \tilde{A} = \hat{A} + (|\tilde{U}_x| + \zeta) \begin{pmatrix} 0 & O(1) \\
O(1) & O(1) \end{pmatrix},
\]
where \( \hat{A} \) is a symmetric matrix obeying the same derivative bounds as described for \( \tilde{A} \), identical to \( A \) in the 11 block and obtained in other blocks \( j,k \) by smoothly interpolating over a bounded interval \([-R, +R]\) between \( \tilde{A}(\tilde{W}_{-\infty})_{jk} \) and \( \tilde{A}(\tilde{W}_{+\infty})_{jk} \). Replacing \( \tilde{A} \) by \( \hat{A} \) in the qth order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as (integrating by parts if necessary)
\[
\langle \partial_x^q O(|\tilde{U}_x| + \zeta) |W|, |\partial_x^{q+1} w^{II}| \rangle
\]
plus lower-order terms, hence absorbed using Young’s inequality to recover the same
Friedrichs-type estimates obtained in the previous case. Thus, we may relax (A1')
to (A1).

The second observation is that, because of the favorable terms

\[-(C_\epsilon \theta/4) (|\bar{U}_x| |\partial_x^2 w|^2, |\partial_x^2 w|^2)\]

occurring in the righthand sides of the Friedrichs-type estimates, we need the
Kawashima-type bound only to control the contribution to $|\partial_x^2 w|^2$ coming from
$x$ near $\pm \infty$; more precisely, we require from this estimate only a favorable term

\[(4.84)\]

\[-\theta (\langle 1 - \mathcal{O}(|\bar{U}_x| + \zeta + \bar{\zeta}) \rangle |\partial_x^2 w|^2, |\partial_x^2 w|^2)\]

rather than $-\theta |\partial_x^2 w|^2$ as in (4.77) and (4.80). But, this may easily be obtained
by substituting for $\tilde{K}$ a skew-symmetric matrix-valued function $\hat{K}$ defined to be
identically equal to $\bar{K}(+\infty)$ and $\bar{K}(-\infty)$ for $|x| > R$, and smoothly interpolating
between $\bar{K}(\pm \infty)$ on $[-R, +R]$, and using the fact that

\[(\bar{K}(A^0)^{-1} \tilde{A} + \tilde{B})_{\pm} \geq \theta > 0,\]

hence

\[(\hat{K}(\hat{A}^0)^{-1} \hat{A} + \hat{B}) \geq \theta (1 - \mathcal{O}(|\bar{U}_x| + \zeta)).\]

Thus, we may relax (A2') to (A2).

Finally, notice that the term $\tilde{G} - \hat{G}$ in the perturbation equation may be Taylor
expanded as

\[(4.85)\]

\[
\begin{pmatrix}
0 \\
n(\bar{W}_x, \bar{U}_x) + g(\bar{U}_x, \bar{W}_x) \\
\mathcal{O}(|W_x|^2)
\end{pmatrix}
\]

The first, linear term on the righthand side may be grouped with term $\hat{A}^0 W_x$
and treated in the same way, since it decays at plus and minus spatial infinity
and vanishes in the 1-1 block. The $(0, \mathcal{O}(|W_x|^2)$ nonlinear term may be treated
as other source terms in the energy estimates Specifically, the worst-case terms
$\langle \partial_x^3 W, K \partial_x^2 \mathcal{O}(|W_x|^2) \rangle$ and $\langle \partial_x^3 W, \partial_x^2 \mathcal{O}(0, \mathcal{O}(|W_x|^2)) \rangle = \langle \partial_x^3 w^{11}, K \partial_x^2 \mathcal{O}(|W_x|^2) \rangle$ may be bounded, respectively, by $|W|_{W^{2, \infty}} |W|_{H3}$ and $|W|_{W^{2, \infty}} |w^{11}|_{H4} |W|_{H3}$. Thus, we
may relax (A3') to (A3), completing the proof of the general case (A1)–(A3) and
the theorem. \(\square\)

**Remark 4.17.** Given $\hat{A}^i, \hat{B}^{jk}, \hat{G} \in C^{2r}$, we may obtain by the same argument
a corresponding result in $H^{2r-1}$ or $H^{2r}$, $r \geq 1$, assuming that $W^{r, \infty}$ remains
sufficiently small.
4.2.4 Nonlinear stability.

We are now ready to carry out our proof of nonlinear stability. We give a simplified version of the basic iteration scheme of [MaZ.1–2, Z.6, Z.4]; for precursors of this scheme, see [Go.2, K.1–2, LZ.1–2, ZH, HZ.1–2]. For this stage of the argument, it will be convenient to work again with the conservative variable

\[ U := \tilde{U} - \bar{U}, \]

writing (4.52) in the more standard form:

\[ U_t - LU = Q(U, U_x) + \dot{\delta}(t)(\bar{U}_x + U_x), \]

where

\[ Q(U, U_x) = O(|U|^2 + |U||U_x|), \]

\[ Q(U, U_x)_x = O(|U|^2 + |U_x|^2 + |U||U_{xx}|), \]

so long as \(|U|, |U_x|\) remain bounded.

By Duhamel’s principle, and the fact that

\[ \int_{-\infty}^{\infty} G(x, t; y)\bar{U}_x(y)dy = e^{Lt}\bar{U}_x(x) = \bar{U}_x(x), \]

we have, formally,

\[
U(x, t) = \int_{-\infty}^{\infty} G(x, t; y)U_0(y)dy \\
+ \int_{0}^{t} \int_{-\infty}^{\infty} G_y(x, t-s; y)(Q(U, U_x) + \dot{\delta}(y, s))dyds \\
+ \delta(t)\bar{U}_x.
\]

Defining, by analogy with the linear case, the nonlinear instantaneous projection:

\[ \phi(x, t) := -\delta(t)\bar{U}_x \\
:= \int_{-\infty}^{\infty} E(x, t; y)U_0(y)dy \\
- \int_{0}^{t} \int_{-\infty}^{\infty} E_y(x, t-s; y)(Q(U, U_x) + \dot{\delta}(y, s))dy,
\]

or equivalently, the instantaneous shock location:

\[ \delta(t) = -\int_{-\infty}^{\infty} e(y, t)U_0(y)dy \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} e_y(y, t-s)(Q(U, U_x) + \dot{\delta}(y, s))dyds,
\]
where $E, e$ are defined as in (4.24), (4.41), and recalling (4.43), we thus obtain the reduced equations:

$$U(x, t) = \int_{-\infty}^{\infty} (H + \tilde{G})(x, t; y)U_0(y)dy$$

$$+ \int_0^t \int_{-\infty}^{\infty} H(x, t - s; y)(Q(U, U_x)_x + \delta U_x)(y, s)dy ds$$

$$- \int_0^t \int_{-\infty}^{\infty} \tilde{G}_x(x, t - s; y)(Q(U, U_x) + \delta U)(y, s)dy ds,$$

and, differentiating (4.92) with respect to $t$,

$$\dot{\delta}(t) = - \int_{-\infty}^{\infty} e_t(y, t)U_0(y)dy$$

$$+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y, t - s)(Q(U, U_x) + \delta U)(y, s)dy ds.$$

Note: In deriving (4.94), we have used the fact that $e_y(y, s) \to 0$ as $s \to 0$, as the difference of approaching heat kernels, in evaluating the boundary term

$$\int_{-\infty}^{+\infty} e_y(y, 0)(Q(U, U_x) + \delta U)(y, t)dy = 0.$$

(Indeed, $|e_y(\cdot, s)|_{L^1} \to 0$; see Remark 2.6, below).

The defining relation $\delta(t)\bar{u}_x := -\varphi$ in (4.91) can be motivated heuristically by

$$\tilde{U}(x, t) - \varphi(x, t) \sim U = U(x + \delta(t), t) - \bar{U}(x)$$

$$\sim \tilde{U}(x, t) + \delta(t)\bar{U}_x(x),$$

where $\tilde{U}$ denotes the solution of the linearized perturbation equations, and $\bar{U}$ the background profile. Alternatively, it can be thought of as the requirement that the instantaneous projection of the shifted (nonlinear) perturbation variable $U$ be zero; see [HZ.1–2].

**Lemma 4.18 [Z.6].** The kernel $e$ satisfies

$$|e_y(\cdot, t)|_{L^p}, |e_t(\cdot, t)|_{L^p} \leq Ct^{-\frac{1}{2}(1-1/p)},$$

$$|e_{yt}(\cdot, t)|_{L^p} \leq Ct^{-\frac{1}{2}(1-1/p)-1/2},$$

for all $t > 0$. Moreover, for $y \leq 0$ we have the pointwise bounds

$$|e_y(y, t)|, |e_t(y, t)| \leq Ct^{-\frac{1}{2}} \sum_{\alpha_k > 0} \left(e^{-\frac{(y+a_k^{-1}t)^2}{Mt}} + e^{-\frac{(y-a_k^{-1}t)^2}{Mt}}\right),$$
\[(4.99) \quad |e_{ty}(y,t)| \leq Ct^{-1} \sum_{a_k > 0} \left( e^{-\frac{(y+a^{-1}_k t)^2}{Mt}} + e^{-\frac{(y-a^{-1}_k t)^2}{Mt}} \right), \]

for \( M > 0 \) sufficiently large, and symmetrically for \( y \geq 0 \).

Proof. For definiteness, take \( y \leq 0 \). Then, (4.41) gives

\[(4.100) \quad e_y(y,t) := \sum_{a_k > 0} [c^0_{k,-}]_k^{-t} (K(y + a_k^- t, t, \beta^-_k) - K(y - a_k^- t, t, \beta^-_k)) \]

\[(4.101) \quad e_t(y,t) := \sum_{a_k > 0} [c^0_{k,-}]_k^{-t} ((K + K_y)(y + a_k^- t, t, \beta^-_k) - (K + K_y)(y - a_k^+ t, t, \beta^-_k)) ,\]

\[(4.102) \quad e_{ty}(y,t) := \sum_{a_k > 0} [c^0_{k,-}]_k^{-t} ((K_y + K_{yy})(y + a_k^- t, t) - (K_y + K_{yy})(y - a_k^- t, t, \beta^-_k)) ,\]

where

\[(4.103) \quad K(y, t, \beta) := \frac{e^{-\frac{y^2}{4\beta t}}}{\sqrt{4\pi\beta t}}\]

denotes a heat kernel with diffusion coefficient \( \beta \). The pointwise bounds (4.98)–(4.99) follow immediately for \( t \geq 1 \) by properties of the heat kernel, in turn yielding (4.96)–(4.97) in this case. The bounds for small time \( t \leq 1 \) follow from estimates (4.104)

\[
|K_y(y + at, t, \beta) - K_y(y - at, t, \beta)| = \left| \int_{y+at}^{y-at} K_{yy}(z, t, \beta) \, dz \right| \\
\leq Ct^{-3/2} \int_{y+at}^{y-at} e^{-\frac{z^2}{4\beta t}} \, dz \leq Ct^{-1/2} e^{-\frac{2y^2}{\pi t}},
\]

and, similarly,

\[
|K_{yy}(y + at, t, \beta) - K_{yy}(y - a, t, \beta)| = \left| \int_{-at}^{at} K_{yyyy}(z, t, \beta) \, dz \right| \\
\leq Ct^{-2} \int_{y+at}^{y-at} e^{-\frac{z^2}{4\beta t}} \, dz \leq Ct^{-1} e^{-\frac{2y^2}{\pi t}}.
\]

The bounds for \( |e_y| \) are again immediate. Note that we have taken crucial account of cancellation in the small time estimates of \( e_t, e_{ty} \). □

Remark 4.19: For \( t \leq 1 \), a calculation analogous to that of (4.104) yields

\[
|e_y(y,t)| \leq Ce^{-\frac{y^2}{4\pi t}}, \quad \text{and thus } |e_y(\cdot, s)|_{L^1} \to 0 \text{ as } s \to 0.
\]
With these preparations, we are ready to carry out our analysis:

**Proof of Theorem 4.9.** Define

\[(4.106) \quad \zeta(t) := \sup_{0 \leq s \leq t, 2 \leq p \leq \infty} \left[ |U(\cdot, s)|_{L^p} (1 + s)^{-\frac{\frac{1}{2}}{1 - \frac{1}{p}}} + |\delta(s)| (1 + s)^{1/2} + |\delta'(s)| \right].\]

We shall establish:

**Claim.** For all \( t \geq 0 \) for which a solution exists with \( \zeta \) uniformly bounded by some fixed, sufficiently small constant, there holds

\[(4.107) \quad \zeta(t) \leq C_2 (|U_0|_{L^1 \cap H^3} + \zeta(t)^2).\]

From this result, it follows by continuous induction that, provided

\[|U_0|_{L^1 \cap H^3} < 1/4C_2^2,\]

there holds

\[(4.108) \quad \zeta(t) \leq 2C_2 |U_0|_{L^1 \cap H^3}\]

for all \( t \geq 0 \) such that \( \zeta \) remains small. For, by standard short-time theory/local well-posedness in \( H^3 \), and the standard principle of continuation, there exists a solution \( U \in H^3(x) \) on the open time-interval for which \( |U|_{H^3} \) remains bounded, and on this interval \( \zeta \) is well-defined and continuous. Now, let \([0, T)\) be the maximal interval on which \( |U|_{H^3(x)} \) remains strictly bounded by some fixed, sufficiently small constant \( \delta > 0 \). By Proposition 4.15, and the one-dimensional Sobolev bound \(|U|_{W^{2, \infty}} \leq C|U|_{H^3}\), we have

\[(4.109) \quad |U(t)|_{H^3}^2 \leq C |U(0)|_{H^3}^2 e^{-\theta t} + C \int_0^t e^{-\theta_2(t-\tau)} (|U|_{L^2}^2 + |\delta|^2)(\tau) \, d\tau \\
\quad \leq C_2 (|U(0)|_{H^3}^2 + \zeta(t)^2)(1 + t)^{-1/2},\]

and so the solution continues so long as \( \zeta \) remains small, with bound (4.108), at once yielding existence and the claimed sharp \( L^p \cap H^3 \) bounds, \( 2 \leq p \leq \infty \).

Thus, it remains only to establish the claim above.

**Proof of Claim.** We must show that each of the quantities \( |U|_{L^p} (1 + s)^{-\frac{\frac{1}{2}}{1 - \frac{1}{p}}} \), \( |\delta|(1 + s)^{1/2} \), and \( |\delta| \) is separately bounded by

\[(4.110) \quad C (|U_0|_{L^1 \cap H^3} + \zeta(t)^2),\]
for some $C > 0$, all $0 \leq s \leq t$, so long as $\zeta$ remains sufficiently small. By (4.93)--(4.94), we have

\[
|U|_{L^p}(t) \leq |\int_{-\infty}^{\infty} (H + \tilde{G})(x,t;y)U_0(y)dy|_{L^p} \\
+ |\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)(Q(U,U_x)_x + \dot{\delta} U_x)(y,s)dy ds|_{L^p} \\
+ |\int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x,t-s;y)(Q(U,U_x) + \dot{\delta} U)(y,s)dy ds|_{L^p} \\
=: I_a + I_b + I_c,
\]

(4.111)

\[
|\dot{\delta}|(t) \leq |\int_{-\infty}^{\infty} e_{t}(y,t)U_0(y)dy| \\
+ |\int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(Q(U,U_x) + \dot{\delta} U(y,s)dy ds| \\
=: II_a + II_b,
\]

(4.112)

and

\[
|\delta|(t) \leq |\int_{-\infty}^{\infty} e(y,t)U_0(y)dy| \\
+ |\int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(Q(U,U_x) + \dot{\delta} U(y,s)dy ds| \\
=: III_a + III_b.
\]

(4.113)

We estimate each term in turn, following the approach of [Z.6, MaZ.1]. The linear term $I_a$ satisfies bound

\[
I_a \leq C|U_0|_{L^1 \cap L^p}(1 + t)^{-\frac{1}{2}(1-1/p)},
\]

as already shown in the proof of Theorem 4.8. Likewise, applying the bounds of Lemma 4.14 together with (4.88), (4.109), and definition (4.106), we have

\[
I_b = |\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)(Q(U,U_x)_x + \dot{\delta} U_x)(y,s)dy ds|_{L^p} \\
\leq C \int_0^t e^{-\eta(t-s)}(|U|_{L^\infty} + |U_x|_{L^\infty} + |\dot{\delta}|)|U|_{W^{2,p}}(s)ds \\
\leq C\zeta(t)^2 \int_0^t e^{-\eta(t-s)}(1 + s)^{-1/2}ds \\
\leq C\zeta(t)^2 (1 + t)^{-1/2},
\]

(4.114)
and (taking \( q = 2 \) in (4.45))

\[
I_c = \left| \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x, t-s; y)(Q(U, U_x) + \dot{\delta}U)(y, s) dy ds \right|_{L^p}
\]

\[
\leq C \int_0^t e^{-\eta(t-s)} (|U|_{L^\infty} + |U_x|_{L^\infty} + |\dot{\delta}|)|U|_{L^p}(s) ds
\]

\[
+ C \int_0^t (t-s)^{-3/4+1/2p}(|U|_{L^\infty} + |\dot{\delta}|)|U|_{H^1}(s) ds
\]

(4.116)

\[
\leq C\zeta(t)^2 \int_0^t e^{-\eta(t-s)} (1+s)^{-\frac{1}{2}(1-1/p)-1/2} ds
\]

\[
+ C\zeta(t)^2 \int_0^t (t-s)^{-3/4+1/2p}(1+s)^{-3/4} ds
\]

\[
\leq C\zeta(t)^2 (1+t)^{-\frac{1}{2}(1-1/p)},
\]

Summing bounds (4.114)–(4.116), we obtain (4.110), as claimed, for \( 2 \leq p \leq \infty \).

Similarly, applying the bounds of Lemma 4.18 together with definition (4.106), we find that

\[
II_a = \left| \int_{-\infty}^{\infty} e_t(y, t)U_0(y)dy \right|
\]

(4.117)

\[
\leq |e_t(y, t)|_{L^\infty(t)}|U_0|_{L^1}
\]

\[
\leq C|U_0|_{L^1}(1+t)^{-1/2}
\]

and

\[
II_b = \left| \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y, t-s)(Q(U, U_x) + \dot{\delta}U)(y, s) dy ds \right|
\]

(4.118)

\[
\leq \int_0^t |e_{yt}|_{L^2}(t-s)(|U|_{L^\infty} + |\dot{\delta}|)|U|_{H^1}(s) ds
\]

\[
\leq C\zeta(t)^2 \int_0^t (t-s)^{-3/4}(1+s)^{-3/4} ds
\]

\[
\leq C\zeta(t)^2 (1+t)^{-1/2},
\]

while

\[
III_a = \left| \int_{-\infty}^{\infty} e(y, t)U_0(y)dy \right|
\]

(4.119)

\[
\leq |e(y, t)|_{L^\infty(t)}|U_0|_{L^1}
\]

\[
\leq C|U_0|_{L^1}
\]
and
\[
III_b = \left| \int_0^t \int_{-\infty}^{+\infty} \epsilon_y(y, t-s) \delta U(y, s) dy ds \right|
\]
\[
\leq \int_0^t |\epsilon_y|_{L^2}(t-s) \left( |U|_{L^\infty} + |\delta| \right) |U|_{H^1}(s) ds
\]
\[
\leq C\zeta(t)^2 \int_0^t (t-s)^{-1/4}(1+s)^{-3/4} ds
\]
\[
\leq C\zeta(t)^2.
\]
(4.120)

Summing (4.117)–(4.118) and (4.119)–(4.120), we obtain (4.110) as claimed.

This completes the proof of the claim, and the result. \(\square\)

**Remark.** (Low norm stability analysis, \(L^p\), \(1 \leq p \leq 2\)) The source term \(\delta U_x\) appearing in the reduced equations is convenient for high norm estimates \(L^p, p \geq 2\), but not for low norm estimates \(L^p, 1 \leq p \leq 2\). To treat low norms, we may redefine \(U := \bar{U}(x, \delta t) - \bar{U}(x - \delta(t))\), which has the effect of replacing \(\delta U_x\) in the reduced equations with “centering errors”

\[
S(U, U_x, \delta)_x := -((A(\bar{U}(x - \delta) - A(\bar{U}(x)))U)_x + ((B(\bar{U}(x - \delta) - B(\bar{U}(x)))U_x)_x,
\]
satisfying
\[
|S|_{L^1} \leq C|U|_{W^{1,\infty}}, \quad |S|_{W^{2,1}} \leq C|U|_{W^{3,\infty}} \leq C|U|_{H^4}.
\]

This does not affect the previously-obtained \(L^p\) and \(H^s\) estimates, since these norms are invariant under spatial shift.

Repeating our high-norm analysis for more regular initial data \(U_0 \in L^1 \cap H^4\), we obtain (using higher-derivative analogs of Green distribution bounds (4.44)–(4.45))
\[
|U|_{W^{1,\infty}} \leq C(1 + t)^{-1/2}|U_0|_{L^1 \cap H^4}, \quad |U|_{H^4} \leq C(1 + t)^{-1/4}|U_0|_{L^1 \cap H^4}.
\]

Estimating \(I_a - I_c\) using this new information, taking \(q = 1\) now in (4.45) and estimating \(|Q|_{W^{1,p}} \leq C|U|^2_{H^4} \leq C_2(1 + t)^{-1/2}|U_0|_{L^1 \cap H^4}\), we obtain the sharp rate of decay
\[
|\bar{U}(x, t) - \bar{U}(x)|_{L^p} \leq C(1 + t)^{-(1/2)(1-1/p)}|U_0|_{L^1 \cap H^4},
\]
for all \(1 \leq p < \infty\). (Indeed, we could have carried out the original nonlinear iteration in these “uncentered” coordinates, for all \(2 \leq p \leq \tilde{p} < \infty\), using \(q = 1\) in (4.45).) We omit the details, as lying outside the scope of our main development.

### 4.3 Proof of the linearized bounds.

Finally, we carry out in this subsection the proof of Proposition 4.10, completing the analysis of the one-dimensional case. For simplicity, we restrict to the case that \(A_\pm\) have distinct eigenvalues; as discussed in [MaZ.3–4], the general case can be treated in the same way, at the expense only of some further bookkeeping. Following the notation of [MaZ.3], we abbreviate by \((D)\) the triple conditions of structural, dynamical, and strong spectral stability.\(^{27}\)

\(^{27}\)This refers to an alternate characterization in terms of the Evans function \(D\).
4.3.1 Low-frequency bounds on the resolvent kernel.

We begin by estimating the resolvent kernel in the critical regime $|\lambda| \to 0$, corresponding to large time behavior of the Green distribution $G$, or global behavior in space and time. From a global perspective, the structure of the linearized equations is that of two nearly constant-coefficient regions separated by a thin shock layer near $x = 0$; accordingly, behavior is essentially governed by the two, limiting far-field equations:

\begin{equation}
U_t = L\pm U,
\end{equation}

coupled by an appropriate transmission relation at $x = 0$. We begin by examining these limiting systems.

**Lemma 4.20**[MaZ.3]. Let dimension $d = 1$ and assume (A1)–(A3), (H0)–(H3), and (P0). Then, for $|\lambda|$ sufficiently small, the eigenvalue equation $(L \pm - \lambda)U = 0$ associated with the limiting, constant-coefficient operator $L\pm$ has an analytic basis of $n + r$ solutions $U^\pm_j$ consisting of $r$ “fast” modes

\begin{equation}
\begin{aligned}
|\partial/\partial x|^\alpha U_j^+(x) &\leq Ce^{-\theta|x|}, \quad x \geq 0, \quad j = 1, \ldots, k_+^0, \\
|\partial/\partial x|^\alpha U_j^+(x) &\leq Ce^{-\theta|x|}, \quad x \leq 0, \quad j = k_+^0 + 1, \ldots, r,
\end{aligned}
\end{equation}

$\theta > 0$, $0 \leq |\alpha| \leq 1$, and $n$ “slow” modes $\bar{U}_{r+j}^\pm = e^{\mu x}V_j^\pm$, $j = 1, \ldots, n$, where

\begin{equation}
\begin{aligned}
\mu_{r+j}^+(\lambda) &:= -\lambda/a_j^+ + \lambda^2 \beta_j^+ / a_j^+ + O(\lambda^3), \\
V_j^+(\lambda) &:= r_j^+ + O(\lambda),
\end{aligned}
\end{equation}

with $a_j^+$, $r_j^+$, and $\beta_j^+$ as in Proposition 4.10. Likewise, the adjoint eigenvalue equation $(L \pm - \lambda)^*\hat{U} = 0$ has an analytic basis consisting of $r$ fast solutions

\begin{equation}
\begin{aligned}
|\partial/\partial x|^\alpha \bar{U}_j^-(x) &\leq Ce^{-\theta|x|}, \quad x \leq 0, \quad j = 1, \ldots, k_-, \\
|\partial/\partial x|^\alpha \bar{U}_j^-(x) &\leq Ce^{-\theta|x|}, \quad x \geq 0, \quad j = k_- + 1, \ldots, r,
\end{aligned}
\end{equation}

$\theta > 0$, $0 \leq |\alpha| \leq 1$, and $n$ slow solutions $\bar{U}_{r+j}^- = e^{-\mu_j^-}V_j^-(\lambda)$,

\begin{equation}
\hat{V}_j^-(\lambda) = l_j^- + O(\lambda),
\end{equation}

$j = 1, \ldots, n$, where $l_j^\pm$ are as in Proposition 4.10. Moreover, for $\theta > 0$, $0 \leq |\alpha| \leq 1$,

\begin{equation}
\begin{aligned}
|\partial/\partial y|^\alpha \sum_{j=1}^k \bar{U}_j(x)\hat{U}_j(y) &\leq Ce^{-\theta|x-y|}, \quad x \geq y, \\
|\partial/\partial y|^\alpha \sum_{j=k+1}^r \bar{U}_j^+(x)\hat{U}^+_j(y) &\leq Ce^{-\theta|x-y|}, \quad x \leq y.
\end{aligned}
\end{equation}
**Proof.** By Lemma 3.3 and Corollary 3.4, there exist analytic bases of claimed dimension for the “strong”, or “fast” unstable and stable subspaces of the coefficient matrix of the eigenvalue ODE written as a first-order system, defined as the total eigenspaces of spectra with real part uniformly bounded away from zero as $|\lambda| \to 0$, and these satisfy the standard constant-coefficient bounds (4.122) and (4.124). The bounds (4.126) may likewise be seen from standard constant-coefficient bounds, together with the fact (see Section 2.4.3, and especially relation (2.92)) that

$$
\left( \sum_{j=1}^{k} \tilde{W}_j(x) \tilde{W}_j(y) \right)_{\pm} = \left( \mathcal{F}_{y-x} \Pi_{\text{fast}} (S^0)_{\pm} \right)
$$

$\tilde{W} := (U, b_{11} U')$, $\tilde{W} := (\tilde{U}, (0, b_{11}^{\text{Tr}}) \tilde{U}')$, where $\mathcal{F}_{y-x}$ is the solution operator of the limiting eigenvalue ODE written as a first-order system $W' = A_{\pm} W$, $\Pi_{\text{fast}_{\pm}}$ is the projection onto the strongly stable subspace of $A_{\pm}$, well-conditioned thanks to spectral separation, and $S^0_{\pm}$ is as defined in (4.25), also well-conditioned by (2.75).

There exists also an $n$-dimensional manifold of slow solutions, of which the stable and unstable manifolds in the general, multidimensional case, are only continuous as $|\lambda| \to 0$. In the one-dimensional case, however, they may be seen to vary analytically, with expansions (4.123) and (4.125), by inversion of the expansions

$$
\lambda_j(i\xi) = -ia_j^{\pm} \xi - \beta_j^{\pm} \xi^2 + O(\xi^3)
$$
carried out in Appendix A.4 for the the dispersion curves near $\xi = 0$, together with the fundamental relation $\mu = i\xi$ between roots of characteristic and dispersion equations (recall (2.67)–(2.68) of Section 3.1). Alternatively, one can carry out directly the associated matrix bifurcation problem as in the proof of Lemma 3.4. A symmetric argument applies to the adjoint problem. $\square$

**Remark 4.21** [MaZ.3]. Under the simplifying assumption that matrices

$$
N_{\pm} := (df_{21} \quad df_{22}) \left( \begin{array}{cc} df_{11} & df_{12} \\ b_1 & b_2 \end{array} \right)^{-1} \left( \begin{array}{c} 0 \\ I_r \end{array} \right) |_{(U_{\pm})}
$$
governing fast modes (see discussion of traveling-wave ODE, Appendix A.2) have distinct eigenvalues, then fast modes may be taken as pure exponential solutions $U = e^{i\mu x} V$, $\tilde{U} = e^{-i\mu x} \tilde{V}$, and (4.126) follows immediately from (4.122) and (4.124).

Our main result of this subsection is then:

**Proposition 4.22** [MaZ.3]. Let $d = 1$ and assume (A1)–(A3), (H0)–(H3), (P0), and (D). Then, for $r > 0$ sufficiently small, the resolvent kernel $G_\lambda$ has a meromorphic extension onto $B(0, r) \subset C$, which may in the Lax or overcompressive case be decomposed as

$$
G_\lambda = E_\lambda + S_\lambda + R_\lambda
$$

(4.127)
where, for \( y \leq 0 \):

\[
E_\lambda(x, t; y) := \lambda^{-1} \sum_{a_k^+ > 0} [c_{k, -}^j] \left( \frac{\partial \widetilde{U}^0}{\partial \delta_j} \right) l_k^- t e^{(\lambda/a_k^+ - \lambda^2 \beta_k^+ / a_k^+ \delta^3) y},
\]

\[
S_\lambda(x, t; y) := \sum_{a_k^+ > 0, a_j^+ > 0} [c_{k, -}^j] r_j^+ l_k^- t e^{(-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^- \delta^3) (x-y) + (\lambda/a_k^- - \lambda^2 \beta_k^- / a_k^- \delta^3) y}
\]

for \( y \leq 0 \leq x \),

\[
S_\lambda(x, t; y) := \sum_{a_k^+ > 0} r_k^- l_k^- t e^{(-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^- \delta^3) (x-y)} + \sum_{a_k^+ > 0, a_j^+ < 0} [c_{k, -}^j] r_j^+ l_k^- t e^{(-\lambda/a_j^- + \lambda^2 \beta_j^- / a_j^- \delta^3) (x-y) + (\lambda/a_j^- - \lambda^2 \beta_j^- / a_j^- \delta^3) y}
\]

for \( y \leq x \leq 0 \), and

\[
S_\lambda(x, t; y) := \sum_{a_k^- < 0} r_k^- l_k^- t e^{(-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^- \delta^3) (x-y)} + \sum_{a_k^+ > 0, a_j^- < 0} [c_{k, -}^j] r_j^+ l_k^- t e^{(-\lambda/a_j^- + \lambda^2 \beta_j^- / a_j^- \delta^3) (x-y) + (\lambda/a_j^- - \lambda^2 \beta_j^- / a_j^- \delta^3) y}
\]

for \( x \leq y \leq 0 \), and \( R_\lambda = R_\lambda^E + R_\lambda^S \) is a faster-decaying residual satisfying

\[
(\partial/\partial y)^\alpha R_\lambda^E = \mathcal{O}(e^{-3|x-y|}) + \sum_{a_k^+ > 0} e^{-3|x|} e^{(\lambda/a_k^+ - \lambda^2 \beta_k^+ / a_k^+ \delta^3) y} \times \left( \lambda^{\alpha-1} \mathcal{O}(e^{O(\lambda^3)} y - 1) + \lambda^{\alpha-1} \mathcal{O}(e^{O(\lambda^3)} x - 1) + \mathcal{O}(\lambda^\alpha) \right),
\]

(\partial/\partial y)^\alpha R_\lambda^S(x, t; y) =

\[
\sum_{a_k^+ > 0, a_j^+ > 0} e^{(-\lambda/a_j^+ + \lambda^2 \beta_j^+ / a_j^+ \delta^3) x} + (\lambda/a_j^- - \lambda^2 \beta_j^- / a_j^- \delta^3) y}
\]

\[
\times \left( \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)} y - 1) + \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)} x - 1) + \lambda^\alpha \mathcal{O}(e^{-3|x|}) + \mathcal{O}(\lambda) \right)
\]
for $y \leq 0 \leq x$,

$$
(\partial/\partial y)^{\alpha} R^S_\lambda(x, t; y) = \sum_{a_k^- > 0} e^{(-\lambda/a_k^- + \lambda^2 \beta^-/a_k^- - 3)(x-y)} \nonumber
\times \left( \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)y} - 1) + \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda) \right) 
+ \sum_{a_k^- > 0, a_j^- < 0} e^{(-\lambda/a_j^- + \lambda^2 \beta^-/a_j^- - 3)x+(\lambda/a_k^- - \lambda^2 \beta^-/a_k^- - 3)y} 
\times \left( \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)y} - 1) + \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda) \right)
$$

(4.134)

for $y \leq x \leq 0$, and

$$
(\partial/\partial y)^{\alpha} R^S_\lambda(x, t; y) = \sum_{a_k^- < 0} e^{(-\lambda/a_k^- + \lambda^2 \beta^-/a_k^- - 3)(x-y)} \nonumber
\times \left( \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)y} - 1) + \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda) \right) 
+ \sum_{a_k^- > 0, a_j^- < 0} e^{(-\lambda/a_j^- + \lambda^2 \beta^-/a_j^- - 3)x+(\lambda/a_k^- - \lambda^2 \beta^-/a_k^- - 3)y} 
\times \left( \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)y} - 1) + \lambda^\alpha \mathcal{O}(e^{O(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda) \right)
$$

(4.135)

for $x \leq y \leq 0$, $0 \leq \alpha \leq 1$, with each $\mathcal{O}(\cdot)$ term separately analytic in $\lambda$, and where $[e_{k,i}^{s,0}, i = -, 0, +$, are constants to be determined later. Symmetric bounds hold for $y \geq 0$. Similar, but more complicated formulae hold in the undercompressive case (see Remark 6.9, [MaZ.1] or Section 3, [Z.6] for further discussion).

**Normal modes (behavior in $x$).** We begin by relating the normal modes of the variable coefficient eigenvalue equation (2.60) to those of the limiting, constant-coefficient equations (4.121).

**Lemma 4.23 [MaZ.3].** Let $d = 1$ and assume (A1)–(A3), (H0)–(H3), and (P0). Then, for $\lambda \in B(0, r)$, $r$ sufficiently small, there exist solutions $U_{r+j}^{\pm}(x; \lambda)$ of (2.60), $C^1$ in $x$ and analytic in $\lambda$, satisfying, for slow modes, the bounds

$$
U_{r+j}^{\pm}(x; \lambda) = V_{r+j}^{\pm}(x; \lambda) e^{\mu_j^{\pm}x}
$$

(4.136)

$$(\partial/\partial \lambda)^k V_{r+j}^{\pm}(x; \lambda) = (\partial/\partial \lambda)^k V_{r+j}^{\pm}(\lambda) + \mathcal{O}(e^{-\tilde{\theta}|x|}|V_{r+j}^{\pm}(\lambda)|), \quad x \geq 0,
$$

for any $k \geq 0$ and $0 < \tilde{\theta} < \theta$, where $\theta$ is the rate of decay given in (2.18), $\mu_j^{\pm}(\lambda)$, $V_{r+j}^{\pm}(\lambda)$ are as in Lemma 4.20 above, and $\mathcal{O}(\cdot)$ depends only on $k$, $\tilde{\theta}$, and, for fast modes, the bounds

$$
|U_{r+j}^{\pm}(x)| \leq Ce^{-\theta|x|}, \quad x \geq 0, \quad j = 1, \ldots, k_+,
$$

$$
|\dot{U}_{r+j}^{\pm}(x)| \leq Ce^{-\theta|x|}, \quad x \geq 0, \quad j = k_+ + 1, \ldots, r,
$$

(4.137)
\[ |U_j^- (x)| \leq C e^{-\theta |x|}, \quad x \leq 0, \quad j = k_- + 1, \ldots, r; \]
\[ |\tilde{U}_j^- (x)| \leq C e^{-\theta |x|}, \quad x \leq 0, \quad j = 1, \ldots, k_-; \]
\[ \left| \sum_{j=1}^{k_+} U_j^+ (x) \tilde{U}_j^+ (y) \right| \leq C e^{-\theta |x-y|}, \quad 0 \leq y \leq x; \]
\[ \left| \sum_{j=k_+ + 1}^{r} U_j^+ (x) \tilde{U}_j^+ (y) \right| \leq C e^{-\theta |x-y|}, \quad 0 \leq x \leq y. \]

\[ \left| \sum_{j=k_- + 1}^{r} U_j^- (x) \tilde{U}_j^- (y) \right| \leq C e^{-\theta |x-y|}, \quad x \leq y \leq 0; \]
\[ \left| \sum_{j=1}^{k_-} U_j^- (x) \tilde{U}_j^- (y) \right| \leq C e^{-\theta |x-y|}, \quad y \leq x \leq 0. \]

**Proof.** An immediate consequence of Lemma 4.20 together with Lemma 2.5. (Note the helpful factorization)

\[ \sum W_j(x) \tilde{W}_j(y) = P(x) \left( \sum \tilde{W}_j(x) W_j(y) \right) (S^0)^{-1} P^{-1} S^0(y) \]
linking solutions \( W \) and \( \tilde{W} \) of the associated first-order systems, where \( P \) is the conjugating matrix of Lemma 2.5 and \( S^0, \tilde{S}^0 \) are as defined in (4.25), both uniformly well-conditioned.) \( \square \)

The bases \( \phi_j^\pm, \psi_j^\pm \) defined in Section 2.4 may evidently be chosen from among \( U_j^\pm \), yielding an analytic choice of bases in \( \lambda \), with the detailed description (4.136). It follows that the dual bases \( \tilde{\phi}_j^\pm, \tilde{\psi}_j^\pm \) defined in Section 2.4 are also analytic in \( \lambda \) and satisfy corresponding bounds with respect to the dual solutions (4.124)–(4.125). With this observation, we have immediately, from Definition (3.1) and Corollary 2.26:

**Corollary 4.24**[MaZ.3]. Let \( d = 1 \) and assume (A1)–(A3), (H0)–(H3). Then, the Evans function \( D(\lambda) \) admits an analytic extension onto \( B(0, r) \), for \( r \) sufficiently small, and the resolvent kernel \( G_\lambda(x, y) \) admits a meromorphic extension, with an isolated pole of finite multiplicity at \( \lambda = 0 \).

**Remark 4.25.** Analytic extendability of the Evans function past the origin (more generally, into the essential spectrum of \( L \)) is a special feature of the one-dimensional, or scalar multidimensional case, and reflects the simple geometry of
hyperbolic propagation in these settings; see [HoZ.3–4, Z.3] for discussion of the scalar multidimensional case. The simple hyperbolic propagation makes possible a much more detailed description of the Green’s function in these cases, for which analytic extension is a key tool.

For general systems in multidimensions, $D(\xi, \lambda)$ has a conical singularity at the origin; see Proposition 3.5.

**Refined derivative bounds.** We have close control on the $(x, y)$ behavior of $G_\lambda$ through the spectral decomposition formulae of Corollary 2.26. For slow, dual modes, the bounds (4.136) (in particular, the consequent bounds on first spatial derivatives) can be considerably sharpened, provided that we appropriately initialize our bases at $\lambda = 0$. This observation will be quite significant in the Lax or overcompressive case. Likewise, fast-decaying forward modes can be well-approximated near $\lambda = 0$ by their representatives at $\lambda = 0$, using only the basic bounds (4.136). These two categories comprise the modes determining behavior of the Green distribution to lowest order.

Specifically, due to the special, conservative structure of the underlying evolution equations, the adjoint eigenvalue equation $L \ast \hat{U} = 0$ at $\lambda = 0$ admits an $n$-dimensional subspace of constant solutions

$$(4.141) \quad \hat{U} \equiv \text{constant};$$

this is equivalent to the observation that integral quantities in variable $U$ are conserved under time evolution for general conservation laws. Thus, at $\lambda = 0$, we may choose, by appropriate change of coordinates if necessary, that slow-decaying dual modes $\hat{\phi}_j^\pm$ and slow-growing dual modes $\hat{\psi}_j^\pm$ be identically constant, Note that this does not interfere with our previous choice in Lemma 4.23, since that concerned only the choice of limiting solutions $\hat{U}_j^\pm$ of the asymptotic, constant coefficient equations at $x \to \pm \infty$, and not the particular representatives $U_j^\pm$ that approach them (which, in the case of slow modes, are specified only up to the addition of an arbitrary fast-decaying mode).

**Remark 4.26.** The prescription of constant dual bases just described requires that we reverse our previous approach, choosing dual bases first using the conjugation Lemma, then defining forward bases using duality. Alternatively, we may rewrite the forward eigenvalue equation at $\lambda = 0$ as

$$(4.142) \quad \left( \begin{array}{cc} A_{11} & A_{12} \\ b_1 & b_2 \end{array} \right) U' = \left( \begin{array}{cc} 0 & 0 \\ A_{12} & A_{22} \end{array} \right) U'$$

to see, after integration from $x = \pm \infty$, that fast-decaying modes satisfy the linearized traveling-wave ODE (cf. Appendix A.2)

$$(4.143) \quad U' = \left( A_{11} b_1 \begin{array}{cc} A_{12} & 0 \\ b_2 & A_{22} \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 0 \\ A_{12} & A_{22} \end{array} \right) U.$$
By duality relation (2.76), requiring slow dual modes to be constant is equivalent to choosing fast-growing (forward) modes as well as fast-decaying modes from among the solutions of (4.143). (Recall: though fast-decaying modes are uniquely determined as a subspace, fast-growing modes are only determined up to the addition of faster decaying modes.)

**Lemma 4.27** [MaZ.1]. In dimension $d = 1$, assuming (A1)–(A3), (H0)–(H3), and (P0), with the above choice of bases at $\lambda = 0$, and for $\lambda \in B(0, r)$ and $r$ sufficiently small, slow modes $\tilde{U}_j^\pm$ satisfy

\[
\tilde{U}_j^\pm (y; \lambda) = e^{-\mu(y)\lambda} \tilde{V}_j^\pm (0) + \lambda \tilde{\Theta}_j^\pm (y; \lambda),
\]

where

\[
|\tilde{\Theta}_j^\pm| \leq C|e^{-\mu(y)\lambda}|
\]

\[
|((\partial/\partial y)\tilde{\Theta}_j^\pm)| \leq C|e^{-\mu(y)\lambda}|||\lambda| + e^{-\theta|y|}|,
\]

$\theta > 0$, as $y \to \pm \infty$, and $\tilde{V}_j^\pm \equiv$ constant.\(^{28}\) Similarly, fast-decaying (forward) modes $U_j^\pm$ satisfy

\[
U_j^\pm (x; \lambda) = U_j^\pm (x; 0) + \lambda \Theta_j^\pm (x; \lambda),
\]

where

\[
|\Theta_j^\pm|, |((\partial/\partial x)\Theta_j^\pm)| \leq Ce^{-\theta|x|}
\]
as $x \to \pm \infty$, for some $\theta > 0$.

**Proof.** Applying Lemma 2.5 to the augmented variables

\[
\tilde{U}_j^\pm (y; \lambda) := \begin{pmatrix} \tilde{U}_j^\pm \\ \tilde{U}_j^\pm \end{pmatrix} (y; \lambda) =: \begin{pmatrix} \tilde{U}_j^\pm \\ \tilde{U}_j^\pm \end{pmatrix} (y; \lambda)
\]

\[
= e^{-\mu(y)\lambda} \begin{pmatrix} \tilde{V}_j^\pm \\ -\mu_j \tilde{V}_j^\pm + \tilde{V}_j^\pm \end{pmatrix} (y; \lambda)
\]

and

\[
U_j^\pm (x; \lambda) := \begin{pmatrix} U_j^\pm \\ U_j^\pm \end{pmatrix} (x; \lambda) =: e^{\mu(x)\lambda} \begin{pmatrix} V_j^\pm \\ \mu_j V_j^\pm + V_j^\pm \end{pmatrix} (x; \lambda)
\]

\[
= e^{-\mu(y)\lambda} \begin{pmatrix} \tilde{V}_j^\pm \\ \mu_j \tilde{V}_j^\pm + \tilde{V}_j^\pm \end{pmatrix} (y; \lambda)
\]

\(^{28}\)In particular, this includes $|((\partial/\partial y)\tilde{U}_j^\pm (y; \lambda)| \leq C|\lambda e^{\mu_j^\pm (\lambda)y}|$ in place of the general spatial-derivative bound $|((\partial/\partial y)\tilde{U}_j^\pm (y; \lambda)| \leq C(|\mu \tilde{V}_j^\pm| + |(\partial/\partial y)\tilde{V}_j^\pm|)e^{\mu_j^\pm (\lambda)y} \sim C|e^{\mu_j^\pm (\lambda)y}|.$
we obtain bounds
\[\hat{U}_j^\pm(x; \lambda) = \hat{V}_j^\pm(x; \lambda)e^{-\mu_j^\pm(\lambda)x}\]
(4.148)
\[\frac{\partial}{\partial \lambda}k\hat{V}_j^\pm(x; \lambda) = \left(\frac{\partial}{\partial \lambda}\right)^k\hat{V}_j^\pm(\lambda) + \mathcal{O}(e^{-\tilde{\theta}|x|\hat{V}_j^\pm(\lambda)}), \quad x \geq 0,\]
and
\[\hat{U}_j^\pm(x; \lambda) = \hat{V}_j^\pm(x; \lambda)e^{\mu_j^\pm(\lambda)x}\]
(4.149)
\[\left(\frac{\partial}{\partial \lambda}\right)^k\hat{V}_j^\pm(x; \lambda) = \left(\frac{\partial}{\partial \lambda}\right)^k\hat{V}_j^\pm(\lambda) + \mathcal{O}(e^{-\tilde{\theta}|x|\hat{V}_j^\pm(\lambda)}), \quad x \geq 0,\]
\(\tilde{\theta} > 0,\) analogous to (4.136), valid for \(\lambda \in B(0, r),\) where
\[\hat{V}_j^\pm(\lambda) = \begin{pmatrix} \hat{V}(\lambda) \\ -\mu_j^\pm\hat{V}(\lambda) \end{pmatrix}\]
and
\[\hat{V}_j^\pm(\lambda) = \begin{pmatrix} V(\lambda) \\ \mu_j^\pm V(\lambda) \end{pmatrix}\]
(Note: an appropriate phase space in which to apply the conjugation lemma is, e.g., \((U, U^*, (bU^*)^*)\) for the forward equations, and similarly for the dual equations.)

By Taylor’s Theorem with differential remainder, applied to \(\hat{V},\) we have:
\[\hat{U}_j^\pm(y, \lambda) = e^{-\mu_j^\pm(\lambda)y}\left(\hat{V}_j^\pm(y; 0) + \lambda(\partial/\partial \lambda)\hat{V}_j^\pm(y; 0) + \frac{1}{2}\lambda^2(\partial/\partial \lambda)^2\hat{V}_j^\pm(y; \lambda_*)\right),\]
for some \(\lambda_*\) on the ray from 0 to \(\lambda,\) where, recall, \((\partial/\partial \lambda)\hat{V}_j^\pm(y; \cdot)\) and \((\partial/\partial \lambda)^2\hat{V}_j^\pm(y; \cdot)\)
are uniformly bounded in \(L^\infty[0, \pm \infty]\) for \(\lambda \in B(0, r).\) Together with the choice
\[\hat{V}_j^\pm(y, 0) \equiv \text{constant},\]
this immediately gives the first (undifferentiated) bound in (4.145).

Applying now the bound (4.149) with \(k = 1,\) we may expand the second coordinate of (4.150) as
\[\frac{\partial}{\partial y}\hat{U}_j^\pm(y, \lambda) = e^{-\mu_j^\pm(\lambda)y}\left(-\mu_j^\pm\hat{V}_j^\pm(y; 0) + \hat{V}_j^\pm(0; y)\right)\]
(4.151)
\[-\lambda((\partial/\partial \lambda)(\mu_j^\pm\hat{V}_j^\pm)(0) + \mathcal{O}(e^{-\theta|y|})) + \mathcal{O}(\lambda^2))\]
\[= e^{-\mu_j^\pm(\lambda)y}\left(-\lambda((\partial/\partial \lambda)(\mu_j^\pm\hat{V}_j^\pm)(0) + \mathcal{O}(e^{-\theta|y|})) + \mathcal{O}(\lambda^2))\right],\]
and subtracting off the corresponding Taylor expansion
\[\left(\frac{\partial}{\partial y}\left(e^{-\mu_j^\pm(\lambda)y}\hat{V}_j^\pm(y, 0)\right)\right)\]
(4.152)
\[= \mu_j^\pm(\lambda)e^{-\mu_j^\pm(\lambda)y}\hat{V}_j^\pm(y, 0)\]
\[= e^{-\mu_j^\pm(\lambda)y}\left(-\mu_j^\pm(0)\hat{V}_j^\pm(y, 0) - \lambda(\partial/\partial \lambda)(\mu_j^\pm(0)\hat{V}_j^\pm(y, 0) + \mathcal{O}(\lambda^2))\right)\]
\[= e^{-\mu_j^\pm(\lambda)y}\left(-\lambda(\partial/\partial \lambda)(\mu_j^\pm(0)\hat{V}_j^\pm(y, 0) + \mathcal{O}(\lambda^2))\right),\]
we obtain

\[(4.153) \quad (\partial/\partial y)\tilde{\tilde{\psi}}_j(y, \lambda) = e^{-\mu_j^\pm(y)} \left( \lambda \mathcal{O}(e^{-\theta|y|}) + \mathcal{O}(\lambda^2) \right), \]

as claimed.

Similarly, we may obtain (4.146)–(4.147) by Taylor’s Theorem with differential remainder applied to \(U = e^{\mu(\lambda)x}\), and the Leibnitz calculation

\[(\partial/\partial \lambda)U = (d\mu/d\lambda)xe^{\mu x}V + e^{\mu x}(\partial/\partial \lambda)V,\]

together with the observation that \(|xe^{-\theta|x|}| \leq Ce^{-|x|/2}|. □

This leaves only the problem of determining behavior in \(\lambda\) through the study of coefficients \(M, d^\pm\). To this end, we make the following further observations in the Lax or overcompressive case, generalizing the corresponding observation of Lemma 4.30, [Z.3], in the strictly parabolic case:

**Lemma 4.28**[MaZ.3]. For transverse (\(\gamma \neq 0\)) Lax and overcompressive shocks, with the above-specified choice of basis at \(\lambda = 0\), fast-growing modes \(\hat{\psi}_j^+, \hat{\psi}_j^-\) are fast-decaying at \(-\infty, +\infty\), respectively. Equivalently, fast-decaying modes \(\tilde{\hat{\psi}}_j^+, \tilde{\hat{\psi}}_j^-\) are fast-growing at \(-\infty, +\infty\): i.e., the only bounded solutions of the adjoint eigenvalue equation are constant solutions.

**Proof.** Noting that the manifold of solutions of the \(r\)-dimensional ODE (4.143) that decay at either \(x \rightarrow +\infty\) or \(x \rightarrow -\infty\) is by transversality, together with Lemma 1.7, exactly \(d^+ + d^- - \ell = r\), we see that all solutions of this ODE in fact decay at least at one infinity. This implies the first assertion, by the alternative characterization of our bases described in Remark 4.26 above; the second follows by duality, (2.76). Since the manifold of fast-decaying dual modes is uniquely determined, independent of the choice of basis, this implies that nonconstant dual modes that are bounded at one infinity must blow up at the other, hence bounded solutions must be constant as claimed. □

**Remark 4.29.** As noted in [LZ.2, ZH, Z.3, Z.6], the property of Lax and overcompressive shocks that the adjoint eigenvalue equation has only constant solutions has the interpretation that the only \(L^1\) time-invariants of the evolution of the one-dimensional linearized equations about \((\bar{U}, \bar{v})(\cdot)\) are given by conservation of mass, see discussion [LZ.2]. This distinguishes then from undercompressive shocks, which do have additional \(L^1\) time-invariants. The presence of additional time-invariants for undercompressive shocks has significant implications for their behavior, (see discussions [LZ.2] Section 3 and [ZH], Section 10): in particular, the time-asymptotic location of a perturbed undercompressive shock (generically) evolves nonlinearly, and is not determinable by any linear functional of the initial perturbation. By contrast, the time-asymptotic location of a perturbed (stable) Lax or overcompressive shock may be determined by the mass of the initial perturbation alone.
Scattering coefficients (behavior in $\lambda$). We next turn to the estimation of scattering coefficients $M_{jk}$, $d_{jk}$ defined in Corollary 2.26. Consider coefficient $M_{jk}$. Expanding (2.101) using Cramer’s rule, and setting $z = 0$, we obtain

$$
M_{jk}^\pm = D^{-1} C_{jk}^\pm,
$$

where

$$
C^+ := (I, 0) \left( \begin{array}{cc} \Phi^+ & \Phi^- \\ \Phi^+ & \Phi^- \end{array} \right)^{\text{adj}} \left( \begin{array}{c} \Psi^- \\ \Psi^- \end{array} \right) |_{z=0}
$$

and a symmetric formula holds for $C^-$. Here, $P^{\text{adj}}$ denotes the adjugate matrix of a matrix $P$, i.e. the transposed matrix of minors. As the adjugate is polynomial in the entries of the original matrix, it is evident that $|C^\pm|$ is uniformly bounded and therefore

$$
|M_{jk}^\pm| \leq C_1 |D^{-1}| \leq C_2 \lambda^{-\ell}
$$

by (D), where $C_1, C_2 > 0$ are uniform constants.

However, the crude bound (4.156) hides considerable cancellation, a fact that will be crucial in our analysis. Relabel the $\{\varphi^\pm\}$ so that, at $\lambda = 0$,

$$
\varphi^+_{n+r-j+1} \equiv \varphi^-_j = (\partial/\partial \delta)(\tilde{u}^\delta) \left( \begin{array}{c} \tilde{v}^\delta \\ \tilde{v}^\delta \end{array} \right), \quad j = 1, \ldots, \ell.
$$

With convention (4.157), we have the sharpened bounds:

**Lemma 4.30** [MaZ.3]. In dimension $d = 1$, let there hold (A1)–(A3), (H0)–(H3), and (D), and let $\phi^\pm_j$ be labeled as in (4.157). Then, for $|\lambda|$ sufficiently small, there hold

$$
|M_{jk}^\pm|, |d_{jk}^\pm| \leq C \begin{cases} \lambda^{-1} & \text{for } j = 1, \ldots, \ell, \\ 1 & \text{otherwise,} \end{cases}
$$

where $M^\pm, d^\pm$ are as defined in Corollary 2.26. Moreover,

$$
\text{Residue } \lambda=0 M_{n+r-j+1,k}^+ = \text{Residue } \lambda=0 d_{j,k}^+,
$$

for $1 \leq j \leq \ell$, all $k$.

That is, blowup in $M_{jk}$ occurs to order $\lambda^{\ell-1} |D^{-1}|$ rather than $|D^{-1}|$, and, more importantly, only in (fast-decaying) stationary modes $(\partial/\partial \delta)(\tilde{u}^\delta, \tilde{v}^\delta)$. Moreover, each stationary mode $(\partial/\partial \delta)(\tilde{u}^\delta, \tilde{v}^\delta)$ has consistent scattering coefficients with dual mode $\tilde{\psi}_k^\pm$, to lowest order, across all of the various representations of $G_{\lambda}$ given in Corollary 2.26.
Proof. Formula (4.155) may be rewritten as
\[
C_{jk}^+ = \det \left( \begin{array}{c} 
\varphi_1^+, \ldots, \varphi_{j-1}^+, \psi_k^- \varphi_j^+, \ldots, \varphi_n^+ \\
\varphi_1^+, \ldots, \varphi_{j-1}^+, \psi_k^- \varphi_j^+, \ldots, \varphi_n^+ 
\end{array} \right),
\]
from which we easily obtain the desired cancellation in \( M^+ = C^+ D^{-1} \). For example, for \( j > \ell \), we have
\[
C_{jk}^+ = \det \left( \begin{array}{c} 
\varphi_1^+ + \lambda \varphi_1^+, \ldots, \varphi_n^+ + \lambda \varphi_n^+, \Phi \\
\varphi_1^+ + \lambda \varphi_1^+, \ldots, \varphi_n^+ + \lambda \varphi_n^+, \Phi 
\end{array} \right)
\]
at $-\infty$, $+\infty$ respectively. It follows that in the righthand side of (4.160), there is at $\lambda = 0$ a linear dependency between columns

$$\varphi^+_1, \cdots, \varphi^+_{j-1}, \psi_k^-, \varphi^+_{j+1}, \varphi^+_{i_+} \text{ and } \varphi^-_j = \varphi^+_j$$

i.e. an $\ell$-fold dependency among columns

$$\varphi^+_1, \cdots, \psi^-_k, \cdots, \varphi^+_{i_+} \text{ and } \varphi^-_1, \cdots, \varphi^-_e.$$  

It follows as in the proof of Lemma 4.30 that $|C_{jk}| \leq C\lambda^\ell$ for $\lambda$ near zero, giving bound (4.162) for $M_{jk}$. If $\varphi^+_j$ is a slow mode, on the other hand, then the same argument shows that there is a linear dependency in columns

$$\varphi^+_1, \cdots, \varphi^-_{j-1}, \psi_k^+, \varphi^+_{j+1}, \varphi^+_{i_+}$$

and an $(\ell + 1)$-fold dependency in (4.165), since the omitted slow mode $\varphi^+_j$ plays no role in either linear dependence; thus, we obtain the bound (4.163), instead. Analogous calculations yield the result for $d_{jk}$, as well. □

**Proof of Proposition 4.22.** The proof of Proposition 4.22 is now just a matter of collecting the bounds of Lemmas 4.23–4.31, and substituting in the representations of Corollary 2.26. More precisely, approximating fast-decaying dual modes $\phi^+_1, \ldots, \phi^+_\ell$ and $\phi^-_{n+r-\ell+1}, \ldots, \phi^-_{n+r}$ by stationary modes $(\partial/\partial \delta_j)(\bar{u}^\delta, \bar{v}^\delta)$, and slow dual modes by $e^{-\mu(\lambda)y}\bar{V}(\lambda)$ as described in Lemma 4.27, truncating $\mu(\lambda)$ at second order, and keeping only the lowest order terms in the Laurent expansions of scattering coefficients $M^\pm$, $d^\pm$ we obtain $E_\lambda$ and $S_\lambda$, respectively; as order $\lambda^{-1}$ and order one terms, except for negligible $O(e^{-\theta|x-y|})$ terms arising through (4.137)–(4.138) and (4.139)–(4.140), which we have accounted for in error term $R^E_\lambda$.

Besides the latter, we have accounted in term $R^S_\lambda$ for: (i) the truncation errors involved in the aforementioned approximations (difference between stationary modes and actual fast modes, and between constant-coefficient approximant and actual slow dual modes, plus truncation errors in exponential rates); and, (ii) neglected fast-decaying forward/slow-decaying dual mode combinations (of order $\lambda$, by Lemma 4.30). A delicate point is the fact that term $O(e^{-\theta|x-y|})O(e^{-\theta|y|})$ arising in the dual truncation of $E$ through the $\lambda O(e^{-\theta|y|})$ portion of estimate (4.145) for $(\partial/\partial y)\tilde{\Theta}^\pm_j$, absorbs in the (leading) $O(e^{-\theta|x-y|})$ term of (4.132). It was to obtain this key reduction that we carried out expansion (4.145) to such high order.

In $R^S_\lambda$, we have accounted for truncation errors in the slow/slow pairings approximated by term $S_\lambda$ (difference between constant-coefficient approximants and actual slow forward modes, corresponding to an additional $e^{-\theta|x|}$ term in the truncation error factor, and between constant-coefficient approximants and actual slow dual modes, plus truncation errors in exponential rates). We have also accounted for the remaining, slow-decaying forward/fast-decaying dual modes, which according to Lemma 4.31 have scattering coefficient of order $\lambda$ and therefore may be grouped with the difference between dual modes and their constant-coefficient approximants (of smaller order by a factor of $e^{-\theta|y|}$) in the term $O(\lambda)$. □
4.3.2 High-frequency bounds on the resolvent kernel.

We now turn to the estimation of the resolvent kernel in the high frequency regime $|\lambda| \to +\infty$. According to the usual duality between spatial and frequency variables, large frequency $\lambda$ corresponds to small time $t$, or local behavior in space and time. Therefore, we begin by examining the frozen-coefficient equations

$$
U_t = L_{x0}U := B(x_0)U_{xx} - A(x_0)U_x
$$

at each value $x_0$ of $x$, for simplicity taking $b(x_0)$ diagonalizable (for motivation only: we make no such assumption in the variable-coefficient case).

**Lemma 4.32** [MaZ.3]. In dimension $d = 1$, let $b_2(x_0)$ be diagonalizable, and let $A_*(U(x_0))$ have distinct, real eigenvalues. Then, for $|\lambda|$ sufficiently large, the eigenvalue equation $(L_{x0} - \lambda)U = 0$ associated with the frozen-coefficient operator $L_{x0}$ at $x_0$ has a basis of $n + r$ solutions

$$
\{\tilde{\phi}_1^+, \ldots, \tilde{\phi}_k^+, \tilde{\phi}_{k+1}^-, \ldots, \tilde{\phi}_{n+r}^-\}(x; \lambda, x_0), \quad \tilde{\phi}_j^\pm = e^{\mu_j(\lambda, x_0)x}V_j(\lambda, x_0),
$$

of which $n - r$ are “hyperbolic modes,” analytic in $1/\lambda$ and satisfying

$$
\mu_j^{\pm}(\lambda, x_0) = -\sqrt{\lambda/\gamma_j(x_0)} + \mathcal{O}(1/\lambda),
$$

$$
V_j(\lambda, x_0) = \left(\begin{array}{c} 0_{n-r} \\ s_j \\ \mu_j s_j \end{array}\right)(x_0) + \mathcal{O}(1/\lambda),
$$

where $\gamma_j(\cdot)$, $s_j(\cdot)$, the eigenvalues and right eigenvectors of $b_2(U(x_0))$. Likewise, the adjoint eigenvalue equation $(L_{x0} - \lambda)^*\tilde{U} = 0$ has a basis of solutions

$$
\{\tilde{\phi}_1^-, \ldots, \tilde{\phi}_k^-, \tilde{\phi}_{k+1}^+, \ldots, \tilde{\phi}_{n+r}^+\}(x; \lambda, x_0), \quad \tilde{\phi}_j^\pm = e^{-\mu_j(\lambda, x_0)x}\tilde{V}_j(\lambda, x_0),
$$

with

$$
\tilde{V}_j(\lambda, x_0) = L_j^*(x_0) + \mathcal{O}(1/\lambda), \quad \tilde{V}_j(\lambda, x_0) = \left(\begin{array}{c} \gamma_j^{-1}b_1^{tr}r_j \\ t_j \\ -\mu_j t_j \end{array}\right)(x_0) + \mathcal{O}(1/\lambda),
$$

respectively, for hyperbolic and parabolic modes, $L_j^*(\cdot)$ as defined in Section 4.2.1 and $t_j(\cdot)$ the left eigenvectors of $b_2(U(x_0))$. 

The expansions (4.168)–(4.170) hold also in the nonstrictly hyperbolic case, with \( \vec{\phi}_j \), \( \tilde{\phi}_j \) and \( R^*_j \), \( L^*_j \) now denoting \( n \times m^*_j \) blocks, and \( \mu_j \), \( \eta_j \) denoting \( m^*_j \times m^*_j \) matrices, \( j = 1, \ldots, J \), where \( m^*_j \) is the multiplicity of eigenvalue \( a^*_j \) of \( A_*(x_0) \).

**Proof.** This follows by inversion of the expansions about \( \xi = \infty \) of the eigenvalues \( \lambda_j(\xi) \) and eigenvectors \( V_j(\xi) \) of the one-dimensional frozen-coefficient Fourier symbol \( -i\xi A(x_0) - \xi^2 B(x_0) \), using the basic relation \( \mu_j = i\xi \) relating characteristic and dispersion equations (2.67) and (2.68). The expansions of the one-dimensional Fourier symbol are carried out in Appendix A.4. \( \square \)

By (2.104), Lemma 4.32 gives an expression for the constant-coefficient resolvent kernel \( G_{\lambda;x_0} \) at \( x_0 \) of, to lowest order, a “hyperbolic” part,

\[
H_{\lambda;x_0} \sim \begin{cases} 
\sum_{j=1}^{k} a^*_j(x_0)^{-1} e^{(-\lambda/a^*_j-\eta_j/a^*_j)(x-y)} R^*_j(x_0)L^*_j(x_0) & x < y, \\
-\sum_{j=k+1}^{J} a^*_j(x_0)^{-1} e^{(-\lambda/a^*_j-\eta_j/a^*_j)(x-y)} R^*_j(x_0)L^*_j(x_0) & x > y,
\end{cases}
\]

with next-order contribution a “parabolic” part \( P_{\lambda;x_0} \), analytic in \( \lambda^{-1/2} \) (i.e., for \( \lambda \) away from the negative real axis) and satisfying

\[
P_{\lambda;x_0} = O(|\lambda|^{-1/2} e^{-\theta|\lambda|^{1/2}|x-y|}), \quad \theta > 0,
\]

on some sector

\[
\Omega_P := \{ \lambda : \Re \lambda \geq -\theta_1 | \Im \lambda + \theta_2 \}, \quad \theta_j > 0.
\]

Our main result of this section will be to establish on an appropriate unbounded subset of the resolvent set \( \rho(L) \) that the variable coefficient solutions \( \phi_{j+}^\pm, \phi_{j-}^\pm \), and thus the resolvent kernel, satisfy analogous formulae, with static quantities replaced by the corresponding dynamical ones defined in Section 4.2.1. More precisely, define

\[
\Omega := \{ \lambda : -\eta_1 \leq \Re \lambda \},
\]

with \( \eta_1 > 0 \) sufficiently small that \( \Omega \setminus B(0, r) \) is compactly contained in the set of consistent splitting \( \Lambda \) (defined in (2.62)), for some small \( r > 0 \) to be chosen later; this is possible, by Lemma 2.21(i).

**Proposition 4.33 [MaZ.3].** In dimension \( d = 1 \), assume (A1)–(A3) and (H0)–(H3), plus strict hyperbolicity of \( A_*(\cdot) \). Then, for \( R > 0 \) sufficiently large, there holds \( \Omega \setminus B(0, R) \subset \Lambda \cap \rho(L) \); moreover, there holds on \( \Omega \setminus B(0, R) \) the decomposition

\[
G_{\lambda}(x, y) = H_{\lambda}(x, y) + P_{\lambda}(x, y) + \Theta_{\lambda}(x, y),
\]

\[
H_{\lambda}(x, y) := \begin{cases} 
-\sum_{j=k+1}^{J} a^*_j(x)^{-1} e^{\int_s^z (-\lambda/a^*_j-\eta_j/a^*_j)(z) dz} R^*_j(x)L^*_j(y) & x > y, \\
\sum_{j=1}^{k} a^*_j(x)^{-1} e^{\int_s^z (-\lambda/a^*_j-\eta_j/a^*_j)(z) dz} R^*_j(x)L^*_j(y) & x < y,
\end{cases}
\]
\[ (4.177) \quad \Theta_\lambda(x, y) = \lambda^{-1} B(x, y; \lambda) + \lambda^{-1} (x - y) C(x, y; \lambda) + \lambda^{-2} D(x, y; \lambda), \]

where
\[ (4.178) \quad B(x, y; \lambda) = \left\{ \begin{array}{ll}
\sum_{j=k+1}^{J} e^{-\int_y^x \frac{\lambda}{a_j^*(s)} ds} b_j^+(x, y) & x > y, \\
\sum_{j=1}^{k} e^{-\int_y^x \frac{\lambda}{a_j^*(s)} ds} b_j^-(x, y) & x < y,
\end{array} \right. \]

\[ (4.179) \quad C(x, y; \lambda) = \left\{ \begin{array}{ll}
-\sum_{i,j=k+1}^{K} \text{mean}_{z \in [x,y]} e^{-\int_y^x \frac{\lambda}{a_j^*(s)} ds - \int_x^y \frac{\lambda}{a_i^*(s)} ds} c_{i,j}^+(x, y; z) & x > y, \\
\sum_{i,j=1}^{k} \text{mean}_{z \in [x,y]} e^{-\int_y^x \frac{\lambda}{a_j^*(s)} ds - \int_x^y \frac{\lambda}{a_i^*(s)} ds} c_{i,j}^- (x, y; z) & x < y,
\end{array} \right. \]

with
\[ (4.180) \quad |b_j^\pm|, |c_{i,j}^\pm| \leq Ce^{-\theta |x-y|} \]

and
\[ (4.181) \quad D(x, y; \lambda) = \mathcal{O}(e^{-\theta (1+\Re \lambda)|x-y|} + e^{-\theta \sqrt{|\lambda||x-y|}}), \]

for some uniform \( \theta > 0 \) independent of \( x, y, z \), each described term separately analytic in \( \lambda \), and \( P_\lambda \) is analytic in \( \lambda \) on a (larger) sector \( \Omega_\delta \) as in (4.173), with \( \theta_1 \) sufficiently small, and \( \theta_2 \) sufficiently large, satisfying uniform bounds
\[ (4.182) \quad (\partial/\partial x)^\alpha (\partial/\partial y)^\beta P_\lambda(x, y) = \mathcal{O}(|\lambda|^{(|\alpha|+|\beta|)-1/2} e^{-\theta |\lambda|^{1/2}|x-y|}), \quad \theta > 0, \]

for \( |\alpha| + |\beta| \leq 2 \) and \( 0 \leq |\alpha|, |\beta| \leq 1 \).

Likewise, there hold derivative bounds
\[ (4.183) \quad (\partial/\partial x) \Theta_\lambda(x, y) = \left( B_x^0(x, y; \lambda) + (x - y) C_x^0(x, y; \lambda) \right) + \lambda^{-1} \left( B_x^1(x, y; \lambda) \right) \\
+ (x - y) C_x^1(x, y; \lambda) + (x - y)^2 D_x^1(x, y; \lambda) + \lambda^{-3/2} E_x(x, y; \lambda) \]

and
\[ (4.184) \quad (\partial/\partial y) \Theta_\lambda(x, y) = \left( B_y^0(x, y; \lambda) + (x - y) C_y^0(x, y; \lambda) \right) + \lambda^{-1} \left( B_y^1(x, y; \lambda) \right) \\
+ (x - y) C_y^1(x, y; \lambda) + (x - y)^2 D_y^1(x, y; \lambda) + \lambda^{-3/2} E_y(x, y; \lambda), \]

where \( B_\beta^0 \) and \( C_\beta^0 \) satisfy bounds of form (4.178) and (4.179), \( D_\beta^1 \) now denotes the iterated integral
\[ (4.185) \quad D_\beta^1(x, y; \lambda) = \left\{ \begin{array}{ll}
-\sum_{h,i,j=k+1}^{K} \text{mean}_{w \leq z \leq x} e^{-\int_y^w \frac{\lambda}{a_h^*(s)} ds - \int_w^x \frac{\lambda}{a_j^*(s)} ds - \int_x^y \frac{\lambda}{a_i^*(s)} ds} \\
\times d_{h,i,j}^\beta(x, y; z), & x > y, \\
\sum_{h,i,j=1}^{k} \text{mean}_{x \leq z \leq w} e^{-\int_y^w \frac{\lambda}{a_h^*(s)} ds - \int_w^x \frac{\lambda}{a_j^*(s)} ds - \int_x^y \frac{\lambda}{a_i^*(s)} ds} \\
\times d_{h,i,j}^\beta(x, y; z), & x < y,
\end{array} \right. \]
with $|e_{h,i,j}^{3,-}| \leq Ce^{-\theta|x-y|}$, and $E_\beta$ satisfies a bound of form (4.181).

The bounds (4.177)–(4.184) hold also in the nonstrictly hyperbolic case, with (4.176) replaced by

\[
H_\lambda(x,y) = \begin{cases} 
-\sum_{j=K+1}^{\infty} a_j^*(x)^{-1} e^{-\int_y^x \lambda/a_j^*(z) \, dz} R_j^*(x) \tilde{\zeta}_j(x,y) L_j^*(y) & x > y, \\
\sum_{j=1}^{K} a_j^*(x)^{-1} e^{-\int_y^x \lambda/a_j^*(z) \, dz} R_j^*(x) \tilde{\zeta}_j(x,y) L_j^*(y) & x < y,
\end{cases}
\]

where $\tilde{\zeta}_j(x,y) \in \mathbb{R}^{m_j^* \times m_j^*}$ denotes a dissipative flow similar to $\zeta_j$ in (4.27), but with respect to variable $x$: i.e.,

\[
d\tilde{\zeta}_j/dx = \eta_j^*(x) \tilde{\zeta}_j(x,y)/a_j^*(x), \quad \tilde{\zeta}_j(y,y) = I,
\]

or, equivalently, $\tilde{\zeta}_j(x,y) = \zeta_j(x,\tau)$ for $\tau$ such that $z_j(x,0) = y$, where $z_j(x,s)$ as in (4.26) denotes the backward characteristic path associated with $a_j^*$, $z_j(x,\tau) := x$; $R_j^*$, $L_j^*$ now denote $n \times m_j^*$ blocks; and $\mu_j$, $\eta_j^*$ denote $m_j^* \times m_j^*$ matrices, $j = 1, \ldots, K$, $j = K+1, \ldots, J$, where $m_j^*$ is the multiplicity of eigenvalue $a_j^*$ of $A_\ast(x_0)$, with

$$a_1^* \leq \cdots \leq a_K^* < 0 < a_{K+1}^* \leq \cdots \leq a_J^*.$$

(Note that we have not here assumed (D)).

**Proof.** Our argument is similar in spirit to the proof of Proposition 4.4 [MaZ.1] in the hyperbolic relaxation case. However, the mixed hyperbolic–parabolic nature of the underlying equations, and the associated presence of multiple scales, makes the analysis more delicate. In accordance with the behavior we seek to identify, it is convenient to express the eigenvalue equation in “local” variables $\tilde{u} := A_\ast u$, $\tilde{v} := b_1 \tilde{u} + b_2 \tilde{v}$ similar to those used in proving Lemma 2.1, yielding

\[
\begin{align*}
\tilde{u}_x &= -\lambda A_\ast^{-1} \tilde{u} - (A_{12} b_2^{-1} \tilde{v})_x, \\
(\tilde{v}_x)_x &= \left[ ((A_{21} - A_{22} b_2^{-1} b_1 + b_2 \partial_x (b_2^{-1} b_1)) A_\ast^{-1} \tilde{u})_x \\
&\quad + (A_{22} + \partial_x (b_2)) b_2^{-1} \tilde{v}_x + \lambda b_2^{-1} b_1 A_\ast^{-1} \tilde{u} + \lambda b_2^{-1} \tilde{v} \right].
\end{align*}
\]

Similarly as in the relaxation case, the substitution of $A_\ast u$ for $u$ converts the hyperbolic part of the equation from divergence to nondivergence form, simplifying slightly the flow along characteristics in the hyperbolic term $H_\lambda$; for related discussion, see Remark 4.12.

Following standard procedure (e.g., [AGJ, GZ, ZH, Z.3]), we perform the rescaling

\[
\tilde{x} := |\lambda| x, \quad \tilde{\lambda} := \lambda / |\lambda|.
\]
suggested by the leading order hyperbolic behavior of \( \mu(x_0) \), (4.168), to obtain after some rearrangement the perturbation equation

\[
Y' = A(\tilde{x}, |\lambda|^{-1})Y, \quad Y := (\tilde{u}, \tilde{v}, \tilde{v}')^T,
\]

where

\[
(4.191) \quad A(\tilde{x}, |\lambda|^{-1}) = A_0(\tilde{x}) + |\lambda|^{-1}A_1(\tilde{x}) + O(|\lambda|^{-2}),
\]

\[
A_0(\tilde{x}) := \begin{pmatrix}
-\lambda A_*^{-1} & 0 & -A_{12}b_2^{-1} \\
0 & 0 & I_r \\
0 & 0 & 0
\end{pmatrix}, \quad A_1(\tilde{x}) := \begin{pmatrix}
0 & \partial_x(A_{12}b_2^{-1}) & 0 \\
0 & 0 & 0 \\
-\lambda d_* A_*^{-2} & -\lambda b_2^{-1} & e_* b_2^{-1}
\end{pmatrix},
\]

\[
(4.193) \quad d_* := A_{21} - A_{22}b_2^{-1}b_1 + b_2^{-1}b_1A_* + b_2\partial_x(b_2^{-1}b_1),
\]

\[
e_* := A_{22} + d_* A_*^{-1}A_{12} + \partial_x(b_2),
\]

and \( \frac{\partial}{\partial \tilde{x}} \) denotes \( \partial / \partial \tilde{x} \). Here, expansion (4.191) is to be considered as a continuous family of one-parameter perturbation equations, indexed by \( \tilde{\lambda} \in S^1 \). The form of \( e_* \) is not important, and is included only for completeness; \( d_* \), on the other hand, may be recognized as the important quantity defined in (4.37).

We seek to develop corresponding expansions of the resolvent kernel in orders of \(|\lambda|^{-1}\), or, equivalently, by representation (2.84), expansions of stable/unstable manifolds of (4.190), and the reduced flows therein. This we will accomplish by the methods described in Sections 2.2.2–2.2.5, first reducing to an approximately block-diagonal system segregating spectrally separated (in particular, stable/unstable) modes, with formal error of a suitably small order, then converting the formal error into rigorous error bounds using Proposition 2.13 together with Corollary 2.14. An interesting aspect of this calculation is the block-diagonalization of parabolic modes, which are not uniformly spectrally separated in this (hyperbolic) scaling, but rather comprise a block-Jordan block of order \( s = 2 \) associated with eigenvalue \( \mu = 0 \). Accordingly, we treat them separately, by the method sketched in Section 2.2.4, after the initial block-diagonalization has been carried out. This results in a slower decaying error series for these blocks, converging in powers of \(|\lambda|^{-1/2}\); parabolic modes are well-behaved, however, and so the loss of accuracy is harmless.

We will carry out in detail the basic estimate (4.177), indicating the extension to derivative bounds (4.183)–(4.184) by a few brief remarks. First, observe that in the modified coordinates \( \tilde{Y} = \tilde{Q}W \), (2.84) just becomes

\[
(4.194) \quad G^\lambda(x, y) = \begin{cases}
(I_n, 0)Q^{-1}(x)F_{Y}^{y-x}Y^{-1}Y^{-1}Q^{-1}(y)(I_n, 0)^T_x & x < y, \\
-(I_n, 0)Q^{-1}(x)F_{Y}^{y-x}Y^{-1}Q^{-1}(y)(I_n, 0)^T_x & x > y,
\end{cases}
\]
where $\Pi_{Y}^{\pm}$ and $F_{Y}^{ur}$ denote projections and flows in $Y$-coordinates, and the lower triangular matrices

\begin{equation}
Q = \begin{pmatrix} I_{n-r} & 0 & 0 \\
0 & I_r & 0 \\
0 & 0 & |\lambda|^{-1}I_r \end{pmatrix},
\end{equation}

\begin{equation}
Q^{-1} = \begin{pmatrix} A_{s}^{-1} & 0 & 0 \\
-b_{2}^{-1}b_{1}A_{s}^{-1} & b_{2}^{-1} & 0 \\
-b_{2}\partial_{x}(b_{2}^{-1}b_{1})A_{s}^{-1} & -\partial_{x}(b_{2}^{-1}) & I_r \end{pmatrix} \begin{pmatrix} I_{n-r} & 0 & 0 \\
0 & I_r & 0 \\
0 & 0 & |\lambda|I_r \end{pmatrix},
\end{equation}

relate $Y$ coordinates to the original coordinates $W = (u, v, b_{1}u_{x} + b_{2}v_{x})^{tr}$ of (2.84):

\begin{equation}
Y = QW, \quad W = Q^{-1}Y.
\end{equation}

It is readily checked that $Q \in C^{p+1}$ for $f, B \in C^{p+2}$, whence $Q \in C^{2}$ under the regularity hypothesis (H0) ($p = 3$).

**Initial diagonalization.** For $f, B \in C^{p+2}$, we may check further that $A_{j} \in C^{p-j+1}$ as required in the theory of Section 2.2.3, suitable for approximate block-diagonalization to formal error $O(|\lambda|^{-p-1})$: formal error $O(|\lambda|^{-4})$ in the present case $p = 3$. Applying the formal diagonalization procedure of Proposition 2.15 to (4.190), and choosing the special initialization (2.47), at the same time taking care as described in Remark 2.16(3) to preserve analyticity with respect to $\lambda^{-1}$ in original coordinates, we obtain the approximately block-diagonalized system

\begin{equation}
Z' = D(\tilde{x}, |\lambda|^{-1})Z, \quad TZ := Y, \quad D := TAT^{-1},
\end{equation}

\begin{equation}
T(\tilde{x}, |\lambda|^{-1}) = T_{0}(\tilde{x}) + |\lambda|^{-1}T_{1}(\tilde{x}) + \cdots + |\lambda|^{-3}T_{3}(\tilde{x}),
\end{equation}

\begin{equation}
D(\tilde{x}, |\lambda|^{-1}) = D_{0}(\tilde{x}) + |\lambda|^{-1}D_{1}(\tilde{x}) + \cdots + D_{2}(\tilde{x})|\lambda|^{-3} + O(|\lambda|^{-4}),
\end{equation}

where without loss of generality (since $T_{0}$ is uniquely determined up to a constant linear coordinate change)

\begin{equation}
T_{0}^{-1} = \begin{pmatrix} L_{s}^{tr} & 0 & -\tilde{\lambda}^{-1}L_{s}^{tr}A_{s}A_{12}b_{2}^{-1} \\
0 & I_{r} & 0 \\
0 & 0 & I_{r} \end{pmatrix}, \quad T_{0} = \begin{pmatrix} R_{s} & 0 & -\tilde{\lambda}^{-1}A_{s}A_{12}b_{2}^{-1} \\
0 & I_{r} & 0 \\
0 & 0 & I_{r} \end{pmatrix},
\end{equation}

$L_{s} := (L_{s}, \ldots, L_{s})$, $R_{s} := (R_{s}, \ldots, R_{s})$, $L_{s}^{tr}$, $R_{s}^{tr}$ as defined in Section 4.2.1, and so

\begin{equation}
D_{0} = \begin{pmatrix} -\text{diag}\{\tilde{\lambda}I_{m_{j}}/\eta_{j}^{2}a_{j}^{2} \} & 0 & 0 \\
0 & 0 & I_{r} \\
0 & 0 & 0 \end{pmatrix}, \quad D_{1} = \begin{pmatrix} -\text{diag}\{\eta_{j}^{2}I_{m_{j}}/\tilde{\lambda}a_{j}^{2} \} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\tilde{\lambda}b_{2}^{-1} \end{pmatrix},
\end{equation}
\(a_j^*\) and \(m_j^*\) as defined in (4.15), and \(\eta_j^*\) as defined in (4.18)–(4.19). Here, we have used the simple block upper-triangular form of \(A_0\) to deduce the form of (4.201), then calculated (4.202) from the relations \(D_0 = T_0^{-1}A_0T_0\), \(D_1 = T_0^{-1}A_1T_0\), the second a consequence of the special normalization (2.47).

**The parabolic block.** At this point, we have diagonalized into \(J m_j^* \times m_j^*\) hyperbolic blocks corresponding to eigenvalues \(\tilde{\mu} = -\tilde{\lambda}/a_j^*\) of \(A_0\), and a \(2r \times 2r\) parabolic block-Jordan block (the lower righthand corner)

\[
(4.203) \quad N := \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix} + |\lambda|^{-1} \begin{pmatrix} 0 & 0 \\ -\tilde{\lambda}b_2^{-1} & * \end{pmatrix} + \mathcal{O}(|\lambda|^{-2}),
\]

corresponding to eigenvalue \(\tilde{\mu} = 0\) of \(A_0\). We now focus on the latter block, applying the procedure of Section 2.2.4. “Balancing” by transformation \(B := \{1, |\lambda|^{1/2}\}\) yields

\[
(4.204) \quad \tilde{M} := B^{-1}NB = |\lambda|^{-1/2}\tilde{M}_1 + \mathcal{O}(|\lambda|^{-1}), \quad \tilde{M}_1 := \begin{pmatrix} 0 & I_r \\ -\tilde{\lambda}b_2^{-1} & 0 \end{pmatrix},
\]

or \(|\lambda|^{-1/2}\) times the standard rescaled expansion of the strictly parabolic case [ZH].

Proceeding as in that case, we observe that \(\sigma(\tilde{M}_1) = \pm \sqrt{\sigma(\tilde{\lambda}b_2^{-1})}\) has a uniform spectral gap of order one, by (H1)(i), on the sector \(\Omega_P\) defined in (4.173), for \(\theta_1 > 0\) sufficiently small. Thus (see, e.g. Proposition A.9, p. 361 of [St]), there exists a well-conditioned coordinate change \(S = S(\tilde{M}_1(y))\) depending continuously on \(\tilde{M}_1\), such that

\[
\tilde{M}_1 = \text{diag} \{\tilde{M}^-_1, \tilde{M}^+_1\}(\tilde{x}) = S^{-1}\tilde{M}S(\tilde{x})
\]

with \(\tilde{M}^-_1\) uniformly negative definite and \(\tilde{M}^+_1\) uniformly positive definite. Applying this coordinate change, and noting that the “dynamic error” \(S^{-1}(\partial/\partial\tilde{x})\tilde{M}_1 = \mathcal{O}(|\lambda|^{-1})\), we obtain, finally, the formal expansion

\[
(4.205) \quad \tilde{M}(\tilde{x}, |\lambda|^{-1}) = |\lambda|^{-1/2} \text{diag} \{\tilde{M}^-_1, \tilde{M}^+_1\} + \mathcal{O}(|\lambda|^{-1}).
\]

On the sector \(\Omega_P\), blocks \pm are exponentially separated to order \(|\lambda|^{-1/2}\), by Remark 2.12. Thus, we may apply Proposition 2.13 to find that, on \(\Omega_P\), there is a further transformation \(\hat{S} = I_{2r} + \mathcal{O}(|\lambda|^{-1/2})\) converting (4.205) to fully diagonalized form

\[
(4.206) \quad M(\tilde{x}, |\lambda|^{-1}) = |\lambda|^{-1/2} \text{diag} \{M^-_1, M^+_1\},
\]

where \(M^\pm_1 = \hat{M}^\pm + \mathcal{O}(|\lambda|^{-1/2})\) are still uniformly positive/negative definite. We will define \(P_\lambda\) to be the part of the Green distribution associated to the flow of this diagonal block. Note that it has been obtained independent of any separation (or lack thereof) from hyperbolic modes, so is analytic on all of \(\Omega_P\).
More precisely, denote by

\[(4.207) \quad Z := TY := \begin{pmatrix} I_{n-r} & 0 \\ 0 & SSB \end{pmatrix} (T_0 + |\lambda|^{-1} T_1) Y \]

the concatenation of coordinate changes made in the course of the above reductions, \(Z = (\zeta_1, \ldots, \zeta_x, \rho_-, \rho_+)^{tr}\). Then, we have, similarly as in (4.194), the representation

\[(4.208) \quad G_\lambda(x, y) = \begin{cases} \quad (I_n, 0) Q^{-1} T^{-1}(x) F_{Z}^{y-x} \Pi_Z T Q S^1(y)(I_n, 0)^{tr} & x < y, \\
- (I_n, 0) Q^{-1} T^{-1}(x) F_{Z}^{y-x} \Pi_{Z} T Q S^1(y)(I_n, 0)^{tr} & x > y, \end{cases} \]

where \(\Pi_Z^\pm\) and \(F_{Z}^{y-x}\) denote projections and flows in \(Z\)-coordinates, and \(Z\) satisfies the approximately decoupled system of equations

\[(4.209) \quad \zeta' = - (\lambda / a_j - |\lambda|^{-1} \eta^2 / a_j^2 + c_j |\lambda|^{-2} + d_j |\lambda|^{-3}) \zeta + O(|\lambda|^{-1}) (\rho, \zeta), \]
\[(4.210) \quad \rho' = |\lambda|^{-1/2} M_{+\pm} \rho \pm + O(|\lambda|^{-7/2}) \zeta. \]

(Note: the degraded error estimate in the \(\rho\)-equations is due to the ill-conditioning of the balancing transformation \(B\).) Approximating \(\Pi_Z^\pm\) and \(F_{Z}^{y-x}\) by the corresponding projections \(\bar{\Pi}_Z^\pm = \bar{\Pi}_Z^{H, \pm} + \bar{\Pi}_Z^{P, \pm}\) and flows \(\bar{F}_Z^{y-x}\) for the associated decoupled equations, where \(\bar{\Pi}_Z^{H, \pm}\) and \(\bar{\Pi}_Z^{P, \pm}\) denote the subprojections on hyperbolic and parabolic blocks, we obtain for \(x < y\) the approximation

\[(4.211) \quad \bar{G}_\lambda(x, y) := (I_n, 0)^{tr} Q^{-1} T^{-1}(x) \bar{F}_Z^{y-x} (\bar{\Pi}_Z^{H,-} + \bar{\Pi}_Z^{H,-}) T Q S^1(y)(I_n, 0),\]

and similarly for \(x > y\). Defining

\[(4.212) \quad P_\lambda(x, y) := (I_n, 0)^{tr} Q^{-1} T^{-1}(x) \bar{F}_Z^{y-x} \bar{\Pi}_Z^{H,-} T Q S^1(y)(I_n, 0)^{tr} = (I_n, 0)^{tr} Q^{-1} T^{-1}(x) \bar{F}_Z^{y-x} \text{diag}\{0_n - r, I_r, 0_n\} T Q S^1(y)(I_n, 0)^{tr},\]

and untangling the various coordinate changes, we find that \(P_\lambda\) satisfies bounds (4.182), similarly as in the strictly parabolic case [ZH]. Specifically, we have by direct calculation that

\[(4.213) \quad (I_n, 0)^{tr} Q^{-1} = \begin{pmatrix} A_s^{-1} & 0 \\ -b_2^{-1} b_1 A_s^{-1} & b_2^{-1} 0 \end{pmatrix} = O(1) \]

and

\[(4.214) \quad Q S^{-1}(I_n, 0)^{tr} = \begin{pmatrix} -I_{n-r} & 0 \\ 0 & 0 \end{pmatrix} \text{O}(|\lambda|^{-1}) |\lambda|^{-1} I_r,].\]
whence (by the special form of $T_0$)

\[(4.214) \quad (T_0 + |\lambda|^{-1}T_1)Q\bar{S}^{-1}(I_n, 0)^{tr} = \begin{pmatrix} O(1) & O(1) \\ 0 & 0 \\ O(|\lambda|^{-1}) & |\lambda|^{-1}I_r \end{pmatrix}\]

and therefore

\[(4.215) \quad \Pi^{P,-}_NQ\bar{S}^{-1}(I_n, 0)^{tr} = O(|\lambda|^{-1/2}).\]

Combining with the evident bound

\[(4.216) \quad |\mathcal{F}^{y-x}_\bar{z}\Pi^{P,-}_N| \leq Ce^{-\theta|\lambda|^{-1/2}||\bar{x}-\bar{y}|} = Ce^{-\theta|\lambda|^{1/2}|x-y|},\]

for some $\theta > 0$, coming from $M_1^{-} \leq -\theta|\lambda|^{-1/2}$, we obtain (4.182) for $|\alpha| = 0$. Derivative bounds then follow from the fact that $x$ and $y$ derivatives of all transformations are at worst order one, while

\[
|\frac{\partial}{\partial(x, y)}^{\alpha}\mathcal{F}^{y-x}_\bar{z}\Pi^{P,-}_N| = |\lambda|^{(1+\text{Re}\lambda)/2}|(x, y)|^{\alpha}\mathcal{F}^{y-x}_\bar{z}\Pi^{P,-}_N| = O(|\lambda|^{-|\alpha/2|})
\]

by inspection of the decoupled equation $\rho'_- = |\lambda|^{-1/2}M_1^{-}\rho_-$. For closely related but much simpler calculations in the strictly parabolic case, see the proof of Proposition 7.3, [ZH] (discussion just below (7.24)).

**Approximation error.** Let us now return to the approximately decoupled equations (4.209), or, converting back to the original spatial coordinates:

\[(4.217) \quad \zeta'_j = -(\lambda/a^+_j - \eta^*_j/a^*_j + c_j\lambda^{-1} + d_j\lambda^{-2})\zeta_j + O(|\lambda|^{-3})(\rho, \zeta), \quad \rho'_\pm = \lambda^{1/2}M_1^{\pm}\rho_\pm + O(|\lambda|^{-5/2})\zeta.\]

By condition (2.52) together with (4.21) to the $\zeta_j$ equations and condition (2.32) to the $\rho_\pm$ equations, we find that the stable and unstable manifolds of the decoupled flow, consisting respectively of $\rho_- \pm\text{ together with } \zeta_j, a^+_j > 0$ and $\rho_+ \pm \text{ together with } \zeta_j, a^*_j > 0$, are exponentially separated to order $\theta \min\{|\lambda|^{1/2}, 1+\text{Re}\lambda\}$, for some $\theta > 0$. Applying Proposition 2.13 together with Corollary 2.14, therefore, yields just the decoupled flows with error terms of order $|\lambda|^{-5/2}(e^{-\theta|\lambda|^{1/2}|x-y|}e^{-\theta(1+\text{Re}\lambda)|x-y|})$, and diagonalization error $O(|\lambda|^{-5/2})$.

Recalling representation (4.208), we find after some rearrangement that $G_\lambda$ is given by the decoupled approximant $G_\lambda$ of (4.210) plus error terms (coming from flow plus diagonalization error) of form $|\lambda|^{-5/2}(e^{-\theta|\lambda|^{1/2}|x-y|}e^{-\theta(1+\text{Re}\lambda)|x-y|})$, absorbable in term $D$. The contribution to $G_\lambda$ from decoupled parabolic modes, as already discussed, is exactly $P_\lambda$. Having separated off the parabolic modes, we have now essentially reduced to the purely hyperbolic case treated in [MaZ.1], and we proceed accordingly. Focusing on the decoupled hyperbolic equations

\[(4.218) \quad \zeta'_j = -(\lambda/a^*_j - \eta^*_j/a^*_j)\zeta_j + (c_j\lambda^{-1} + d_j\lambda^{-2})\zeta_j,\]
and applying Corollary 2.14 once again, considering the second term of the right-hand side as a perturbation, we find, substituting into (4.210), that the contribution to $G_\lambda$ from decoupled hyperbolic modes is $H_\lambda$ plus terms of form $\Theta_\lambda$ as claimed. We omit the details of this latter calculation, which are entirely similar to those of the relaxation case. We note in passing that exponential separation implies that the stable/unstable manifolds of (4.217) are uniformly transverse for $\lambda \in \Omega$, $|\lambda|$ sufficiently large, yielding $\Omega \setminus B(0, R) \subset \rho(L) \cap \Lambda$, for $R$ sufficiently large, as claimed.

**Derivative estimates.** Derivative estimates (4.183)–(4.184) now follow in straightforward fashion, by differentiation of (4.208), noting from the approximately decoupled equations that differentiation of the flow brings down a factor (to absorbable error) of $1/2$ in hyperbolic modes, $1$ in parabolic modes.

### 4.3.3 Extended spectral theory.

Finally, we cite without proof the extended spectral theory of [ZH, MaZ.3] that we shall need in order to establish necessity of condition $(D)$. In the case that $(D)$ holds, this reduces to the simple relations (4.159). Let $C^j_\eta$ denote the space of $C^j$ functions $f(x)$ satisfying

$$|(d/dx)^k f(x)e^{-\eta|x|}| \leq C, \quad 0 \leq k \leq j. \quad (4.219)$$

**Definition 4.34.** Let $L$ be a (possibly degenerate type) $r$th order linear ordinary differential operator for which the associated eigenvalue equation may be written as a first-order system with bounded $C^q$ coefficients (so that $L : C^{q+r}_\eta \to C^q_\eta$), and let $G_\lambda$ denote the Green distribution of $L - \lambda I$, $G_\lambda \in C^{q+1}_{-\eta/2}(x, y)$ away from $y = x$. Further, let $\Omega$ be an open, simply connected domain intersecting the resolvent set of $L$, on which $G_\lambda$ has a (necessarily unique) meromorphic extension considered as a function from $\Omega$ to $L^\infty_{loc}$. Then, for $\lambda_0 \in \Omega$, we define the effective eigenprojection $P_{\lambda_0} : C^q_\eta \to C^{q-\eta/2}_\eta$ by

$$P_{\lambda_0}f(x) = \int_{-\infty}^{+\infty} P_{\lambda_0}(x, y)f(y)dy,$$

where

$$P_{\lambda_0}(x, y) = \text{Res}_{\lambda_0} G_\lambda(x, y) \quad (4.220)$$

and $\text{Res}_{\lambda_0}$ denotes residue at $\lambda_0$. (See [ZH], p. 44, for a proof that $P_{\lambda_0} : C^q_\eta \to C^{q-\eta/2}_\eta$.) We will refer to $P_{\lambda_0}(x, y)$ as the projection kernel. Likewise, we define the effective eigenspace $\Sigma'_{\lambda_0}(L) \subset C^q_\eta$ by

$$\Sigma'_{\lambda_0}(L) = \text{Range } (P_{\lambda_0}),$$
and the effective point spectrum $\sigma_\eta^\prime(L)$ of $L$ in $\Omega$ to be the set of $\lambda \in \Omega$ such that $\dim \Sigma_{\lambda_0}^\prime(L) \neq 0$.

**Definition 4.35.** Let $L$, $\Omega$, $\lambda_0$ be as above, and $K$ be the order of the pole of $(L - \lambda I)^{-1}$ at $\lambda_0$. For $\lambda_0 \in \Omega$, and $k$ any integer, we define $Q_{\lambda_0,k} : C^q_{\eta} \to C^q_{-\eta/2}$ by

$$Q_{\lambda_0,k}(x, y) = \text{Res}_{\lambda_0}(\lambda - \lambda_0)^k G_\lambda(x, y).$$

For $0 \leq k \leq K$, we define the effective eigenspace of ascent $k$ by

$$\Sigma_{\lambda_0,k}^\prime(L) = \text{Range } (Q_{\lambda_0,k} + I)$$

With the above definitions, we obtain the following, modified Fredholm Theory.

**Proposition 4.36 [ZH, MaZ.3].** Let $L$, $\lambda_0$, $\Omega$ be as in Definition 4.34, and $K$ be the order of the pole of $G_\lambda$ at $\lambda_0$. Then,

(i) The operators $P_{\lambda_0} : Q_{\lambda_0,k} : C^q_{\eta} \to C^q_{-\eta}$ are $L$-invariant, with

$$Q_{\lambda_0,k+1} = (L - \lambda_0 I)Q_{\lambda_0,k} = Q_{\lambda_0,k}(L - \lambda_0 I)$$

for all $k \neq -1$, and

$$Q_{\lambda_0,k} = (L - \lambda_0 I)^k P_{\lambda_0}$$

for $k \geq 0$.

(ii) The effective eigenspace of ascent $k$ satisfies

$$\Sigma_{\lambda_0,k}^\prime(L) = (L - \lambda_0 I)\Sigma_{\lambda_0,k+1}(L).$$

for all $0 \leq k \leq K$, with

$$\{0\} = \Sigma_{\lambda_0,0}(L) \subset \Sigma_{\lambda_0,1}(L) \subset \cdots \subset \Sigma_{\lambda_0,K}(L) = \Sigma_{\lambda_0}^\prime(L).$$

Moreover, each containment in (4.224) is strict.

(iii) On $P_{\lambda_0}^{-1}(C^q_{\eta})$, $P_{\lambda_0}$, $Q_{\lambda_0,k}$ all commute ($k \geq 0$), and $P_{\lambda_0}$ is a projection. More generally, $P_{\lambda_0} f = f$ for any $f \in C^{Kr}_{\eta}$ such that $(L - \lambda_0)^K f = 0$, hence each $\Sigma_{\lambda_0,k}^\prime(L)$ contains all $f \in C^{kr}_{\eta}$ such that $(L - \lambda_0)^k f = 0$.

(iv) The multiplicity of the eigenvalue $\lambda_0$, defined as $\dim \Sigma_{\lambda_0}^\prime(L)$, is finite and bounded by $Kn$, where $n$ is the dimension of the phase space for the first-order eigenvalue ODE. Moreover, for all $0 \leq k \leq K$,

$$\dim \Sigma_{\lambda_0,k}^\prime(L) = \dim \Sigma_{\lambda_0,k}^\prime(L^*).$$
Further, the projection kernel can be expanded as

\[(4.226) \quad P_{\lambda_0} = \sum_j \varphi_j(x)\pi_j(y),\]

where \(\{\varphi_j\}, \{\pi_j\}\) are bases for \(\Sigma'_{\lambda_0}(L), \Sigma'_{\lambda_0}(L^*),\) respectively.

**Proof.** Residue calculations similar in spirit to those of [Kat] in the classical case of an isolated eigenvalue of finite multiplicity, but substituting everywhere the resolvent kernel for the resolvent. See [ZH] for details. \(\Box\)

**Remark 4.37.** For \(\lambda_0\) in the resolvent set, \(P_{\lambda_0}\) agrees with the standard definition, hence \(\Sigma'_{\lambda_0,k}(L)\) agrees with the usual \(L^p\) eigenspace \(\Sigma_{\lambda_0,k}(L)\) of generalized eigenfunctions of ascent \(\leq k\), for all \(p < \infty\), since \(C_0^\infty\) is dense in \(L^p\), \(p > 1\), and \(\Sigma'_{\lambda_0,k}(L)\) is closed. In the context of stability of traveling waves (more generally, whenever the coefficients of \(L\) exponentially approach constant values at \(\pm \infty\)), \(\Sigma'_{\lambda_0,k}(L)\) lies between the \(L^p\) subspace \(\Sigma_{\lambda_0,k}(L)\) and the corresponding \(L^p_{\text{Loc}}\) subspace.

Now, suppose further (as holds in the case under consideration) that, on \(\Omega\): (i) \(L\) is a differential operator for which both the forward and adjoint eigenvalue equations may be expressed in appropriate phase spaces as nondegenerate first-order ODE of the same order, and for which the reduced quadratic form \(S\) of Lemma 2.22 is invertible; and, (ii) there exists an analytic choice of bases for stable/unstable manifolds at \(+\infty/-\infty\) of the associated eigenvalue equation. In this case, we can define an analytic Evans function \(D(\lambda)\) as in (3.1), which we assume does not vanish identically. It is easily verified that the order to which \(D\) vanishes at any \(\lambda_0\) is independent of the choice of analytic bases \(\Phi^\pm\). The following result established in [ZH, MaZ.3] generalizes to the extended spectral framework the standard result of Gardner and Jones [GJ.1–2] in the classical setting.

**Proposition 4.38 [ZH, MaZ.3].** Let \(L, \lambda_0\) be as above. Then,

(i) \(\dim \Sigma'_{\lambda_0}(L)\) is equal to the order \(d\) to which the Evans function \(D_L\) vanishes at \(\lambda_0\).

(ii) \(P_{\lambda_0}(L) = \sum_j \varphi_j(x)\pi_j(y),\) where \(\varphi_j\) and \(\pi_j\) are in \(\Sigma'_{\lambda_0}(L)\) and \(\Sigma'_{\lambda_0}(L^*),\) respectively, with ascents summing to \(\leq K + 1\), where \(K\) is the order of the pole of \(G_\lambda\) at \(\lambda_0\).

**Proof.** Direct calculation using the augmented resolvent kernel formula (2.88) of Remark 2.24, Section 2.4.2, and working in a Jordan basis similarly as in [GJ.1]. See [ZH] for details. \(\Box\)

### 4.3.4 Bounds on the Green distribution.

With these preparations, we may now establish pointwise bounds on the Green distribution \(G\) and consequently sufficiency and necessity of condition \((D)\) for linearized orbital stability.
Proposition 4.39. In dimension $d = 1$, assuming (A1)–(A3), (H0)–(H3), and (D), the Green distribution $G(x, t; y)$ associated with (2.1) may be decomposed as in Theorem 4.10, where, for $y \leq 0$:

\begin{align}
R(x, t; y) &= \mathcal{O}(e^{-\eta t} e^{-|x-y|^2/Mt}) \\
&\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1} e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1} e^{-(x-y-a_k^- t^2)/Mt} \\
&\quad + \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{|a_k^- t| \geq |y|\}} O((t+1)^{-1} t^{-1/2}) e^{-(x-a_j^- (t-|y/a_k^-|))}/Mt e^{-\eta x^+}, \\
&\quad + \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{|a_k^- t| \geq |y|\}} O((t+1)^{-1} t^{-1/2}) e^{-(x-a_j^+ (t-|y/a_k^-|))}/Mt e^{-\eta x^-},
\end{align}

(4.227)

\begin{align}
R_y(x, t; y) &= \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t}) \delta_{a_j^+} t (-y) + \mathcal{O}(e^{-\eta t} e^{-|x-y|^2/Mt}) \\
&\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1} e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1} e^{-(x-y-a_k^- t^2)/Mt} \\
&\quad + \sum_{a_k^- > 0, a_j^- < 0} \chi_{\{|a_k^- t| \geq |y|\}} O((t+1)^{-1} t^{-1}) e^{-(x-a_j^- (t-|y/a_k^-|))}/Mt e^{-\eta x^+} \\
&\quad + \sum_{a_k^- > 0, a_j^+ > 0} \chi_{\{|a_k^- t| \geq |y|\}} O((t+1)^{-1} t^{-1/2}) e^{-(x-a_j^+ (t-|y/a_k^-|))}/Mt e^{-\eta x^-},
\end{align}

(4.228)

for some $\eta, M > 0$, where $a_j^+, \bar{a}_j^+$ are as in Theorem 4.10, $x^\pm$ denotes the positive/negative part of $x$, and indicator function $\chi_{\{|a_k^- t| \geq |y|\}}$ is one for $|a_k^- t| \geq |y|$ and zero otherwise. Symmetric bounds hold for $y \geq 0$.

**Proof.** Our starting point is the representation

\begin{equation}
G(x, t; y) = \frac{1}{2\pi i} \text{P.V.} \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_{\lambda}(x, y) d\lambda,
\end{equation}

valid by Proposition 2.3 for any $\eta$ sufficiently large.

**Case I.** $|x - y|/t$ large. We first treat the simpler case that $|x - y|/t \geq S$, $S$ sufficiently large. Fixing $x, y, t$, set $\lambda = \eta + i\xi$. Applying Proposition 4.33, we
obtain, for \( \eta > 0 \) sufficiently large, the decomposition
\begin{equation}
(4.230)
G(x, t; y) = \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} H_\lambda(x, y) d\lambda + \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} \Theta^H_\lambda(x, y) d\lambda \\
+ \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} P_\lambda(x, y) d\lambda + \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} \Theta^P_\lambda(x, y) d\lambda =: I + II + III + IV,
\end{equation}
where
\begin{equation}
(4.231)
H_\lambda(x, y) = \begin{cases} 
- \sum_{j=K+1}^{J} a^*_j(y)^{-1} e^{-\int_y^x \frac{\lambda}{a^*_j(z)} dz} R^*_j(x) \zeta^*_j(x, y) L^t_j(y) & x > y, \\
\sum_{j=1}^{K} a^*_j(y)^{-1} e^{-\int_y^x \frac{\lambda}{a^*_j(z)} dz} R^*_j(x) \zeta^*_j(x, y) L^t_j(y) & x < y,
\end{cases}
\end{equation}
and
\begin{equation}
(4.232)\Theta^H_\lambda(x, y) := \lambda^{-1} B(x, y; \lambda) + \lambda^{-1} (x - y) C(x, y; \lambda),
\end{equation}
and
\begin{equation}
(4.233)\Theta^P_\lambda(x, y) := \lambda^{-2} D(x, y; \lambda),
\end{equation}
with \( \zeta_j, B, C, D \) as defined in Proposition 4.33. For definiteness taking \( x > y \), we estimate each term in turn.

\textit{Term I.} Term I of (4.230) contributes to integral (4.229) the explicitly evaluable quantity
\begin{equation}
(4.234)
\frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} H_\lambda(x, y) d\lambda = \sum_{j=K+1}^{J} \left( a^*_j(x)^{-1} \text{P.V.} \int_{-\infty}^{+\infty} e^{\xi \left( t - \int_y^x \frac{1}{a^*_j(z)} dz \right)} d\xi \right) \\
\times e^{\eta \left( t - \int_y^x \frac{1}{a^*_j(z)} dz \right)} r_j(x) \tilde{\zeta}_j(x, y) l^t_j(y) \\
= \sum_{j=K+1}^{J} \left( a^*_j(x)^{-1} \delta(t - \int_y^x \frac{1}{a^*_j(z)} dz) \right) r_j(x) \tilde{\zeta}_j(x, y) l^t_j(y) \\
= \sum_{j=K+1}^{J} a^*_j(x)^{-1} a^*_j(y) \delta_{x-a^*_j t}(-y) r_j(x) \zeta^*_j(y, t) l^t_j(y) \\
= H(x, t; y),
\end{equation}
where \( a^*_j, \zeta^*_j, \) and \( H \) are as defined in Theorems 4.10 and 4.39. Note: in the second to last equality of (4.234) we have used the general fact that
\begin{equation}
(4.235) f_y(x, y, t) \delta(f(x, y, t)) = \delta_{h(x, t)}(y),
\end{equation}
provided \( f_y \neq 0 \), where \( h(x, t) \) is defined by \( f(x, h(x, t), t) = 0 \). This term clearly vanishes for \( x \) outside \([z_1(y, t), z_J(y, t)]\), hence makes no contribution for \( S \) sufficiently large.

**Term II.** Similar calculations show that the “hyperbolic error term” II in (4.230) also vanishes. For example, the term

\[
\frac{1}{2\pi i} \sum_{j=K+1}^{K} \left( \text{P.V.} \int_{-\infty}^{+\infty} (\eta + i\xi)^{-1} e^{\xi (t-f_y^* (1/a_j^*(z)) dz)} \right) e^{\eta (t-f_y^* (1/a_j^*(z)) dz)} b_j(x, y).
\]

The factor

\[
(\text{4.237}) \quad \text{P.V.} \int_{-\infty}^{+\infty} (\eta + i\xi)^{-1} e^{\xi (t-f_y^* (1/a_j^*(z)) dz)} d\xi,
\]

though not absolutely convergent, is integrable and uniformly bounded as a principal value integral, for all real \( \eta \) bounded from zero, by explicit computation. On the other hand,

\[
(\text{4.238}) \quad e^{\eta (t-f_y^* (1/a_j^*(z)) dz)} \leq Ce^{-\eta d(x, [z_1(y, t), z_J(y, t)])/\min_j |a_j^*(x)|} \to 0
\]
as \( \eta \to +\infty \), for each \( K+1 \leq j \leq J \), since \( a_j^* > 0 \) on this range. Thus, taking \( \eta \to +\infty \), we find that the product (4.236) goes to zero, giving the result. Likewise, the contributions of terms \( e^{\lambda} C(x, y; \lambda) \) and \( e^{\lambda} D(x, y; \lambda) \) split into the product of a convergent, uniformly bounded integral in \( \xi \), a (constant) factor depending only on \( (x, y) \), and a factor \( \alpha(x, y, t, \eta) \) going to zero as \( \eta \to 0 \) at rate (4.238). Thus, each of these terms vanishes also as \( \eta \to +\infty \), as claimed.

**Term III.** The parabolic term III may be treated exactly as in the strictly parabolic case [ZH]. Namely, using analyticity of \( P_\lambda \) within the sector \( \Omega_\lambda \) defined in (4.173), we may for any fixed \( \eta \) deform the contour in the principal value integral to

\[
(\text{4.239}) \quad \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda} P_\lambda(x, y) d\lambda,
\]

where \( \Gamma_1 := \partial B(0, R) \cap \Omega_\lambda \) and \( \Gamma_2 := \partial \Omega_\lambda \setminus B(0, R) \) plus an error term of order not greater than

\[
|\xi|^{-1/2} \int_{\eta}^{-\infty} e^{w_{12} dw} \to 0
\]
as \( |\xi| \to \infty \), where

\[
(\text{4.240}) \quad R := \alpha^2, \quad \bar{\alpha} := \frac{\theta|x-y|}{2t},
\]
By estimate (4.182) we have for all \( \lambda \in \Gamma_1 \cup \Gamma_2 \) that
\[
(4.241) \quad |P_\lambda(x, y)| \leq C|\lambda^{-1/2}|e^{-\theta|\lambda^{1/2}|x-y}|
\]
Further, we have
\[
(4.242) \quad \text{Re} \lambda \leq R(1 - \eta \omega^2), \quad \lambda \in \Gamma_1,
\]
\[
\text{Re} \lambda \leq \text{Re} \lambda_0 - \eta(|\text{Im} \lambda| - |\text{Im} \lambda_0|), \quad \lambda \in \Gamma_2
\]
for \( R \) sufficiently large, where \( \omega \) is the argument of \( \lambda \) and \( \lambda_0 \) and \( \lambda_0^* \) are the two points of intersection of \( \Gamma_1 \) and \( \Gamma_2 \), for some \( \eta > 0 \) independent of \( \bar{\alpha} \).

Combining (4.241), (4.242), and (4.240), we obtain
\[
|\int_{\Gamma_1} e^{\lambda t} P_\lambda d\lambda| \leq \int_{\Gamma_1} C|\lambda^{-1/2} e^{\text{Re} \lambda t - \theta|\lambda^{1/2}|x-y}| d\lambda
\]
\[
\leq C e^{-\bar{\alpha} t} \int_{-L}^{+L} R^{-1/2} e^{-R\eta \omega^2 t} Rd\omega
\]
\[
\leq C t^{-1/2} e^{-\bar{\alpha}^2 t}.
\]
Likewise,
\[
|\int_{\Gamma_2} e^{\lambda t} P_\lambda d\lambda| \leq \int_{\Gamma_2} C|\lambda^{-1/2} |Ce^{\text{Re} \lambda t - \theta|\lambda^{1/2}|x-y}| d\lambda
\]
\[
\leq C e^{\text{Re} \lambda_0 t - \theta|\lambda_0^{1/2}|x-y} \int_{\Gamma_2} |\lambda^{-1/2} e^{(\text{Re} \lambda) - \text{Re} \lambda_0 t}| d\lambda| \leq C e^{-\bar{\alpha}^2 t} \int_{\Gamma_2} |\text{Im} \lambda|^{-1/2} e^{-\eta|\text{Im} \lambda - \text{Im} \lambda_0|t} |d \text{Im} \lambda|
\]
\[
\leq C t^{-1/2} e^{-\bar{\alpha}^2 t}.
\]
Combining these last two estimates, and recalling (4.240), we have
\[
(4.244) \quad III \leq C t^{-1/2} e^{-\bar{\alpha}^2 t/2} e^{-\theta^2(x-y)^2/8t} \leq C t^{-1/2} e^{-\eta t e^{-\theta^2/4M t}},
\]
for \( \eta > 0, M > 0 \) independent of \( \bar{\alpha} \). Observing that
\[
\frac{|x - at|}{2t} \leq \frac{|x - y|}{t} \leq \frac{2|x - at|}{t}
\]
for any bounded \( a \), for \( |x - y|/t \) sufficiently large, we find that \( III \) can be absorbed in the residual term \( \mathcal{O}(e^{-\eta t e^{-|x-y|^2/4M t}}) \) for \( t \geq \epsilon \), any \( \epsilon > 0 \), and by any summand \( \mathcal{O}(t^{-1/2}(t + 1)^{-1/2} e^{-(x-y-a_{\epsilon}^2 t)^2/M t})e^{-\eta x \pm e^{-\eta y \pm}} \) for \( t \) small.
Likewise, differentiating within the absolutely convergent integral (4.239), we find that $|III_y|$ can be absorbed in the corresponding summand of (4.228).

Note. A delicate point about the derivative bounds to first move the contour to (4.239) and then differentiate. Differentiating within the original principal value integral yields an integral that is not obviously convergent, and for which it is not clear that the contour may be so deformed.

Term IV. Similarly as in the treatment of term III, the principal value integral for the “parabolic error term” IV may be shifted to $\eta = R = \bar{\alpha}^2$, $\bar{\alpha}$ as in (4.240), at the cost of an error term that vanishes as $|\xi| \to \infty$. But, this yields an estimate

$$|IV| \leq C e^{-\bar{\alpha}^2t} \int_{-\infty}^{+\infty} |\eta_0 + i\xi|^{-2} d\xi \leq e^{-\bar{\alpha}^2t}$$

that by (4.244) may be absorbed in $O(e^{-\eta t} e^{-|x-y|^2/Mt})$ for all $t$. Derivatives $IV_x$ and $IV_y$ may be similarly absorbed, while $IV_{xy}$ may be absorbed in

$$(\partial/\partial y)O(e^{-\eta t} e^{-|x-y|^2/Mt}).$$

Case II. $|x - y|/t$ bounded. We now turn to the critical case that $|x - y|/t \leq S$ for some fixed $S$. In this regime, note that any contribution of order $e^{\theta t}$, $\theta > 0$, may be absorbed in the residual (error) term $R$ (resp. $R_x$, $R_y$); we shall use this observation repeatedly.

Decomposition of the contour. We begin by converting contour integral (4.229) into a more convenient form decomposing high, intermediate, and low frequency contributions.

Observation 4.40. In dimension $d = 1$, assuming (A1)–(A3), (H0)–(H3), there holds the representation

$$G(x, t; y) = I_a + I_b + I_c + II_a + II_b$$

$$:= \frac{1}{2\pi i} \text{P.V.} \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x, y) d\lambda$$

$$+ \frac{1}{2\pi i} \text{P.V.} \left( \int_{-\eta_1 - i\infty}^{-\eta_1 + i\infty} + \int_{-\eta_1 + i\infty}^{-\eta_1 - i\infty} \right) e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x, y) d\lambda$$

$$+ \frac{1}{2\pi i} \oint_{\Gamma_2} e^{\lambda t} P_\lambda(x, y) d\lambda$$

$$+ \frac{1}{2\pi i} \oint_{\Gamma_3} e^{\lambda t} G_\lambda(x, y) d\lambda - \frac{1}{2\pi i} \int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} H_\lambda(x, y) d\lambda,$$

(4.246) $\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR]$,
\( \Gamma_2 := \partial \Omega_p \setminus \Omega \)

\( \Omega_p \) as defined in (4.173), for any \( \eta > 0 \) such that (4.229) holds, \( R \) sufficiently large, and \( -\eta_1 < 0 \) as in (4.174). (Note that we have not here assumed (D)).

**Proof.** We first observe that, by Proposition 4.33, \( L \) has no spectrum on the portion of \( \Omega \) lying outside of the rectangle

\( \mathcal{R} := \{ \lambda : -\eta_1 \leq \text{Re} \lambda \leq \eta, -R \leq \text{Im} \lambda \leq R \} \)

for \( \eta > 0, R > 0 \) sufficiently large, hence \( G_\lambda \) is analytic on this region. Since, also, \( H_\lambda \) is analytic on the whole complex plane, contours involving either one of these contributions may be arbitrarily deformed within \( \Omega \setminus \mathcal{R} \) without affecting the result, by Cauchy’s Theorem. Likewise, \( P_\lambda \) is analytic on \( \Omega_p \setminus \mathcal{R} \), and so contours involving this contribution may be arbitrarily deformed within this region.

Recalling, further, that

\[ |G_\lambda - H_\lambda - P_\lambda| = \mathcal{O}(\lambda^{-1}) \]

by Proposition 4.33, we find that

\[ \frac{1}{2\pi i} \mathrm{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x, y) d\lambda \]

may be deformed to

\[ \frac{1}{2\pi i} \mathrm{P.V.} \oint_{\partial(\Omega \setminus \mathcal{R})} e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x, y) d\lambda. \]

And, similarly as in the treatment of term III of the large \( |x - y|/t \) case, we find using bounds (4.182) that

\[ \frac{1}{2\pi i} \mathrm{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} P_\lambda(x, y) d\lambda \]

may be deformed to the absolutely convergent integral

\[ \frac{1}{2\pi i} \oint_{\partial(\Omega_p \setminus \mathcal{R})} e^{\lambda t} P_\lambda(x, y) d\lambda. \]

Noting, finally, that

\[ +\frac{1}{2\pi i} \oint_{\partial \mathcal{R}} e^{\lambda t} H_\lambda(x, y) d\lambda = 0, \]
by Cauchy’s Theorem, we obtain, finally,
\begin{equation}
\frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} G(x, y) d\lambda =
\end{equation}
\begin{align*}
\frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} H(x, y) d\lambda + \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x, y) d\lambda \\
+ \frac{1}{2\pi i} \oint_{\partial(\Omega \setminus \mathcal{R})} e^{\lambda t} P_\lambda(x, y) d\lambda
\end{align*}
\begin{align*}
= \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} H(x, y) d\lambda + \frac{1}{2\pi i} \text{P.V.} \oint_{\partial(\Lambda \setminus \mathcal{R})} e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x, y) d\lambda \\
+ \frac{1}{2\pi i} \oint_{\partial \mathcal{R}} e^{\lambda t} H(x, y) d\lambda + \frac{1}{2\pi i} \oint_{\partial(\Omega \setminus \mathcal{R})} e^{\lambda t} P_\lambda(x, y) d\lambda,
\end{align*}
from which the result follows by combining the second third and fourth contour integrals along their common edges \( \Gamma \). \( \square \)

**Observation 4.41.** In dimension \( d = 1 \), assuming (A1)–(A3), (H0)–(H3), and (D), we may replace (4.245) by
\begin{equation}
G(x, t; y) = I_a + I_b + I_c + II_\tilde{a} + II_b + III,
\end{equation}
where \( I_a, I_b, I_c, \) and \( II_b \) are as in (4.245), and
\begin{equation}
II_\tilde{a} := \frac{1}{2\pi} \left( \int_{-\eta_1-iR}^{-\eta_1+ir/2} + \int_{-\eta_1+ir/2}^{-\eta_1+ir} \right) e^{\lambda t} G(x, y) dx,
\end{equation}
\begin{equation}
III := \frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} G(x, y) d\lambda,
\end{equation}
and
\begin{equation}
\tilde{\Gamma} := [-\eta_1 - ir/2, \eta - ir/2] \cup [\eta - ir/2, \eta + ir/2] \cup [\eta + ir/2, -\eta_1 + ir/2],
\end{equation}
for any \( \eta, r > 0 \), and \( \eta_1 \) sufficiently small with respect to \( r \).

**Proof.** By assumption (D), \( L \) has no spectrum on the region between contour \( \Gamma \) and the union of contour \( \tilde{\Gamma} \) and the contour of term \( II_\tilde{a} \), hence \( G_\lambda \) is analytic on that region, and
\begin{equation}
II_a = II_\tilde{a} + III
\end{equation}
by Cauchy’s Theorem, giving the result. \( \square \)
Using the final decomposition (4.250), we shall estimate in turn the high frequency contributions \( I_a, I_b, \) and \( I_c \), the intermediate frequency contributions \( II_a \) and \( II_b \), and the low frequency contributions \( III \).

**High frequency contribution.** We first carry out the straightforward estimation of the high-frequency terms \( I_a, I_b, \) and \( I_c \). The principal term \( I_a \) has already been computed in (4.234) to be \( H(x, t; y) \). Likewise, calculations similar to those of (4.236)–(4.238) show that the error term

\[
I_b = \frac{1}{2\pi i} \text{P.V.} \left( \int_{-\eta_1-i\infty}^{-\eta_1-R} \int_{-\eta_1+iR}^{-\eta_1+i\infty} \right) e^{s\lambda} \Theta_\lambda(x, y) d\lambda
\]

is time-exponentially small.

For example, for \( x > y \) the term \( e^{s\lambda} \lambda^{-1} B(x, y; \lambda) \) contributes

\[
\sum_{j=K+1}^{\infty} \frac{1}{2\pi i} \text{P.V.} \left( \int_{-\infty}^{-R} \int_{R}^{+\infty} \right) (-\eta_1 + i\xi)^{-1} e^{i\xi \left( t - \int_y^z (1/a_j^s(z)) \right) dz} d\xi
\]

\[
\times e^{-\eta_1 \left( t - \int_y^z (1/a_j^s(z)) \right) dz} b_j^+ (x, y),
\]

where

\[
\frac{1}{2\pi i} \text{P.V.} \left( \int_{-\infty}^{-R} \int_{R}^{+\infty} \right) (\eta + i\xi)^{-1} e^{i\xi \left( t - \int_y^z (1/a_j^s(z)) \right) dz} d\xi < \infty,
\]

and

\[
e^{\eta_1 \int_y^z (1/a_j^s(z)) dz} b(x, y) \leq C_1 e^{\eta_1 \int_y^z (1/a_j^s(z)) dz - \theta |x - y|} \leq C_2,
\]

for \( \eta_1 \) sufficiently small. This may be absorbed in the first term of \( R \), (4.227). Likewise, the contributions of terms \( e^{s\lambda} C(x, y; \lambda) \) and \( e^{s\lambda} D(x, y; \lambda) \) split into the product of a convergent, uniformly bounded integral in \( \xi \), a bounded factor analogous to (4.254), and the factor \( e^{-\eta_1 t} \), giving the result.

The term \( I_c \) may be estimated exactly as was term III in the large \( |x - y|/t \) case, to obtain contribution \( \mathcal{O}(t^{-1/2} e^{-\eta t}) \) absorbable again in the residual term \( \mathcal{O}(e^{-\eta t} e^{-|x-y|^2/4Mt}) \) for \( t \geq \epsilon \), any \( \epsilon > 0 \), and by any summand \( \mathcal{O}(t^{-1/2}(t + 1)^{-1/2} e^{-|x-y-a_x^s|^2/4Mt}) e^{-\eta x \pm e^{-\eta y} \pm} \) for \( t \) small.

**Derivative bounds.** Derivatives \( (\partial/\partial y) I_b \) may be treated in identical fashion using (4.184) to show that they are absorbable in the estimates given for \( R_y \). We point
out that the integral arising from term $B_0$ of (4.184) corresponds to the first, delta-function term in (4.228), while the integral arising from term $(x-y)C_0^y$ vanishes, as the product of $(x-y)$ and a delta function $\delta_z(y)$ with $z \neq x$. Derivatives of term $I_c$ may be treated exactly as were derivatives of term III in the large $|x-y|/t$ case.

**Intermediate frequency contribution.** Error term $II_b$ is time-exponentially small for $\eta_1$ sufficiently small, by the same calculation as in (4.252)–(4.254), hence negligible. Likewise, term $II_a$ by the basic estimate (2.85) is seen to be time-exponentially small of order $e^{-\eta_1 t}$ for any $\eta_1 > 0$ sufficiently small that the associated contour lies in the resolvent set of $L$.

**Low frequency contribution.** It remains to estimate the low-frequency term $III$, which is of essentially the same form as the low-frequency contribution analyzed in [ZH, Z.3, Z.6] in the strictly parabolic case, in that the contour is the same and the resolvent kernel $G_\lambda$ satisfies identical bounds in this regime. Thus, we may conclude from these previous analyses that $III$ gives contribution $E + S + R$, as claimed, exactly as in the strictly parabolic case. For completeness, we indicate the main features of the argument here.

Case $t \leq 1$. First observe that estimates in the short-time regime $t \leq 1$ are trivial, since then $|e^{\lambda t}G_\lambda(x, y)|$ is uniformly bounded on the compact set $\tilde{\Gamma}$, and we have $|G(x, t; y)| \leq C \leq e^{-\delta t}$ for $\theta > 0$ sufficiently small. But, likewise, $E$ and $S$ are uniformly bounded in this regime, hence time-exponentially decaying. As observed previously, all such terms are negligible, being absorbable in the error term $R$. Thus, we may add $E + S$ and subtract $G$ to obtain the result.

Case $t \geq 1$. Next, consider the critical (long-time) regime $t \geq 1$. For definiteness, take $y \leq x \leq 0$; the other two cases are similar. Decomposing

$$G(x, t; y) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} e^{\lambda t} E_\lambda(x, y) d\lambda + \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} e^{\lambda t} S_\lambda(x, y) d\lambda + \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} e^{\lambda t} R_\lambda(x, y) d\lambda,$$

with $E_\lambda$, $S_\lambda$, and $R_\lambda$ as defined in Proposition 4.22, we consider in turn each of the three terms on the righthand side.

**$E_\lambda$ term.** Let us first consider the dominant term

$$\frac{1}{2\pi i} \oint_{\tilde{\Gamma}} e^{\lambda t} E_\lambda(x, y) d\lambda,$$

which, by (4.128), is given by

$$\sum_{a_k^- > 0} [c_{k,-}^0] (\partial \tilde{U}/\partial \delta)(x) I_{\lambda}^{-t} \alpha_k(x, t; y),$$

where

$$\alpha_k(x, t; y) := \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} \lambda^{-1} e^{\lambda t} e^{(\lambda/a_k^- - \lambda^2 \beta_k^- / a_k^- 3)y} d\lambda.$$
Using Cauchy’s Theorem, we may move the contour \( \tilde{\Gamma} \) to obtain (4.259)
\[
\alpha_k(x, t; y) = \frac{1}{2\pi} \text{P.V.} \int_{-r/2}^{+r/2} (i\xi)^{-1} e^{i\xi t} e^{(i\xi/a_k^+) + (\xi^2/\beta_k^+/a_k^{-3})y} \, d\xi
\]
\[
+ \frac{1}{2\pi i} \left( \int_{-\eta_1 - ir/2}^{-ir/2} + \int_{ir/2}^{+\eta_1 + ir/2} \right) \lambda^{-1} e^{\lambda t} e^{(\lambda/a_k^+) - (\lambda^2/\beta_k^+/a_k^{-3})y} \, d\lambda
\]
\[
+ \frac{1}{2} \text{Residue } \lambda = 0 e^{\lambda t} e^{(\lambda/a_k^+) - (\lambda^2/\beta_k^+/a_k^{-3})y},
\]
or, rearranging and evaluating the final, residue term:
(4.260)
\[
\alpha_k(x, t; y) = \left( \frac{1}{2\pi} \text{P.V.} \int_{-\infty}^{+\infty} (i\xi)^{-1} e^{i\xi (t+y/a_k^+)} e^{\xi^2 (\beta_k^-/a_k^{-3})y} \, d\xi + \frac{1}{2} \right)
\]
\[
- \frac{1}{2\pi} \left( \int_{-\infty}^{-r/2} + \int_{r/2}^{+\infty} \right) (i\xi)^{-1} e^{i\xi (t+y/a_k^+)} e^{\xi^2 (\beta_k^-/a_k^{-3})y} \, d\xi
\]
\[
+ \frac{1}{2\pi i} \left( \int_{-\eta_1 - ir/2}^{-ir/2} + \int_{ir/2}^{+\eta_1 + ir/2} \right) \lambda^{-1} e^{\lambda t} e^{(\lambda/a_k^+) - (\lambda^2/\beta_k^+/a_k^{-3})y} \, d\lambda.
\]

The first term in (4.260) may be explicitly evaluated to give
(4.261)
\[
\text{errfn} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^- |y/a_k^-|}} \right),
\]
where
(4.262)
\[
\text{errfn} (z) := \frac{1}{2\pi} \int_{-\infty}^{z} e^{-y^2} \, dy,
\]
whereas the second and third terms are clearly time-exponentially small for \( t \leq C|y| \) and \( \eta_1 \) sufficiently small relative to \( r \) (see detailed discussion of similar calculations below, under \( R_\lambda \) term). In the trivial case \( t \geq C|y|, C > 0 \) sufficiently large, we can simply move the contour to \([-\eta_1 - ir/2, -\eta_1 + ir/2]\) to obtain (complete) residue 1 plus a time-exponentially small error corresponding to the shifted contour integral, which result again agrees with (4.261) up to a time-exponentially small error.

Expression (4.261) may be rewritten as
(4.263)
\[
\text{errfn} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^- |y/a_k^-|}} \right)
\]
plus error

\[
\text{erf} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^-}} \right) - \text{erf} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^-}} \right)
\]

(4.264)

\[
\sim \text{erf} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^-}} \right) \left(-\frac{4\beta_k^-}{2} (y + a_k^- t)^2 (4\beta_k^- t)^{-3/2}\right)
\]

\[= \mathcal{O}(t^{-1} e^{(y+a_k^- t)^2/Mt}),\]

for \(M > 0\) sufficiently large, and similarly for \(x\)- and \(y\)-derivatives. Multiplying by \([c_{j,k}^{i,0}](\partial\bar{U}^\delta / \partial \delta)(x) l_k^{-t}\)

we find that term (4.263) gives contribution

(4.265)

\[ [c_{k,-}^{j,0}](\partial\bar{U}^\delta / \partial \delta)(x) l_k^{-t} \text{erf} \left( \frac{y + a_k^- t}{\sqrt{4\beta_k^-}} \right), \]

whereas term (4.264) gives a contribution absorbable in \(R\) (resp. \(R_x, R_y\)); see Remark 7.5 below for detailed discussion of a similar calculation.

Finally, observing that

(4.266)

\[ [c_{k,-}^{j,0}] l_k^{-t} \text{erf} \left( \frac{y - a_k^- t}{\sqrt{4\beta_k^-}} \right) \]

is time-exponentially small for \(t \geq 1\), since \(a_k^- > 0\), \(y < 0\), and \((\partial / \partial j)(\bar{u}^\delta, \bar{v}^\delta) \leq Ce^{-\theta|x|}\), \(\theta > 0\), we may subtract and add this term to (4.265) to obtain a total of \(E(x, t; y)\) plus terms absorbable in \(R\) (resp. \(R_x, R_y\)).

**S\(_\lambda\) term.** Next, consider the second-order term

(4.267)

\[
\frac{1}{2\pi i} \oint_{\Gamma \delta} e^{\lambda t} S_\lambda(x, y) d\lambda,
\]

which, by (4.130), is given by

(4.268)

\[
\sum_{a_k^- > 0} r_k^- l_k^- t \alpha_k(x, t; y) + \sum_{a_k^- > 0, a_j^- < 0} [c_{k,j}^{i,0}] r_j^- l_j^- t \alpha_{jk}(x, t; y)
\]

where

(4.269)

\[
\alpha_k(x, t; y) := \frac{1}{2\pi i} \oint_{\Gamma \delta} e^{\lambda t} (e^{-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^- 3}(x-y)) d\lambda.
\]
and
\begin{equation}
\alpha_{jk}(x, t; y) := \frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} e^{-(\lambda/a_j^- + \lambda^2 \beta_j^-/a_j^-^3)x + (\lambda/a_k^- - \lambda^2 \beta_k^-/a_k^-^3)y} d\lambda.
\end{equation}

Similarly as in the treatment of the $E_\lambda$ term, just above, by deforming the contour $\tilde{\Gamma}$ to
\begin{equation}
\Gamma' := [-\eta_1 - ir/2, -ir/2] \cup [-ir/2, +ir/2] \cup [+ir/2, -\eta_1 + ir/2],
\end{equation}
these may be transformed modulo time-exponentially decaying terms to the elementary Fourier integrals
\begin{equation}
\frac{1}{2\pi} \text{P.V.} \int_{-\infty}^{+\infty} e^{i\xi(t-(x-y)/a_k^-)} e^{\xi^2(-\beta_k^-/a_k^-^3)(x-y)} d\xi
= (4\pi \beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t}
\end{equation}
and
\begin{equation}
\frac{1}{2\pi} \text{P.V.} \int_{-\infty}^{+\infty} e^{i\xi((t-|y/a_k^-|)-x/a_j^-)} e^{\xi^2((\beta_k^-/a_k^-^3)|y|+(\beta_j^-/a_j^-^3)|x|)} d\xi
= (4\pi \beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\beta_{jk}^- t},
\end{equation}
respectively, where $\beta_{jk}^- + -$ and $z_{jk}^- \pm$ are as defined in (4.29) and (4.28). These correspond to the first and third terms in expansion (4.25), the latter of which has an additional factor $e^{-x}/(e^x + e^{-x})$. Noting that the second and fourth terms of (4.25) are time-exponentially small for $t \geq 1$, $y \leq x \leq 0$, and that
\begin{align*}
|(4\pi \beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\beta_{jk}^- t}(1 - e^{-x}/(e^x + e^{-x}))| &
\leq |(4\pi \beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\beta_{jk}^- t} e^{-\theta|x|}|
\end{align*}
for some $\theta > 0$, so is absorbable in error term $R$, we find that the total contribution of this term, modulo terms absorbable in $R$, is $S$.

$R_\lambda$ term. Finally, we briefly discuss the estimation of error term
\begin{equation}
\frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} R_\lambda(x, y) d\lambda,
\end{equation}
which decomposes into the sum of integrals involving the various terms of $R^E_\lambda$ and $R^S_\lambda$ given in (4.132) and (4.134). Since each of these are separately analytic, they may be split up and estimated sharply via the Riemann saddlepoint method (method of steepest descent), as described at great length in [ZH, HoZ.4]. That is,
for each summand \( \alpha(x, y) \sim e^{\beta(x+y+\gamma)y} \) in \( R \) we deform the contour \( \Gamma \) to a new, contour in \( \Lambda \) that is a mini-max contour for the modulus

\[
m_\alpha(x, y, \lambda) := |e^{\lambda \alpha(x, y)}| = e^{Re \lambda t + Re \beta x + Re \gamma y},
\]

passing through an appropriate saddlepoint/critical point of \( m_\alpha(x, y, \cdot) \): necessarily lying on the real axis, by the underlying complex symmetry resulting from reality of operator \( L \).

Since terms of each type appearing in \( R \) have been sharply estimated in [ZH], we shall omit the details, only describing two sample calculations to illustrate the method:

**Example 1:** \( e^{-\theta|x-y|} \). Contour integrals of form

\[
\frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} e^{-\theta|x-y|} d\lambda,
\]

arising through the pairing of fast forward and fast dual modes, may be deformed to

\[
\frac{1}{2\pi i} \int_{-\eta_1+ir/2}^{\eta_1+ir/2} e^{\lambda t} e^{-\theta|x-y|} d\lambda
\]

and estimated as \( O(e^{-\eta_1 t - \theta|x-y|}) \), a negligible, time-exponentially decaying contribution. □

**Example 2:** \( \lambda^r e^{(-\lambda/a_k^{-3} + \lambda^2 \beta_k^{-1}/a_k^{-3})(x-y)} \). Contour integrals of form

\[
\frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} \lambda^r e^{(-\lambda/a_k^{-3} + \lambda^2 \beta_k^{-1}/a_k^{-3})(x-y)} d\lambda,
\]

\( a_k^{-} > 0 \), arising through the pairing of slow forward and slow dual modes, may be deformed to contour

\[
\Gamma' := [-\eta_1 - ir/2, \eta_1 - ir/2] \cup [\eta_1 - ir/2, \eta_1 + ir/2] \cup [\eta_1 + ir/2, -\eta_1 + ir/2],
\]

where saddlepoint \( \eta_1 \) is defined as

\[
\eta_1(x, y, t) := \begin{cases} \frac{\alpha}{p} & \text{if } |\frac{\alpha}{p}| \leq \varepsilon \\ \pm \varepsilon & \text{if } |\frac{\alpha}{p}| \geq \varepsilon, \end{cases}
\]

with

\[
\bar{\alpha} := \frac{x - y - a_k^{-} t}{2t}, \quad p := \frac{\beta_k^{-} (x - y)}{(a_k^{-})^2 t} > 0,
\]
and \( \eta_1, \varepsilon > 0 \) are chosen sufficiently small with respect to \( r \), to yield, modulo time-exponentially decaying terms, the estimate

\[
e^{-|x-y-a_k^- t|^2/4\beta_2 - t} \int_{-\infty}^{+\infty} \mathcal{O}(|\eta_s|^r + |\xi|^r) e^{-\theta |\xi|^2 t} d\xi = \mathcal{O}(t^{-(r+1)/2} e^{-|x-y-a_k^- t|^2/M t})
\]

if \( |\frac{\alpha}{p}| \leq \varepsilon \), and

\[
e^{-\varepsilon t/M} \int_{-\infty}^{+\infty} \mathcal{O}(|\eta_s|^r + |\xi|^r) e^{-\theta |\xi|^2 t} d\xi = \mathcal{O}(t^{-(r+1)/2} e^{-\eta t}).
\]

if \( |\frac{\alpha}{p}| \geq \varepsilon, \theta > 0 \). In either case, the main contribution lies along the central portion \([\eta_s - ir/2, \eta_s + ir/2]\) of contour \( \Gamma' \).

To see this, note that \(|x-y|\) and \( t \) are comparable when \(|\alpha/p|\) is bounded, whence the evident spatial decay \( e^{-\theta |\xi|^2|x-y|} \) of the integrand along contour \( \lambda = \eta_s + i\xi \) may be converted to the \( e^{-\xi^2 t} \) decay displayed in (4.281)–(4.282); likewise, temporal growth in \( e^{rt} \) on the horizontal portions of the contour, of order \( \leq e^{(|\eta_1|+|\varepsilon|)t} \), is dominated by the factor \( e^{-\theta |\xi|^2} = e^{-\theta r^2/4} \), provided \(|\eta_1| + |\varepsilon| \) is sufficiently small with respect to \( r^2 \). For \(|\alpha/p|\) large, on the other hand, \( \eta_s \) is uniformly negative, and also \(|x-y|\) is negligible with respect to \( t \), whence the estimate (4.282) holds trivially.

We point out that \( \eta_s \) is easily determined, as the minimal point on the real axis of the quadratic function

\[
f_{x,y,t}(\lambda) := \lambda t (-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^- 3)(x-y),
\]

the argument of the integrand of (4.277).

Other terms may be treated similarly: All “constant-coefficient” terms \( \phi_k \tilde{\phi}_k^* \) or \( \psi_k \bar{\psi}_k^* \) are of either the form treated in Example 1 (fast modes) or in example 2 (slow modes). Scattering pairs involving slow forward and slow dual modes from different families (i.e., terms with coefficients \( M_{jk}, d_{jk}^\pm \)) may be treated similarly as in Example 2. Scattering pairs involving two fast modes are of the form already treated in Example 1, since both modes of a scattering pair are decaying. Scattering pairs involving one fast and one slow mode may be factored as the product of a term of the form treated in example 2 and a term that is uniformly exponentially decaying in either \( x \) or \( y \); factoring out the exponentially decay, we may treat such terms as in Example 2.

**Scattering coefficients.** The relation (4.30) may now be deduced, a posteriori, from conservation of mass in the linearized flow (2.1). For, by inspection, all terms save \( E \) and \( S \) in the decomposition (4.22) of \( G \) decay in \( L^1 \), whence these terms must, time-asymptotically, carry exactly the mass in \( U \) of the initial perturbation \( \delta_y(x)I_n, \) i.e.,

\[
\lim_{t \to \infty} \int_{-\infty}^{+\infty} (E(x,t;y) + S(x,t;y)) \, dx = I_n.
\]
Taking $y \leq 0$ for definiteness, and right-multiplying both sides of (4.284) by $r_k^-$, we thus obtain the result from definitions (4.24)–(4.25).

Moreover, rewriting (4.30) as

\begin{equation}
(r_1^-, \ldots, r_{n-i}^-, r_{i_+}^+, \ldots, r_n^+, m_1, \ldots, m_\ell) = r_k^-, \\
\begin{pmatrix}
[c_{k_-, -}^{1,-}] \\
\vdots \\
[c_{k_-, -}^{n-i,-}] \\
[c_{k_-, -}^{i_+-}] \\
\vdots \\
[c_{k_-, -}^{n,+}] \\
[c_{k_-, -}^{0,0}] \\
\vdots \\
[c_{k_-, -}^{0,0}]
\end{pmatrix}
\end{equation}

we find that, under assumption $(D)$, it uniquely determines the scattering coefficients $[c_{k_-, -}^{j, \pm}]$. For, the determinant

\begin{equation}
m_j := \int_{-\infty}^{+\infty} (\partial/\partial \delta_j) \tilde{U}^\delta(x) \, dx,
\end{equation}

of the matrix on the righthand side of (4.285) is exactly the inviscid stability coefficient $\Delta$ defined in [ZS, Z.3], hence is nonvanishing by the equivalence of $(D2)$ and $(D2)$ (recall discussion of Section 1.2, just below Theorem 4.8). Note that (4.287) reduces to the inviscid stability determinant (1.36) in the Lax case, $\ell = 1$, with $\tilde{U}^\delta(x)$ parametrized as $\tilde{U}^\delta(x) := \tilde{U}(x - \delta_1 x)$, for which $m_1$ reduces to $[U]$.

Relation (4.31) follows from the observation that $P_{\lambda}(x, y) := \text{Residue } \lambda=0 G_{\lambda}(x, y) = \sum_{j=1}^{\ell} (\partial/\partial \delta_j) \tilde{U}^\delta(x) \pi_j,$

with $\pi_j$ defined by the first, or second expressions appearing in (4.31), according as $y$ is less than or equal to/greater than or equal to zero. For, the extended spectral theory of Section 4.3.3 then implies that $\pi_j$ are the left effective eigenfunctions associated with right eigenfunctions $(\partial/\partial \delta_j) \tilde{U}^\delta$. Alternatively, (4.31) may be deduced by linear-algebraic manipulation directly from (4.285) and its counterpart for $y \geq 0$ (the same identity with $k, -$ everywhere replaced by $k, +$).\hfill\Box
Remark 4.42 (The undercompressive case). In the undercompressive case, the result of Lemma 4.28 is false, and consequently the estimates of Lemma 4.31 do not hold. This fact has the implication that shock dynamics are not governed solely by conservation of mass, as in the Lax or overcompressive case, but by more complicated dynamics of front interaction; for related discussion, see [LZ.2, Z.6]. At the level of Proposition 4.22, it means that the simple representations of \( E_\lambda \) and \( S_\lambda \) in terms of slow dual modes alone (corresponding to characteristics that are incoming to the shock) are no longer valid in the undercompressive case, and there appear new terms involving rapidly decaying dual modes \( \sim e^{-\theta |y|} \) related to inner layer dynamics. Though precise estimates can nonetheless be carried out, we have not found a similarly compact representation of the resulting bounds as that of the Lax/overcompressive case, and so we shall not state them here. We mention only that this rapid variation in the \( y \)-coordinate precludes the \( L^p \) stability arguments used here and in [Z.6], requiring instead detailed pointwise bounds as in [HZ.2]. See [LZ.1–2, ZH, Z.6] for further discussion of this interesting case.

4.3.5. Inner layer dynamics.

Similarly as in [ZH], we now investigate dynamics of the inner shock layer, in the case that \((\mathcal{D})\) does not necessarily hold, in the process establishing the necessity of \((\mathcal{D})\) for linearized orbital stability.

Proposition 4.43. In dimension \( d = 1 \), given \((A1)-(A3)\) and \((H0)-(H3)\), there exists \( \eta > 0 \) such that, for \( x, y \) restricted to any bounded set, and \( t \) sufficiently large,

\[
G(x, y; t) = \sum_{\lambda \in \sigma'_L(L) \cap \{\text{Re}(\lambda) \geq 0\}} e^{\lambda t} \sum_{k \geq 0} t^k (L - \lambda I)^k P_\lambda(x, y) + O(e^{-\eta t}),
\]

where \( P_\lambda(x, y) \) is the effective projection kernel described in Definition 4.34, and \( \sigma'_L(L) \) the effective point spectrum.

Proof. Similarly as in the proof of Proposition 4.39, decompose \( G \) into terms \( I_a, I_b, I_c, II_a, \) and \( II_b \) of (4.245). Then, the same argument (Case II. \( |x - y|/t \) bounded) yields that \( I_a = H(x, t; y) \), while \( I_b \) and \( II_b \) are time-exponentially small. Observing that \( H = 0 \) for \( x, y \) bounded, and \( t \) sufficiently large, we have reduced the problem to the study of

\[
II_a := \frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda,
\]

where, recall,

\[
\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR].
\]
Note that the high-frequency bounds of Proposition 4.33 imply that $L$ has no point spectrum in $\Omega$ outside of the rectangle $R$ enclosed by $\Gamma \cup [-\eta_1 - iR, -\eta + iR]$. Choosing $\eta_1$ sufficiently small, therefore, we may ensure that no effective point spectrum lies within the strip $-\eta_1 \leq \text{Re} \lambda < 0$, by compactness of $R$ together with the fact that effective eigenvalues are isolated from one another, as zeroes of the analytic Evans function.

Recall that $G_\lambda$ is meromorphic on $\Omega$. Thus, we may express (4.289), using Cauchy’s Theorem, as

$$(4.291) \quad \frac{1}{2\pi i} \int_{-\eta_1 - iR}^{\eta_1 + iR} e^{\lambda t} G_\lambda(x, y) d\lambda + \text{Residue}_{\lambda \in R} e^{\lambda t} G_\lambda(x, t; y).$$

By Definition 4.34 and Proposition 4.36,

$$(4.292) \quad \text{Res}_{\lambda \in \{\text{Re} \lambda \geq 0\}} e^{\lambda t} G_\lambda(x, y) = \sum_{\lambda_0 \in \sigma_p'(L) \cap \{\text{Re} \lambda \geq 0\}} e^{\lambda_0 t} \text{Res}_{\lambda_0} e^{(\lambda - \lambda_0)t} G_\lambda(x, y)$$

$$= \sum_{\lambda_0 \in \sigma_p'(L) \cap \{\text{Re} \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k / k!) \text{Res}_{\lambda_0} (\lambda - \lambda_0)^k G_\lambda(x, y)$$

$$= \sum_{\lambda_0 \in \sigma_p'(L) \cap \{\text{Re} \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k / k!) Q_{\lambda_0, k}(x, y)$$

$$= \sum_{\lambda_0 \in \sigma_p'(L) \cap \{\text{Re} \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k / k!) (L - \lambda_0 I)^k P_{\lambda_0}(x, y)).$$

On the other hand, for $x, y$ bounded and $t$ sufficiently large, $t$ dominates $|x|$ and $|y|$ and we obtain from Propositions 2.23 and 4.22 that

$$|G_\lambda(x, t; y)| \leq C$$

for all $\lambda \in \Omega$, for $\eta_1$ sufficiently small, hence

$$(4.293) \quad \frac{1}{2\pi i} \int_{-\eta_1 - iR}^{\eta_1 + iR} e^{\lambda t} G_\lambda(x, y) d\lambda \leq 2C \text{Re}^{-\eta_1 t}.$$

Combining (4.291), (4.292), and (4.293), we obtain the result. □

**Corollary 4.44.** Let dimension $d = 1$, and assume (A1)–(A3) and (H0)–(H3). Then, $\mathcal{D}$ is necessary for linearized orbital stability with respect to compactly supported initial data, as measured in any $L^p$ norm, $1 \leq p \leq \infty$.

**Proof:** From (4.288), we find that $\bar{U} (\cdot)$ is linearly orbitally stable only if $P_\lambda = 0$ for all $\lambda \in \{\text{Re} \lambda \geq 0\} \setminus \{0\}$ and Range $P_0 = \text{Span} \{((\partial / \partial \delta_j) \bar{U}^k)\}$. By Propositions 4.36 and 4.38, this is equivalent to $\mathcal{D}$. □

This completes our treatment of the one-dimensional case.
5. MULTI-DIMENSIONAL STABILITY.

We now focus on the multi-dimensional case, establishing results 4 and 5 of the introduction.

5.1. Necessary conditions.

Define the reduced Evans function as
\[
\bar{\Delta}(\varepsilon, \lambda) := \lim_{\rho \to 0} \rho^{-\ell} D(\rho \varepsilon, \rho \lambda).
\]

By the results of the previous section, the limit \( \bar{\Delta} \) exists and is analytic, with
\[
\bar{\Delta} = \gamma \Delta,
\]
for shocks of pure type (indeed, such a limit exists for all types). Evidently, \( \bar{\Delta}(\cdot, \cdot) \) is homogeneous, degree \( \ell \).

Recall that the restriction
\[
D(\lambda) := D(0, \lambda)
\]
is the one-dimensional Evans function considered in [GZ, ZH].

**Lemma 5.1** [ZS]. Let \( \bar{\Delta}(0, 1) \neq 0 \). Then, near any root \((\bar{\xi}_0, \lambda_0)\) of \( \bar{\Delta}(\cdot, \cdot) \), there exists a continuous branch \( \lambda(\bar{\xi}) \), homogeneous degree one, of solutions of
\[
\bar{\Delta}(\bar{\xi}, \lambda(\bar{\xi})) = 0
\]
defined in a neighborhood \( V \) of \( \bar{\xi}_0 \), with \( \lambda(\bar{\xi}_0) = \lambda_0 \). Likewise, there exists a continuous branch \( \lambda_*(\bar{\xi}) \) of roots of
\[
D(\bar{\xi}, \lambda_*(\bar{\xi})) = 0,
\]
defined on a conical neighborhood \( V_{\rho_0} := \{ \bar{\xi} = \rho \bar{\eta} : \bar{\eta} \in V, \; 0 < \rho < \rho_0 \} \), \( \rho_0 > 0 \) sufficiently small, “tangent” to \( \lambda(\cdot) \) in the sense that
\[
|\lambda_*(\bar{\xi}) - \lambda(\bar{\xi})| = o(|\bar{\xi}|)
\]
as \( |\bar{\xi}| \to 0 \), for \( \bar{\xi} \in V_{\rho_0} \).

**Proof.** The statement (5.4) follows by Rouche’s Theorem, provided \( \bar{\Delta}(\bar{\xi}_0, \cdot) \neq 0 \), since \( \bar{\Delta}(\bar{\xi}_0, \cdot) \) are a continuous family of analytic functions. For, otherwise, restricting \( \lambda \) to the positive real axis, we have by homogeneity that
\[
0 = \lim_{\lambda \to +\infty} \bar{\Delta}(\bar{\xi}_0, \lambda) = \lim_{\lambda \to +\infty} \bar{\Delta}(\bar{\xi}_0/\lambda, 1) = \bar{\Delta}(0, 1),
\]

\[29\]Here, and elsewhere, homogeneity is with respect to the positive reals, as in most cases should be clear from the context. Recall that \( \Delta \) (and thus \( \bar{\Delta} \)) is only defined for real \( \xi \), and \( \text{Re} \lambda \geq 0 \).
in contradiction with the hypothesis. Clearly we can further choose \( \lambda(\cdot) \) homogeneous degree one, by homogeneity of \( \bar{\Delta} \). Similar considerations yield existence of a branch of roots \( \bar{\lambda}(\bar{\xi}, \rho) \) of the family of analytic functions

\[
(5.7)\quad g^{\bar{\xi}, \rho}(\lambda) := \rho^{-\ell}D(\rho \bar{\xi}, \rho \lambda),
\]

for \( \rho \) sufficiently small, since \( g^{\bar{\xi}, 0} = \bar{\Delta}(\bar{\xi}, \cdot) \). Setting \( \lambda_*(\bar{\xi}) := |\bar{\xi}|\bar{\lambda}(\bar{\xi}/|\bar{\xi}|, |\bar{\xi}|) \), we have

\[
D(\bar{\xi}, \lambda_*(\bar{\xi})) = |\bar{\xi}|^\ell g^{\bar{\xi}/|\bar{\xi}|, |\bar{\xi}|}(\lambda_*) \equiv 0,
\]

as claimed. “Tangency,” in the sense of (5.6), follows by continuity of \( \bar{\lambda} \) at \( \rho = 0 \), the definition of \( \lambda_* \), and the fact that \( \bar{\lambda}(\bar{\xi}) = |\bar{\xi}|\lambda(\bar{\xi}/|\bar{\xi}|) \), by homogeneity of \( \bar{\Delta} \). \( \square \)

**Corollary 5.2** [ZS]. Given one-dimensional inviscid stability, \( \bar{\Delta}(0, 1) \neq 0 \), weak refined dynamical stability (Definition 1.21) is necessary for viscous multi-dimensional weak spectral stability, and thus for multi-dimensional linearized viscous stability with respect to test function \( \mathcal{C}_0^\infty \) initial data.

**Proof.** By the discussion just preceding Section 2.1, if is sufficient to prove the first assertion, i.e., that failure of weak refined dynamical stability implies existence of a zero \( D(\bar{\xi}, \lambda) = 0 \) for \( \bar{\xi} \in \mathbb{R}^{d-1}, \text{Re} \lambda > 0 \). Failure of weak inviscid stability, or \( \bar{\Delta}(\bar{\xi}, \lambda) = 0 \) for \( \bar{\xi} \in \mathbb{R}^{d-1}, \text{Re} \lambda > 0 \), implies immediately the existence of such a root, by tangency of the zero-sets of \( D \) and \( \bar{\Delta} \) at the origin, Lemma 5.1. Thus, it remains to consider the case that weak inviscid stability holds, but there exists a root \( D(\bar{\xi}, i\tau) \) for \( \bar{\xi}, \tau \) real, at which \( \Delta \) is analytic, \( \Delta_\lambda \neq 0 \), and \( \beta(\bar{\xi}, i\tau) < 0 \), where \( \beta \) is defined as in (4.17).

Recalling that \( D(\rho \bar{\xi}, \rho \lambda) \) vanishes to order \( \ell \) in \( \rho \) at \( \rho = 0 \), we find by L’Hôpital’s rule that

\[
(\partial/\partial \rho)^{\ell+1}D(\rho \bar{\xi}, \rho \lambda)|_{\rho=0, \lambda=i\tau} = (1/\ell!)(\partial/\partial \rho)g^{\bar{\xi}, i\tau}(0)
\]

and

\[
(\partial/\partial \lambda)D(\rho \bar{\xi}, \rho \lambda)|_{\rho=0, \lambda=i\tau} = (1/\ell!)(\partial/\partial \lambda)g^{\bar{\xi}, i\tau}(0),
\]

where \( g^{\bar{\xi}, \rho}(\lambda) := \rho^{-\ell}D(\rho \bar{\xi}, \rho \lambda) \) as in (5.7), whence

\[
(5.8)\quad \beta = \frac{(\partial/\partial \rho)g^{\bar{\xi}, \lambda}(0)}{(\partial/\partial \lambda)g^{\bar{\xi}, \lambda}(0)},
\]

with \( (\partial/\partial \lambda)g^{\bar{\xi}, \lambda}(0) \neq 0 \).

By the (analytic) Implicit Function Theorem, therefore, \( \lambda(\bar{\xi}, \rho) \) is analytic in \( \bar{\xi}, \rho \) at \( \rho = 0 \), with

\[
(5.9)\quad (\partial/\partial \rho) \lambda(\bar{\xi}, 0) = -\beta,
\]
where $\lambda(\tilde{\xi}, \rho)$ as in the proof of Lemma 5.1 is defined implicitly by $g^{\tilde{\xi}}(\rho) = 0$, $\lambda(\tilde{\xi}, 0) := i\tau$. We thus have, to first order,

\begin{equation}
(5.10) \quad \lambda(\tilde{\xi}, \rho) = i\tau - \beta \rho + O(\rho^2).
\end{equation}

Recalling the definition $\lambda_0(\tilde{\xi}) := |\tilde{\xi}|\lambda(\tilde{\xi}/|\tilde{\xi}|, |\tilde{\xi}|)$, we have then, to second order, the series expansion

\begin{equation}
(5.11) \quad \lambda_0(\rho\tilde{\xi}) = i\rho\tau - \beta \rho^2 + O(\rho^3),
\end{equation}

where $\lambda_0(\tilde{\xi})$ is the root of $D(\tilde{\xi}, \lambda) = 0$ defined in Lemma 5.1. It follows that there exist unstable roots of $D$ for small $\rho > 0$ unless $\text{Re} \beta \geq 0$. \hfill \Box

**Remarks 5.3.** When $\Delta \lambda \neq 0$, $\lambda = i\tau$ is a simple root of $\Delta(\tilde{\xi}, \cdot)$ and, likewise, $\lambda_0(\tilde{\xi})$ defined in (5.11) represents an isolated branch of the zeroes of $D(\tilde{\xi}, \cdot)$. The quantity $-\beta$ gives the curvature of the zero level set of $D$ tangent to the level set $\{\rho\tilde{\xi}, \rho i\tau\}$ for $\Delta$. The value $\beta$ also represents the effective diffusion coefficient for the transverse traveling waves associated with this frequency; see [Z.3], Section 3.3.

### 5.2. Sufficient Conditions for Stability.

Result 5 of the introduction is subsumed in the following two theorems, to be established throughout the remainder of the section.

**Theorem 5.4.** (*Linearized stability*) Under assumptions (A1)–(A3), (H0)–(H5), structural and strong refined dynamical stability together with strong spectral stability are sufficient for linearized viscous stability in $L^2$ with respect to initial perturbations $U_0 \in L^1 \cap L^2$, or in $L^p$, $p \geq 2$, with respect to initial perturbations $U_0 \in L^1 \cap H^{(d-1)/2}$, for all dimensions $d \geq 2$, with rate of decay

\begin{equation}
(5.12) \quad \|U(t)\|_{L^2} \leq C(1 + t)^{-\frac{(d-1)}{2} + \beta \epsilon|U_0|_{L^1 \cap L^2}},
\end{equation}

\begin{equation}
(5.13) \quad \|U(t)\|_{L^p} \leq C(1 + t)^{-\frac{(d-1)}{2} (1-1/p) + \beta \epsilon|U_0|_{L^1 \cap H^{(d-1)/2} + 2}}
\end{equation}

for all $t \geq 0$, where

\begin{equation}
(5.14) \quad \beta = \begin{cases} 0 & \text{for uniformly inviscid stable (U.I.S.) shocks}, \\ 1 & \text{for weakly inviscid stable (W.I.S.) shocks}, \end{cases}
\end{equation}

$\epsilon > 0$ is arbitrary, and $C = C(\epsilon)$ is independent of $p$.

**Theorem 5.5.** (*Nonlinear stability*) Under assumptions (A1)–(A3), (H0)–(H5), structural and strong refined dynamical stability together with strong spectral stability are sufficient for nonlinear viscous stability in $L^p \cap H^{s-1}$, $p \geq 2$ with respect to initial perturbations $U_0 := \tilde{U}_0 - \tilde{U}$ that are sufficiently small in $L^1 \cap H^{s-1}$, where
s ≥ s(d) is as defined in (H0), Section 1.1, for dimensions d ≥ 2 in the case of a uniformly inviscid stable Lax or overcompressive shock; d ≥ 3 in the case of a weakly inviscid stable Lax or overcompressive shock, and d ≥ 4 in the case of an undercompressive shock, with rate of decay

$$\|\tilde{U}(\cdot, t) - \tilde{U}\|_{L^p} \leq C(1 + t)^{-\frac{d-1}{2}(1-1/p) + \beta \epsilon} \|U_0\|_{L^1 \cap H^{s-1}},$$

$$\|\tilde{U}(\cdot, t) - \tilde{U}\|_{H^{s-1}} \leq C(1 + t)^{-\frac{d-1}{4} + \beta \epsilon} \|U_0\|_{L^1 \cap H^{s-1}}$$

for all t ≥ 0, where ε > 0 is arbitrary, C = C(ε), and β = 1 if dimension d = 2 and p > 2, or in the weakly inviscid stable (W.I.S.) case, and otherwise β = 0. (In particular, β = 0 for uniformly inviscid stable Lax or overcompressive shocks, in dimension d ≥ 3 or in dimension d = 2 with p = 2).

**Remarks 5.6.**

1. Stability here is in the usual sense of asymptotic stability, and not only bounded or orbital stability as in the one-dimensional case, with rate of decay in dimension d equal to that of a (d−1)-dimensional heat kernel, corresponding to transverse diffusion along the front. This reflects the fact that the class of $L^1$ perturbations is much more restrictive in multiple than in single dimensions; in particular, conservation of mass precludes convergence of an $L^1$ perturbation of a planar shock front to any translate of the shock front other than the initial front itself [Go.3]. Recall that our necessary stability results concerned the still more restrictive class of test function initial data, so that the problems are consistent.

2. The diffusive decay rate given here is sharp, neglecting error term β, for the weakly inviscid stable case; indeed, the basic iteration scheme [Z.3] was motivated by the scalar analysis of [Go.3] (recall: scalar shocks are always weakly inviscid stable). However, uniformly inviscid stable shocks, for which transverse front disturbances are strongly damped, are expected to decay at the faster rate governing far-field behavior, of a $d$-dimensional heat kernel. That is, our analysis here as in [Z.3] is focused on the W.I.S. regime, and the transition from viscous stability to instability; see again Remark 1.22.2.

### 5.2.1. Linearized estimates.

Theorems 5.4 and 5.5 are obtained using the following $L^{q} \to L^{p}$ bounds on the linearized solution operator, analogous to the one-dimensional bounds described in Lemma 4.14 (proof deferred to Section 5.3). Note that we no longer require detailed Green distribution bounds as described in Proposition 4.10, nor, because of the more complicated geometry of characteristic surfaces in multi-dimensions, does there exist such a simple description of the propagation of solutions; see, e.g., [HoZ.1–2] for further discussion in the constant-coefficient case.

**Proposition 5.7.** Let (A1)–(A3), (H0)–(H5) hold, together with structural, strong refined dynamical, and strong spectral stability. Then, solution operator $S(t) := e^{Lt}$
of the linearized equations (2.1) may be decomposed into \( S = S_1 + S_2 \) satisfying
\[
\|S_1(t)\partial_t^2 F\|_{L^p(x)} \leq C(1 + t)^{-\frac{(d-1)}{4}(1-\frac{1}{p}) - (1-\alpha)\frac{\beta}{2} + \beta \|F\|_{L^1(x)}^p}
\]
for \( 0 \leq |\delta| \leq 1, 2 \leq p \leq \infty, \) and \( t \geq 0, \) where \( \alpha = 0 \) for Lax and overcompressive shocks and \( \alpha = 1 \) for undercompressive shocks, \( \beta = 0 \) for uniformly inviscid stable (U.I.S.) shocks and \( \beta = 1 \) for weakly inviscid stable (W.I.S.) shocks, \( \epsilon > 0 \) is arbitrary, and \( C = C(d, \epsilon) \) is independent of \( p, \) and, for \( 0 \leq |\gamma_1| \leq 1, 0 \leq |\hat{\gamma}|, \)
\[
\|\partial_{x_1}^\gamma \partial_{x}^\tilde{\gamma} S_2(t)F\|_{L^2} \leq Ce^{-\eta t}\|F\|_{H^{|\gamma_1|+|\hat{\gamma}|}}.
\]

Remarks 5.8. 1. Operators \( S_1 \) and \( S_2 \) represent low- and high-frequency parts of \( S. \) Though we do not state them, \( S_2 \) satisfies refined bounds analogous to (4.32)–(4.33) in the one-dimensional case.

2. The time-asymptotic decay rate of solutions is determined by the bounds on \( S_1, \) which may be recognized as those for a \((d - 1)\)-dimensional heat equation. In contrast to the situation for the heat equation, \( x \)-differentiation of solution \( U = SF \) does not improve the rate of decay for the variable-coefficient shock problem, due to commutator terms arising in the differentiated equations (that is, terms for which derivatives fall on coefficients), nor in general does differentiation with respect to source \( F. \) A single derivative of the source does improve the rate of decay in the Lax or overcompressive case, but further derivatives would not (and are not needed in our later argument). This is a slight difference from the constant-coefficient case considered in, e.g., [Kaw, HoZ.1–2], and requires some modification in the arguments. However, in spirit our nonlinear stability analysis in dimension \( d \) follows that of [Kaw, HoZ.1–2] in one lower dimension, \( d - 1.\)

Proof of Theorem 5.4. Applying (5.15)–(5.16), we have
\[
|U(t)|_{L^2} \leq |S_1(t)U_0|_{L^2} + |S_2(t)U_0|_{L^2} \leq C(1 + t)^{-\frac{(d-1)}{4} + \beta \epsilon}|U_0|_{1} + Ce^{-\eta t}|U_0|_{L^2} \leq C(1 + t)^{-\frac{(d-1)}{4} + \beta \epsilon}|U_0|_{L^1 \cap L^2}.
\]
Likewise, we have, using the Sobolev embedding
\[
|f|_{L^p} \leq |f|_{L^2 \cap L^\infty} \leq |f|_{H^{(d-1)/2} + 1(\hat{x};H^1(x_1))}
\]
for \( 2 \leq p \leq \infty, \) that
\[
|U(t)|_{L^p} \leq |S_1(t)U_0|_{L^p} + |S_2(t)U_0|_{H^{(d-1)/2} + 1(\hat{x};H^1(x_1))} \leq C(1 + t)^{-\frac{(d-1)}{4}(1-\frac{1}{p}) + \beta \epsilon}|U_0|_{1} + Ce^{-\eta t}|U_0|_{H^{[d/2] + 2}} \leq C(1 + t)^{-\frac{(d-1)}{4}(1-\frac{1}{p}) + \beta \epsilon}|U_0|_{L^1 \cap H^{[d/2] + 2}}.
\]
5.2.2. Auxiliary energy estimate.

In our analysis of nonlinear stability, we shall make use also of the following auxiliary energy estimate, analogous to that of the one-dimensional case.

**Proposition 5.9.** Under the hypotheses of Theorem 5.5, suppose that, for \(0 \leq t \leq T\), the \(H^{s-1}\) norm of the perturbation \(U := \tilde{U} - \bar{U} = (u^I, u^{II})^T\) remains bounded by a sufficiently small constant, where \(\tilde{U} = \bar{U} + U\) denotes a solution of (1.1). Then, for all \(0 \leq t \leq T\),

\[
|U(t)|_{H^{s-1}}^2 \leq C|U(0)|_{H^{s-1}}^2 e^{-\theta t} + C \int_0^t e^{-\theta_2(t-\tau)} |U|_{L^2}^2(\tau) d\tau.
\]

**Proof of Proposition 5.9.** The proof follows almost exactly the argument of the one-dimensional case; see Remark 4.17. Indeed, it is somewhat simpler, since there are no front location terms \(\tilde{\delta} \tilde{U}_x\). We therefore omit most details, pointing out only two small points in which the argument must be modified for the multi-dimensional case.

The first is the standard issue that “good” terms

\[
-\sum_{jk} \langle \partial_x^f W_{x_j}, \tilde{B}^{jk} \partial_x^f W_{x_k} \rangle_\alpha = -\sum_{jk} \langle \partial_x^f w^{II}_{x_j}, \tilde{b}^{jk} \partial_x^f w^{II}_{x_k} \rangle_\alpha
\]

arising in the \(\ell\)th order Friedrichs estimate must now be estimated indirectly, using Gårding inequality

\[
\sum_{jk} \langle \partial_x^f w^{II}_{x_j}, \tilde{b}^{jk} \partial_x^f w^{II}_{x_k} \rangle_\alpha \geq \theta |\partial_x^{\ell+1} w^{II}|_{L^2}^2 - C |\partial_x^f w^{II}|_{L^2}^2
\]

(a consequence of uniform ellipticity, (1.6)). Recall that \(\langle f, g \rangle_\alpha := \langle f, \alpha g \rangle\), where \(\alpha\) is a scalar weight function. Recall that \(O(|\partial_x^\alpha|_{L^2}^2)\) terms already appear in the \(\ell\)th order estimate with arbitrarily large constant, and so the second, lower-order error term in (5.18) is harmless.

A second, related issue, is the treatment of Kawashima estimates. In the case (as in gas dynamics and MHD) that there exists a linear (i.e., differential) compensating matrix \(\tilde{K}(\xi) := \sum_j \tilde{K}^j \xi_j\) for \(\tilde{A}(\xi) := \sum_j \tilde{A}^j \xi_j\) and \(\tilde{B}(\xi) := \sum_{jk} \tilde{B}^{jk} \xi_j \xi_k\) in the sense that

\[
(\tilde{K}(\xi)(\tilde{A}^0)^{-1} \tilde{A}(\xi) + \tilde{B}(\xi)) \geq \theta > 0,
\]

the \(\ell\)th order Kawashima estimate takes the form

\[
(d/dt)^{\ell/2} \langle \partial_x^{\ell-1} W, -\sum_j \tilde{K} \partial_{x_j} \partial_x^{\ell-1} W \rangle_\alpha,
\]
and yields a “good” term
\[ \langle \partial_x^{-1} W, \sum_{j,k} \tilde{K}(\tilde{A}^0)^{-1} \tilde{A}^k \partial x_j \partial x_k \partial_x^{-1} W \rangle_\alpha, \]
which may be again estimated by an appropriate Gårding inequality as
\[ -\theta |\partial_x^\ell w^I|^2_\alpha + C |\partial_x^\ell w^{II}|^2_\alpha + C(C_*) |\partial_x^\ell^{-1} W|^2_{L^2} \]
for some uniform \( \theta, C > 0 \), and \( C(C_*) \) depending on the details of weight \( \alpha \), and used as before to compensate the Friedrichs estimate near plus and minus spatial infinity. Recall that lower-order derivative terms are harmless for the Kawashima estimate, whether in \( |w^I|_{H^{s-1}} \) or \( |w^{II}|_{H^{s-1}} \).

Of course, the compensating matrix \( \tilde{K}(\xi) \) in general depends on \( \xi \) in pseudodifferential rather than differential fashion (that is, it is homogeneous degree one and not linear in \( \xi \)), and in this case the argument must be modified. A convenient method is to consider the variables \( W^\pm := \chi_\pm^{1/2} W \), where \( \chi_\pm = \chi_\pm(x_1) \) are smooth, scalar cutoff functions supported on \( x_1 \geq L \) and \( x_1 \leq -L \), respectively. For \( L \) sufficiently large, the equations for \( W^\pm \) have coefficients that are constant up to an arbitrarily small error. Taking the Fourier transform now in \( x \), we may perform an energy estimate in the frequency domain, treating coefficient errors and nonlinear terms alike as a small source, to obtain a similar \( \alpha \)-weighted Kawashima estimate, this time explicitly localized near plus and minus spatial infinity (essentially, the pseudodifferential Gårding inequality in a simple case). We omit the details, which may be found in [Z.4].

Finally, we estimate nonlinear source terms more systematically using the fact that \( H^r \) is an algebra, \( |fg|_{H^r} \leq |f|_{H^r} |g|_{H^r} \) for \( r > d/2 \), in particular for \( r = s - 1 \) (Moser’s inequality and Sobolev embedding; see, e.g. [M.1–4, Kaw, GMWZ.2, Z.4]). With these changes, the argument goes through as before, yielding the result. See [Z.4] for further details.

5.2.3. Nonlinear stability.

Using the linearized bounds of Section 5.2.1 together with the energy estimate of Section 5.2.2, it is now straightforward to establish \( L^1 \cap H^{s-1} \rightarrow L^p \cap H^{s-1} \) asymptotic nonlinear stability, \( p \geq 2 \).

Proof of Theorem 5.5. We present in detail the case of a uniformly inviscid stable Lax or overcompressive shock in dimension \( d \geq 3 \). Other cases follow similarly; see, e.g., [Z.3–4].

H^{s-1} stability. We first carry out a self-contained stability analysis in \( H^{s-1} \). Afterwards, we shall establish sharp \( L^\infty \) decay rates (and thus, by interpolation, \( L^p \) rates for \( 2 \leq p \leq \infty \)) by a bootstrap argument. Defining
\[ U := \tilde{U} - \bar{U}, \]
and Taylor expanding as usual, we obtain the nonlinear perturbation equation

\[ U_t - LU = \sum_j Q^j(U, \partial_x U)_{x_j}, \]  

where

\[ Q^j(U, \partial_x U) = \mathcal{O}(|U|^2 + |U||\partial_x U|) \]
\[ \partial_x Q^j(U, \partial_x U) = \mathcal{O}(|U||\partial_x U| + |U||\partial^2_x U| + |\partial_x U|^2) \]

so long as \(|U|\) remains bounded by some fixed constant. Applying Duhamel’s principle, and integrating by parts, we can thus express

\[ U(x, t) = S(t)U(0) + \int_0^t S(t-s) \sum_j \partial_{x_j} Q^j(s) ds. \]

Define now

\[ \zeta(t) := \sup_{0 \leq s \leq t} \|U(s)\|_{L^2} (1 + s)^{(d-1)/4}. \]

By standard, short-time existence theory (see, e.g., [Kaw], or [Z.4]) and the principle of continuation, there exists a solution \( U \in H^{s-1}(x), U_t \in H^{s-3}(x) \subset L^2(x) \) on the open time-interval for which \(|U|_{H^{s-1}}\) remains bounded. On this interval, \( \zeta \) is well-defined and continuous.

Now, let \([0, T)\) be the maximal interval on which \(|U|_{H^{s-1}(x)}\) remains strictly bounded by some fixed, sufficiently small constant \( \delta > 0 \). By Proposition 5.9,

\[ |U(t)|_{H^{s-1}}^2 \leq C|U(0)|_{H^{s-1}}^2 e^{-\delta t} + C \int_0^t e^{-\theta_2(t-\tau)} |U(\tau)|_{L^2}^2 d\tau \]
\[ \leq C_2 (|U(0)|_{H^{s-1}}^2 + \zeta(t)^2) (1 + \tau)^{-(d-1)/2}. \]

Combining this with the bounds of Proposition 5.7 and \(|Q|_{L^1 \cap H^1} \leq |U|_{H^{s-1}}^2, and
recalling that $d \geq 3$ and $|U(0)|_{H^{s-1}}$ is small, we thus obtain (5.27)
\begin{align*}
|U(t)|_{L^2} & \leq |S(t)U(0)|_{L^2} + \int_0^t \sum_j (S_1(t-s) \partial_{x_j})Q^j(s)\,ds \\
& \quad + \int_0^t S_2(t-s) (\sum_j \partial_{x_j} Q^j(s))\,ds \\
& \leq C(1+t)^{-\frac{d+1}{4}} |U(0)|_{L^1 \cap L^2} + \int_0^t (1+t-s)^{-\frac{d+1}{4} - \frac{1}{2}} |Q^j(s)|_{L^1} \,ds \\
& \quad + \int_0^t e^{-\theta(t-s)} (\sum_j \partial_{x_j} Q^j(s))_{L^2} \,ds.
\end{align*}
\begin{align*}
& \leq C(1+t)^{-\frac{d+1}{4}} |U(0)|_{L^1 \cap L^2} + \int_0^t (1+t-s)^{-\frac{d+1}{4} - \frac{1}{2}} |Q^j(s)|_{L^1 \cap H^1} \,ds \\
& \leq C(1+t)^{-\frac{d+1}{4}} |U(0)|_{L^1 \cap L^2} \\
& \quad + C_2 (|U(0)|^2_{H^{s-1}} + \zeta(t)^2) \int_0^t (1+t-s)^{-\frac{d+1}{4} - \frac{1}{2} (1+s)^{-\frac{d+1}{4}}} \,ds \\
& \leq C_3 (1+t)^{-\frac{d+1}{4}} \left(|U(0)|_{L^1 \cap H^{s-1}} + \zeta(t)^2\right),
\end{align*}
or, dividing by $(1+t)^{-\frac{d+1}{4}}$,
\begin{equation}
(5.28) \quad \zeta(t) \leq C_2 (|U(0)|_{L^1 \cap H^{s-1}} + \zeta(t)^2).
\end{equation}

Bound (5.28) together with continuity of $\zeta$ implies that
\begin{equation}
(5.29) \quad \zeta(t) \leq 2C_2 |U(0)|_{L^1 \cap H^{s-1}}
\end{equation}
for $t \geq 0$, provided $|U(0)|_{L^1 \cap H^{s-1}} < 1/4C_2^2$. With (5.26), this gives
\begin{equation}
|U(t)|_{H^{s-1}} \leq 4C_2 |U(0)|_{L^1 \cap H^{s-1}}
\end{equation}
and so we find that the maximum time of existence $T$ is in fact $+\infty$, and (5.29) holds globally in time. Definition (5.25) then yields
\begin{equation}
(5.30) \quad \|U(t)\|_{L^2} \leq 2C_2 \zeta(1+t)^{-\left(\frac{d+1}{4}\right)} \leq C_3 \zeta(t) \\
\leq 2C_2 |U(0)|_{L^1 \cap H^{s-1}} (1+t)^{-\left(\frac{d+1}{4}\right)}
\end{equation}
as claimed.

$L^\infty$ stability. We now carry out the $L^\infty$ estimate. Together with the $L^2$ result already obtained, this yields rates (5.14) by interpolation, completing the proof.
Denoting by $U_1$ and $U_2$ the low- and high-frequency parts

$$U_j(t) := S_1(t)U_0 + \int_0^t S_j(t-s)(\sum_j \partial_x^j Q^j(s))ds,$$

\[ j = 1, 2 \text{ of } U, \]

we have, using the bounds of Proposition 5.7 together with the previously obtained $H^{s-1}$ bounds, Sobolev bound $|f|_{L^\infty} \leq C|\partial_{x_1}^\gamma \partial_{x_2}^\tilde{\gamma} f|_{H^\gamma}$ for $\gamma_1 = 1$, $\tilde{\gamma} = \lceil (d-1)/2 \rceil + 1 \leq s - 4$, and the fact that $H^r$ is an algebra, $|fg|_{H^r} \leq |f|_{H^r} |g|_{H^r}$ for $r > d/2$ (Moser’s inequality and Sobolev embedding; see, e.g., [M.1–4, Kaw, GMWZ.2, Z.4]), that

\[ (5.32) \]

$$|U_1(t)|_{L^\infty} \leq |S_1(t)U_0|_{L^1} + \int_0^t (S_1(t)\partial_x Q(s))ds|_{L^\infty}$$

\[ \leq C(1 + t)^{\frac{d-1}{2}}|U_0|_{L^1} + C \int_0^t (1 + t - s)^{-\frac{d-1}{2} - \frac{1}{2}}|U(s)|_{H^1}^2 \, ds 
\leq C\left( (1 + t)^{\frac{d-1}{2}} + \int_0^t (1 + t - s)^{-\frac{d-1}{2} - \frac{1}{2}}(1 + s)^{-\frac{d-1}{2}} \, ds \right)|U_0|_{L^1 \cap H^{s-1}}$$

\[ \leq C(1 + t)^{\frac{d-1}{2}}|U_0|_{L^1 \cap H^{s-1}} \]

and

$$|U_2(t)|_{L^\infty} \leq C|\partial_{x_1}^\gamma \partial_{x_2}^\tilde{\gamma} U_2(t)|_{L^2}$$

\[ \leq \int_0^t \partial_{x_1}^\gamma S_2(t)(\partial_x Q(s)) \, ds|_{L^2} \]

\[ \leq C e^{-\theta t}|U_0|_{H^\gamma} + C \int_0^t e^{-\theta(t-s)}|\partial_x Q(s)|_{H^{\gamma_1}} \, ds 
\leq C\left( e^{-\theta t} + |\int_0^t e^{-\theta(t-s)}(1 + s)^{-\frac{d+1}{2}} \, ds \right)|U(0)|_{H^{\gamma_1 + 2}} 
\leq C(1 + t)^{-\frac{d-1}{2}}|U(0)|_{H^{s-1}}, \]

giving the claimed bound. \( \square \)

**Remarks.** 1. Results of [HoZ.1] show that $\|G\|_{L^p}$ for $p \leq 2$ degrade for systems in multi-dimensions, with blowup at $p = 1$. So, $p \geq 2$ is the optimal result.

2. The restriction to $d \geq 3$ in the case of weakly inviscid stable Lax or overcompressive shocks results from the degraded Green distribution bounds (5.15) obtained in dimension $d = 2$, as indicated by the presence of error terms $\beta$. These, in turn, result from our nonsharp estimation of “pole” terms of order $\int_{\Gamma_{\xi}} (\lambda - \lambda_\ast(\xi))^{-1}$ occurring in the inverse Laplace transform of the resolvent via modulus estimates $\int_{\Gamma_{\xi}} |\lambda - \lambda_\ast(\xi)|^{-1}$; see (5.154)–(5.158), Section 5.3.3. To obtain optimal estimates, one should rather treat this as a residue term similarly as in the one-dimensional case, identifying cancellation in the nonintegrable $1/\lambda$ singularity; see, e.g., the treatment of the (always weakly inviscid stable) scalar case in [HoZ.3–4, Z.3].
5.3 Proof of the linearized estimates.

Finally, we carry out in this subsection the proof of Proposition 5.7, completing the analysis of the multi-dimensional case.

5.3.1. Low-frequency bounds on the resolvent kernel.

Consider the family of elliptic Green distributions $G_{\xi,\lambda}(x_1, y_1)$,

\begin{equation}
G_{\xi,\lambda}(\cdot, y_1) := (L_\xi - \lambda I)^{-1}\delta_{y_1}(\cdot),
\end{equation}

associated with the ordinary differential operators $(L_\xi - \lambda I)$, i.e. the resolvent kernel of the Fourier transformed operator $L_\xi$. The function $G_{\xi,\lambda}(x_1, y_1)$ is the Laplace–Fourier transform in variables $\tilde{x} = (x_2, \ldots, x_d)$ and $t$, respectively, of the time-evolutionary Green distribution

\begin{equation}
G(x, t; y) = G(x_1, \tilde{x}, t; y_1, \tilde{y}) := e^{Lt}\delta_y(x)
\end{equation}

associated with the linearized evolution operator ($\partial/\partial t - L$). Restrict attention to the surface

\begin{equation}
\Gamma_{\tilde{\xi}} := \partial \Lambda_{\tilde{\xi}} = \{ \lambda : \text{Re } \lambda = -\theta_1(\| \text{Im } \lambda \|^2 + |\tilde{\xi}|^2) \}
\end{equation}

$\theta_1 > 0$ sufficiently small.

Then, our main result, to be proved in the remainder of the subsection, is

**Proposition 5.10.** Under the hypotheses of Theorem 5.5, for $\lambda \in \Gamma_{\tilde{\xi}}$ (defined in (5.36)) and $\rho := |(\xi, \lambda)|$, $\theta_1 > 0$, and $\theta > 0$ sufficiently small, there hold:

\begin{equation}
|G_{\xi,\lambda}(x_1, y_1)| \leq C\gamma_2\gamma_1(\rho^{-1}e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + e^{-\theta\rho^2|x_1-y_1|}),
\end{equation}

and

\begin{equation}
|((\partial/\partial y_1)G_{\xi,\lambda}(x_1, y_1)| \leq C\gamma_2\gamma_1[\rho^{-1}e^{-\theta|x_1|}(\rho e^{-\theta\rho^2|y_1|} + \alpha e^{-\theta|y_1|}) + e^{-\theta\rho^2|x_1-y_1|}(\rho + \alpha e^{-\theta|y_1|})],
\end{equation}

where

\begin{equation}
\gamma_1(\tilde{\xi}, \lambda) := \begin{cases} 
1 \text{ in U.I.S. case}, \\
1 + \sum_{j}[\rho^{-1} \text{ Im } \lambda - i\tau_j(\tilde{\xi}) + \rho]^{-1} \text{ in W.I.S. case},
\end{cases}
\end{equation}

\begin{equation}
\gamma_2(\tilde{\xi}, \lambda) := 1 + \sum_{j, \pm}[\rho^{-1} \text{ Im } \lambda - \eta_j^{\pm}(\tilde{\xi}) + \rho]^{1/s_j-1},
\end{equation}

\begin{equation}
\eta_j^{\pm}(\tilde{\xi}) := \begin{cases} 
\frac{1}{2} \text{ in U.I.S. case}, \\
\frac{1}{2} \text{ in W.I.S. case}.
\end{cases}
\end{equation}
\(\eta_j(\cdot), s_j(\cdot) \geq 1\) as in (H5) and \(\tau_j(\cdot)\) as in (1.43), and

\[
\alpha := \begin{cases} 
0 & \text{for Lax or overcompressive case,} \\
1 & \text{for undercompressive case.}
\end{cases}
\]

(Here, as above, U.I.S. and W.I.S. denote “uniform inviscid stable” and “weak inviscid stable,” as defined, respectively, in (1.32), Section 1.4, and (1.28), Section 1.3; the classification of Lax, overcompressive, and undercompressive types is given in Section 1.2). More precisely, \(t := 1 - 1/K_{\text{max}}\), where \(K_{\text{max}} := \max s_j^\pm, s_j^\pm\) is the maximum among the orders of all branch singularities \(\eta_j^\pm(\cdot), s_j^\pm\) and \(\tau_j^\pm\) defined as in (H5); in particular, \(t = 1/2\) in the (generic) case that only square-root singularities occur.\(^{30}\)

**Corollary 5.11.** Under the hypotheses of Theorem 5.5, for \(\lambda \in \Gamma_\xi\) (defined in (5.36)) and \(\rho := |(\xi, \lambda)|, \theta_1 > 0, \) and \(\theta > 0\) sufficiently small, there holds the resolvent bound

\[
|(L_\xi - \lambda)^{-1} \partial_{x_1}^\beta f|_{L^p(x_1)} \leq C \gamma_1 \gamma_2 \rho^{(1-\alpha)|\beta|-1} |f|_{L^1(x_1)}
\]

for all \(2 \leq p \leq \infty, 0 \leq |\beta| \leq 1, \) where \(\gamma_j, \alpha\) are as defined in (5.39)–(5.41).

**Proof.** Direct integration, using the triangle inequality bound

\[
|(L_\xi - \lambda)^{-1} \partial_{x_1}^\alpha f|_{L^p(x_1)} \leq \sup_{y_1} |\partial_{y_1}^\alpha G_{\xi,\lambda}(\cdot, y_1)|_{L^p(x_1)} |f|_{L^1(x_1)}.
\]

\(\square\)

**Remark 5.12.** By standard considerations (see, e.g. [He, ZH, Z.1]), condition (H3) implies that surface \(\Gamma_\xi\) strictly bounds the essential spectrum of operator \(L_\xi\) to the right. That is, in this analysis, we shall work entirely in the resolvent set of \(L_\xi\). This is in marked contrast to our approach in the one-dimensional case, wherein bounds on the resolvent kernel \(G_\lambda\) within the essential spectrum were used to obtain sharp pointwise estimates on \(G\) in the one-dimensional case. The nonoptimal bounds we obtain here could perhaps be sharpened by a similar analysis; for further discussion, see [Z.3, HoZ.3–4].

**Remark.** Terms \(\rho^{-1}e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|}\) and \(e^{-\theta\rho^2|x_1-y_1|}\) in (5.37), respectively encoding near and far-field behavior with respect to the front location \(x_1 = 0\), are.

\(^{30}\)Bound (5.37) corrects a minor error in [Z.3], where it was mistakenly stated as

\[
|G_{\xi,\lambda}(x_1, y_1)| \leq C \gamma_2(\gamma_1 \rho^{-1}e^{-\theta|x_1|}e^{-\theta\rho^2|y_1|} + e^{-\theta\rho^2|x_1-y_1|}).
\]

The source of this error was the normalization (4.137) of [Z.3] that \(\varphi^\pm_1\) agree to first order in \(\rho\) at \(\rho = 0\), which is incompatible with the assumption (used in the proof of the low-frequency expansion) that fast modes decay at \(x_1 = \pm \infty\).
roughly speaking, associated respectively with point and essential spectrum of the 
operator $L$. In particular, the simple pole $\rho^{-1}$ in the first term corresponds to a 
semisimple eigenvalue at $\rho = 0$, and the factors $e^{-\theta|x_1|}$ and $e^{-\theta|y_1|}$ to associated 
right and left eigenfunctions. However, this is clearly not the usual point spectrum 
encountered in the case of an operator with positive spectral gap. For example, 
ote the lack of spatial decay/localization in the “left eigenfunction” term $e^{-\theta \rho^2|y_1|}$ 
at $\rho = 0$. Also, though it is not apparent from the simple bound (5.37), it is a fact 
that the “left eigenspace” at $\rho = 0$ (defined by continuation along rays from $\rho > 0$ 
following the approach of Section 4.3.3) depends on direction $(\tilde{\xi}_0, \lambda_0)$; that is, there 
is a conical singularity at the origin in the spectral projection, reflecting that of the Evans function.

**Remark 5.13.** The $\alpha$-terms appearing in (5.37)-(5.38) are not technical artifacts, but in fact reflect genuinely new effects arising in the undercompressive case, related to $L^1$ time-invariants and conservation of mass. See [Z.6] for a detailed discussion in the one-dimensional case.

**Normal modes.** Paralleling our one-dimensional analysis, we begin by identifying normal modes of the limiting, constant-coefficient equations $(L_{\tilde{\xi}_\pm} - \lambda)U = 0$.
The key estimates of slow and super-slow mode decay will be seen to reduce to a 
matrix perturbation problem closely related to the one arising in the corresponding 
inviscid theory; the crucial issue here, as there, is the careful treatment of branch 
singularities, corresponding to inviscid glancing modes.

Introduce the curves

$$(5.43) \quad (\tilde{\xi}, \lambda)(\rho, \tilde{\xi}_0, \tau_0) := \left(\rho \tilde{\xi}_0, \rho i \tau_0 - \theta_1 \rho^2\right),$$

where $\tilde{\xi}_0 \in \mathbb{R}^{d-1}$ and $\tau_0 \in \mathbb{R}$ are restricted to the unit sphere $S^d : |\tilde{\xi}_0|^2 + |\tau_0|^2 = 1$. Evidently, as $(\tilde{\xi}_0, \tau_0, \rho)$ range in the compact set $S^d \times [0, \delta]$, $(\tilde{\xi}, \lambda)$ traces out the portion of the surface $\Gamma_{\tilde{\xi}}$ contained in the set $|\tilde{\xi}|^2 + |\lambda|^2 \leq \delta$ of interest. As before, fixing $\tilde{\xi}_0, \tau_0$, we denote by $v^\pm_j(\rho)$, the solutions of the limiting, constant coefficient equations at $(\tilde{\xi}, \lambda)(\rho)$ from which $\varphi^\pm_j(\rho), \psi^\pm_j(\rho)$, etc. are constructed; it is these solutions that we wish to estimate.

Making as usual the Ansatz

$$(5.44) \quad v^\pm := e^{\mu x_1} v,$$

and substituting $\lambda = i \rho \tau_0 - \theta_1 \rho^2$ into (3.4), we obtain the characteristic equation

$$(5.45) \quad \left[ \mu^2 B^{11}_\pm + \mu(-A^{1}_\pm + i \rho \sum_{j \neq 1} B^{1j}_\pm \xi_{0j} + i \rho \sum_{k \neq 1} B^{1k}_\pm \xi_{0k}) 
\right.
\left. - (i \rho \sum_{j \neq 1} A^j \xi_{0j} + \rho^2 \sum_{j \neq k \neq 1} B^{jk}_\pm \xi_{0j} \xi_{0k} + (\rho i \tau_0 - \theta_1 \rho^2) I) \right] v = 0.$$
Note that this agrees with (3.4) up to first order in \( \rho \), hence the (first-order) matrix bifurcation analyses of Lemma 3.3 applies for any \( \theta \).

As in past sections, we first separate normal modes (5.44) into fast and slow modes, the former having exponential growth/decay rate with real part bounded away from zero, and the latter having growth/decay rate close to zero (indeed, vanishing for \( \rho = 0 \)). The fast modes have \( \mathcal{O}(1) \) decay rate, and are spectrally separated by assumption (A1), hence admit a straightforward treatment. We now focus on the slow modes, which we will further subdivide into intermediate- and super-slow types, corresponding respectively to elliptic and hyperbolic/glancing modes in the inviscid terminology.

Positing the Taylor expansion
\[
\begin{aligned}
\mu &= 0 + \mu^1 \rho + \cdots, \\
v &= v^0 + \cdots
\end{aligned}
\]
(or Puiseux expansion, in the case of a branch singularity), and matching terms of order \( \rho \) in (5.45), we obtain
\[
(-\mu^1 A_\pm^1 - i \sum_{j \neq 1} A^i \xi_{0j} - i \tau_0 \mathbf{I})v^0 = 0,
\]
just as in (3.6), or equivalently
\[
[(A^1)^{-1}(i \tau_0 + i A^{\xi_0}) - \alpha_0 \mathbf{I}]v^0 = 0,
\]
with \( \mu^1 =: -\alpha_0 \), which can be recognized as the equation occurring in the inviscid theory on the imaginary boundary \( \lambda = i \tau_0 \). For eigenvalues \( \alpha_0 \) of nonzero real part, denoted as intermediate-slow modes, we have growth or decay at rate \( \mathcal{O}(\rho) \), and spectral separation by assumption (A2); thus, they again admit straightforward treatment.

It remains to study super-slow modes, corresponding to pure imaginary eigenvalues \( \alpha_0 =: i \xi_{01} \); here, we must consider quadratic order terms in \( \rho \), and the viscous and inviscid theory part ways. Using \( \mu = -i \rho \xi_{01} + o(\rho) \), and dividing (5.45) by \( \rho \), we obtain the modified equation
\[
[(A^1)^{-1}(i \tau_0 + i A^{\xi_0} + \rho(B^{\xi_0} - \theta_1) + o(\rho)) - \alpha \mathbf{I}]v = 0,
\]
where \( \alpha := -\mu/\rho \) and \( v \) denote exact solutions; that is, we expand the equations rather than the solutions to second order. (Note that this derivation remains valid near branch singularities, since we have only assumed continuity of \( \mu/\rho \) and not analyticity at \( \rho = 0 \).) Here, \( B^{\xi_0} \) as usual denotes \( \sum B^{jk} \xi_{0j} \xi_{0k} \), where \( \xi_0 := (\xi_{01}, \xi_0) \). Equation (5.48) generalizes the (all-orders) perturbation equation
\[
[(A^1)^{-1}(i \tau_0 + i A^{\xi_0} + \rho I) - \alpha \mathbf{I}]v = 0,
\]
\( \rho \to 0^+ \), arising in the inviscid theory near the imaginary boundary \( \lambda = i\tau_0 \) [Kr,Mé.5].

Note that \( \tau_0 \) is an eigenvalue of \( A^{\xi_0} \), as can be seen by substituting \( \alpha_0 = i\xi_0 \) in (5.47), hence \( |\tau_0| \leq C|\xi_0| \) and therefore (since clearly also \( |\xi_0| \geq |\xi_0| \))

\[
|\xi_0| \geq (1/C)(|\xi_0|) = 1/C.
\]

Thus, for \( B \) positive definite, and \( \theta_1 \) sufficiently small, perturbation \( \rho(B^{\xi_0\xi_0} - \theta_1) \), roughly speaking, enters (5.48) with the same sign as does \( \rho I \) in (5.49). Indeed, for identity viscosity \( B^{jk} := \delta_k^j I \), (5.48) reduces for fixed \( (\xi_0, \tau_0) \) exactly to (5.49), by the rescaling \( \rho \to \rho/(|\xi_0|^2 - \theta_1) \).

For \( (\xi_0, \tau_0) \) bounded away from the set of branch singularities \( \cup(\tilde{\xi}, \eta_j(\tilde{\xi})) \), we may treat (5.48) as a continuous family of single-variable matrix perturbation problem in \( \rho \), indexed by \( (\xi_0, \tau_0) \); the resulting continuous family of analytic perturbation series will then yield uniform bounds by compactness. For \( (\xi_0, \tau_0) \) near a branch singularity, on the other hand, we must vary both \( \rho \) and \( (\xi_0, \tau_0) \), in general a complicated multi-variable perturbation problem. Using homogeneity, however, and the uniform structure assumed in (H6), this can be reduced to a two-variable perturbation problem that again yields uniform bounds. For, noting that \( \eta_j(\tilde{\xi}) \equiv 0 \), we find that \( \xi_0 \) must be bounded away from the origin at branch singularities; thus, we may treat the direction \( \xi_0/|\xi_0| \) as a fixed parameter and vary only \( \rho \) and the ratio \( |\tau_0|/|\xi_0| \). Alternatively, relaxing the restriction of \( (\xi_0, \tau_0) \) to the unit sphere, we may fix \( \xi_0 \) and vary \( \rho \) and \( \tau_0 \), obtaining after some rearrangement the rescaled equation

\[
(A^1)^{-1}(i\tau_0 + iA^{\xi_0} + i\sigma + \rho(B^{\xi_0\xi_0} - \theta_1(|\xi_0|^2 + |\tau_0|^2) + O(|\rho, \sigma|))) - \alpha I v = 0,
\]

where \( \sigma \) denotes variation in \( \tau_0 \).

We shall find it useful to introduce at this point the wedge, or exterior algebraic product of vectors, determined by the properties of multilinearity and alternation. We fix once and for all a basis, thus determining a norm. (This allows us to conveniently factorize into minors, below.)

We make also the provisional assumption (P0) of Section 4, that either the inviscid system is strictly hyperbolic, or else \( A^j \) and \( B^{jk} \) are simultaneously symmetrizable. This is easily removed by working with vector blocks in place of individual modes.

**Lemma 5.14** [Z.3]. Under the hypotheses of Theorem 5.5, let \( \alpha_0 = i\xi_0 \) be a pure imaginary root of the inviscid equation (5.47) for some given \( \xi_0, \tau_0 \), i.e. \( \det(A^{\xi_0} + \tau_0) = 0 \). Then, associated with the corresponding root \( \alpha \) in (5.48), we have the following behavior, for some fixed \( \epsilon, \theta > 0 \) independent of \( (\xi_0, \tau_0) \):

(i) For \( (\xi_0, \tau_0) \) bounded distance \( \epsilon \) away from any branch singularity \( (\tilde{\xi}, \eta_j(\tilde{\xi})) \) involving \( \alpha \), the root \( \alpha(\rho) \) in (5.48) such that \( \alpha(0) := \alpha_0 \) bifurcates smoothly into
m roots $\alpha_1, \ldots, \alpha_m$ with associated vectors $v_1, \ldots, v_m$, where $m$ is the dimension of $\ker(A^{\xi_0} + \tau_0)$, satisfying

$$|\text{Re} \, \alpha_j| \geq \theta \rho$$

and

$$|v_1 \wedge \cdots \wedge v_m| \geq \theta > 0$$

for $0 < \rho \leq \epsilon$, where the modulus of the wedge product is evaluated with respect to its coordinatization in some fixed reference basis.

(ii) For $(\xi_0, \tau_0) \text{ lying at a branch singularity } (\tilde{\xi}, \eta_j(\tilde{\xi})) \text{ involving } \alpha$, the root $\alpha(\rho, \sigma)$ in (5.50) such that $\alpha(0,0) = \alpha_0$ bifurcates (nonsmoothly) into $m$ groups of $s$ roots each:

$$\{\alpha_1^1, \ldots, \alpha_s^1\}, \ldots, \{\alpha_1^m, \ldots, \alpha_s^m\},$$

with associated vectors $v_k^j$, where $m$ is the dimension of $\ker(A^{\xi_0} + \tau_0)$ and $s$ is some positive integer, such that, for $0 \leq \rho \leq \epsilon$ and $|\sigma| \leq \epsilon$,

$$\alpha_k^j = \alpha + \pi_k^j + o(|\sigma| + |\rho|)^{1/s},$$

and, in appropriately chosen coordinate system,

$$v_k^j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \pi_k^j \\ (\pi_k^j)^2 \\ \vdots \\ (\pi_k^j)^{m-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + o(|\sigma| + |\rho|)^{1/s},$$

where

$$\pi_k^j := \varepsilon^k i (p\sigma - iq_j \rho)^{1/s},$$

$\varepsilon := 1^{1/s}$, and the functions $p(\xi_0)$ and $q_j(\xi_0)$ are real-valued and uniformly bounded both above and away from zero, with $\text{sgn} \, p = \text{sgn} \, q$. Moreover,

$$|\text{Re} \, \alpha_k^j| \geq \theta \rho$$
and the “group vectors” $v^j$ satisfy

$$|v^1 \wedge \cdots \wedge v^m| \geq \theta > 0$$

with respect to some fixed reference basis.

**Proof.** Case (H2)(i): We first treat the considerably simpler strictly hyperbolic case, which permits a direct and relatively straightforward treatment. In this case, the dimension of $\ker(A^{\xi_0} + \tau_0)$ is one, hence $m$ is simply one and (5.52) and (5.58) are irrelevant. Let $l(\xi_0, \tau_0)$ and $r(\xi_0, \tau_0)$ denote left and right zero eigenvectors of $(A^{\xi_0} + \tau_0) = 1$, spanning co-kernel and kernel, respectively; these are necessarily real, since $(A^{\xi_0} + \tau_0)$ is real. Clearly $r$ is also a right (null) eigenvector of $(A^1)^{-1}(i\tau_0 + iA^{\xi_0})$, and $lA^1$ a left eigenvector.

Branch singularities are signalled by the relation

$$lA^1 r = 0,$$

which indicates the presence of a single Jordan chain of generalized eigenvectors of $(A^1)^{-1}(i\tau_0 + iA^{\xi_0})$ extending up from the genuine eigenvector $r$; we denote the length of this chain by $s$.

**Observation 5.15** [Z.3]. Bound (2.55) implies that

$$lB^{\xi_0} r \geq \theta > 0,$$

uniformly in $\bar{\xi}$.

**Proof of Observation.** In our present notation, (2.55) can be written as

$$\Re \sigma(-iA^{\xi_0} - \rho B^{\xi_0}) \leq -\theta_1 \rho,$$

for all $\rho > 0$, some $\theta_1 > 0$. (Recall: $|\xi_0| \geq \theta_2 > 0$, by previous discussion). By standard matrix perturbation theory [Kat], the simple eigenvalue $\gamma = i\tau_0$ of $-iA^{\xi_0}$ perturbs analytically as $\rho$ is varied around $\rho = 0$, with perturbation series

$$\gamma(\rho) = i\tau_0 - \rho lB^{\xi_0} r + o(\rho).$$

Thus,

$$\Re \gamma(\rho) = -\rho lB^{\xi_0} r + o(\rho) \leq -\theta_1 \rho,$$

yielding the result. □

In case (i), $\alpha(0) = \alpha_0$ is a simple eigenvalue of $(A^1)^{-1}(i\tau_0 + iA^{\xi_0})$, and so perturbs analytically in (5.48) as $\rho$ is varied around zero, with perturbation series

$$\alpha(\rho) = \alpha_0 + \rho \alpha^1 + o(\rho),$$
where \( \alpha^1 = \tilde{l}(A^1)^{-1}\tilde{r}, \tilde{l}, \tilde{r} \) denoting left and right eigenvectors of \((A^1)^{-1}(i\tau_0 + A\tilde{\xi}_0)\). Observing by direct calculation that \( \tilde{r} = r, \tilde{l} = lA^1/lA^1r \), we find that

\[
\alpha^1 = lB^{\xi_0\xi_0}r/lA^1r
\]

is real and bounded uniformly away from zero, by Observation 5.15, yielding the result (5.51) for any fixed \((\xi_0, \tau_0)\), on some interval \(0 \leq \rho \leq \epsilon\), where \(\epsilon\) depends only on a lower bound for \(\alpha^1\) and the maximum of \(\gamma''(\rho)\) on the interval \(0 \leq \rho \leq \epsilon\). By compactness, we can therefore make a uniform choice of \(\epsilon\) for which (5.51) is valid on the entire set of \((\xi_0, \tau_0)\) under consideration.

In case (ii), \(\alpha(0, 0) = \alpha_0\) is an \(s\)-fold eigenvalue of \((A^1)^{-1}(i\tau_0 + iA\tilde{\xi}_0)\), corresponding to a single \(s \times s\) Jordan block. By standard matrix perturbation theory, the corresponding \(s\)-dimensional invariant subspace (or “total eigenspace”) varies analytically with \(\rho\) and \(\sigma\), and admits an analytic choice of basis with arbitrary initialization at \(\rho, \sigma = 0\) [Kat]. Thus, by restricting attention to this subspace we can reduce to an \(s\)-dimensional perturbation problem; moreover, up to linear order in \(\rho, \sigma\), the perturbation may be calculated with respect to the fixed, initial cooridination at \(\rho, \sigma = 0\).

Choosing the initial basis as a real, Jordan chain reducing the restriction (to the subspace of interest) of \((A^1)^{-1}(i\tau_0 + iA\tilde{\xi}_0)\) to \(i\) times a standard Jordan block, we thus reduce (5.50) to the canonical problem

\[
(iJ + i\sigma M + \rho N + o(|(\rho, \sigma)| - (\alpha - \alpha_0))v_I = 0,
\]

where

\[
J := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

\(v_I\) is the coordinate representation of \(v\) in the \(s\)-dimensional total eigenspace, and \(M\) and \(N\) are given by

\[
M := \tilde{L}(A^1)^{-1}\tilde{R}
\]

and

\[
N := \tilde{L}(A^1)^{-1}(B^{\xi_0\xi_0} - \theta_1)\tilde{R},
\]

respectively, where \(\tilde{R}\) and \(\tilde{L}\) are the initializing (right) basis, and its corresponding (left) dual.

Now, we have only to recall that, as may be readily seen by the defining relation

\[
\tilde{L}(A^1)^{-1}(i\tau_0 + iA\tilde{\xi}_0)\tilde{R} = J,
\]
or equivalently \((A^1)^{-1}(\tau_0 + iA\xi_0)\bar{R} = \bar{R}J\) and \(\bar{L}(A^1)^{-1}(\tau_0 + iA\xi_0) = J\bar{L}\), the first column of \(\bar{R}\) and the last row of \(\bar{L}\) are genuine left and right eigenvectors \(\bar{r}\) and \(\bar{l}\) of \((A^1)^{-1}(i\tau_0 + iA\xi_0)\), hence without loss of generality

\[
\bar{r} = r, \quad \bar{l} = plA^1
\]

as in the previous (simple eigenvalue) case, where \(p\) is an appropriate nonzero real constant. Applying again Observation 5.15, we thus find that the crucial \(s, 1\) entries of the perturbations \(M, N\), namely \(p\) and \(pl(B^{\xi_0\xi_0} - \theta_1)r = q\), respectively, are real, nonzero and of the same sign. Recalling, by standard matrix perturbation theory, that this entry when nonzero is the only significant one, we have reduced finally (modulo \(o(|\sigma| + |\rho|)^{1/\epsilon}\) errors) to the computation of the eigenvalues/eigenvectors of

\[
(i \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p\sigma - iq\rho & 0 & 0 & \cdots & 0 \end{pmatrix}),
\]

from which results (5.54)–(5.56) follow by an elementary calculation, for any fixed \((\xi_0, \tau_0)\), and some choice of \(\epsilon > 0\); as in the previous case, the corresponding global results then follow by compactness. Finally, bound (5.57) follows from (5.54) and (5.56) by direct calculation. (Note that the addition of further \(O(|\sigma| + |\rho|)\) perturbation terms in entries other than the lower lefthand corner of (5.66) does not affect the result.) This completes the proof in the strictly hyperbolic case.

Case (\(H2\))/(ii): We next turn to the more complicated symmetrizable, constant-multiplicity case; here, we make essential use of recent results of M´ etivier [M´ e.4] concerning the spectral structure of matrix \((A^1)^{-1}(i\tau + iA\xi)\). Without loss of generality, take \(A^\xi, B^\xi\xi\xi\) (but not necessarily \(A^1\)) symmetric; this may be achieved by the change of coordinates \(A^\xi \rightarrow A^1_{1/2}A^\xi A^{-1/2}_{1/2}, B^\xi\xi\xi \rightarrow A^{1/2}_{1/2}B^\xi\xi\xi A^{-1/2}_{1/2}\). From (H3), we find, additionally, that \(B^\xi\xi\xi \geq \theta|\xi|^2\) for all \(\xi \in \mathbb{R}^d\).

With these assumptions, the kernel and co-kernel of \((A^{\xi_0} + \tau_0)\) are of fixed dimension \(m\), not necessarily equal to one, and are spanned by a common set of zero-eigenvectors \(r_1, \ldots, r_m\). Vectors \(r_1, \ldots, r_m\) are necessarily right zero-eigenvectors of \((A^1)^{-1}(i\tau_0 + iA^{\xi_0})\) as well. Branch singularities correspond to the existence of one or more Jordan chains of generalized zero-eigenvectors extending up from genuine eigenvectors in their span, which by the argument of Lemma 3.4 is equivalent to

\[
\det (r_j^t A^1 r_k) = 0.
\]

In fact, as pointed out by M´ etivier [M´ e.4], the assumption of constant multiplicity implies considerable additional structure.
Observation 5.16 [Mé.4]. Let \((\xi_0, \tau_0)\) lie at a branch singularity involving root \(\alpha_0 = i\xi_0\) in (5.47), with \(\tau_0\) an \(m\)-fold eigenvalue of \(A^{\xi_0}\). Then, for \((\tilde{\xi}, \tau)\) in the vicinity of \((\xi_0, \tau_0)\), the roots \(\alpha\) bifurcating from \(\alpha_0\) in (5.47) consist of \(m\) copies of \(s\) roots \(\alpha_1, \ldots, \alpha_s\), where \(s\) is some fixed positive integer.

Proof of Observation. Let \(a(\xi, \alpha)\) denote the unique eigenvalue of \(A^\xi\) lying near \(-\tau_0\), where, as usual, \(-i\xi_1 := \alpha\); by the constant multiplicity assumption, \(a(\cdot, \cdot)\) is an analytic function of its arguments. Observing that

\[
\det [(A^1)^{-1}(i\tau + iA^\xi) - \alpha] = \det i(A^1)^{-1}\det (\tau + A^\xi)
\]

\[
= e(\tilde{\xi}, \tau, \alpha)(\tau + a(\tilde{\xi}, \alpha))^m,
\]

where \(e(\cdot, \cdot, \cdot)\) does not vanish for \((\tilde{\xi}, \tau, \alpha)\) sufficiently close to \((\xi_0, \tau_0, \alpha_0)\), we see that the roots in question occur as \(m\)-fold copies of the roots of

\[
(5.68) \quad \tau + a(\tilde{\xi}, \alpha) = 0.
\]

But, the lefthand side of (5.68) is a family of analytic functions in \(\alpha\), continuously varying in the parameters \((\tilde{\xi}, \tau)\), whence the number of zeroes is constant near \((\xi_0, \tau_0)\).

Observation 5.17 [Z.3]. The matrix \((r_j^1 A^1 r_k)\), \(j, k = 1, \ldots, m\) is a real multiple of the identity,

\[
(5.69) \quad (r_j^1 A^1 r_k) = (\partial a/\partial \xi_1) I_m,
\]

where \(a(\xi)\) denotes the (unique, analytic) \(m\)-fold eigenvalue of \(A^\xi\) perturbing from \(-\tau_0\).

More generally, if

\[
(5.70) \quad (\partial a/\partial \xi_1) = \cdots = (\partial^{s-1} a/\partial \xi_1^{s-1}) = 0, \quad (\partial^s a/\partial \xi_1^s) \neq 0
\]

at \(\xi_0\), then, letting \(r_1(\tilde{\xi}), \ldots, r_m(\tilde{\xi})\) denote an analytic choice of basis for the eigenspace corresponding to \(a(\tilde{\xi})\), orthonormal at \((\xi_0, \tau_0)\), we have the relations

\[
(5.71) \quad (A^1)^{-1}(\tau_0 + A^{\xi_0}) r_{j,p} = r_{j,p-1},
\]

for \(1 \leq p \leq s - 1\), and

\[
(5.72) \quad (r_{j,0}^1 A^1 r_{k,p-1}) = p! \left(\partial^p a/\partial \xi_1^p\right) I_m,
\]

for \(1 \leq p \leq s\), where

\[
r_{j,p} := (-1)^p p (\partial^p r_j/\partial \xi_1^p).
\]
In particular,
\[ \{r_{j,0}, \ldots, r_{j,s-1}\}, \quad j = 1, \ldots, m \]
is a right Jordan basis for the total zero eigenspace of \((A^1)^{-1}(\tau_0 + A^{\xi_0})\), for which the genuine zero-eigenvectors \(\tilde{l}_j\) of the dual, left basis are given by
\begin{equation}
\tilde{l}_j = \frac{1}{s!}(\partial^s a / \partial \xi_1^s) A^1 r_j.
\end{equation}

**Proof of Observation.** Considering \(A^\xi\) as a matrix perturbation in \(\xi_1\), we find by standard spectral perturbation theory that the bifurcation of the \(m\)-fold eigenvalue \(\tau_0\) as \(\xi_1\) is varied is governed to first order by the spectrum of \((r_j^t A^1 r_k)\). Since these eigenvalues in fact do not split, it follows that \((r_j^t A^1 r_k)\) has a single eigenvalue. But, also, \((r_j^t A^1 r_k)\) is symmetric, hence diagonalizable, whence we obtain result (5.69).

Result (5.72) may be obtained by a more systematic version of the same argument. Let \(R(\xi_1)\) denote the matrix of right eigenvectors
\[ R(\xi_1) := (r_1, \ldots, r_m)(\xi_1). \]
Denoting by
\begin{equation}
a(\xi_1 + h) =: a^0 + a^1 h + \cdots + a^p h^p + \cdots
\end{equation}
and
\begin{equation}
R(\xi_1 + h) =: R^0 + R^1 h + \cdots + R^p h^p + \cdots
\end{equation}
the Taylor expansions of functions \(a(\cdot)\) and \(R(\cdot)\) around \(\xi_0\) as \(\xi_1\) is varied, and recalling that
\[ A^\xi = A^{\xi_0} + h A^1, \]
we obtain in the usual way, matching terms of common order in the expansion of the defining relation \((A - a)R = 0\), the hierarchy of relations:
\begin{align}
(A^{\xi_0} - a^0) R^0 &= 0, \\
(A^{\xi_0} - a^0) R^1 &= -(A^1 - a^1) R^0, \\
(A^{\xi_0} - a^0) R^2 &= -(A^1 - a^1) R^1 + a^2 R^0, \\
&\quad \vdots \\
(A^{\xi_0} - a^0) R^p &= -(A^1 - a^1) R^{p-1} + a^2 R^{p-2} + \cdots + a^p R^0.
\end{align}
Using \(a^0 = \cdots = a^{s-1} = 0\), we obtain (5.71) immediately, from equations \(p = 1, \ldots, s-1\), and \(R^p = (1/p!)(r_{1,p}, \ldots, r_{m,p})\). Likewise, (5.72), follows from equations
$p = 1, \ldots, s$, upon left multiplication by $L^0 := (R^0)^{-1} = (R^0)^t$, using relations $L^0(A^{\xi_0} - a^0) = 0$ and $a^p = (\partial^p a/\partial \xi^*_1)/p!$.

From (5.71), we have the claimed right Jordan basis. But, defining $\tilde{l}_j$ as in (5.73), we can rewrite (5.72) as

$$\begin{align*}
(\tilde{l}_j r_{k,p-1}) &= \begin{cases} 
0 & 1 \leq p \leq s - 1, \\
I_m, & p = s;
\end{cases}
\end{align*}$$

these $ms$ criteria uniquely define $\tilde{l}_j$ (within the $ms$-dimensional total left eigenspace) as the genuine left eigenvectors dual to the right basis formed by vectors $r_{j,p}$ (see also exercise just below).

Observation 5.17 implies in particular that Jordan chains extend from all or none of the genuine eigenvectors $r_1, \ldots, r_m$, with common height $s$. As suggested by Observation 5.16 (but not directly shown here), this uniform structure in fact persists under variations in $\tilde{\xi}, \tau$, see [Mé.4]. Observation 5.17 is a slightly more concrete version of Lemma 2.5 in [Mé.4]; note the close similarity between the argument used here, based on successive variations in basis $r_j$, and the argument of [Mé.4], based on variations in the associated total projection.

With these preparations, the result goes through essentially as in the strictly hyperbolic case. Set

$$p := 1/(s!(\partial^s a/\partial \xi^*_1)).$$

In the result of Observation 5.17, we are free to choose any orthonormal basis $r_j$ at $(\tilde{\xi}_0, \tau_0)$; choosing a basis diagonalizing the symmetric matrix $B^{\tilde{\xi}_0, \tilde{\xi}_0}$, we define

$$pR^t B^{\tilde{\xi}_0, \tilde{\xi}_0} R =: \text{diag}\{q_1, \ldots, q_m\}.$$

Note, as claimed, that $p \neq 0$ by assumption $(\partial^s a/\partial \xi^*_1) \neq 0$ in Observation 5.17, and $\text{sgn} \ q_j = \text{sgn} \ p$ by positivity of $B^{\tilde{\xi}_0, \tilde{\xi}_0}$ (an easy consequence of (2.55)).

Then, working in the Jordan basis defined in Observation 5.17, we find similarly as in the strictly hyperbolic case that the matrix perturbation problem (5.50) reduces to an $ms \times ms$ system consisting of $ms \times s$ equations

$$\begin{align*}
(iJ + i\sigma M_j + \rho N_j + \alpha(\rho, \sigma))v_{I_j} &= \sum_{k \neq j} (i\sigma M_{jk} + \rho N_{jk})v_{I_k},
\end{align*}$$

($J$ denoting the standard Jordan block (5.63)), weakly coupled in the sense that lower lefthand corner elements vanish in the coupling blocks $M_{jk}, N_{jk}$, but in the diagonal blocks $\sigma iM_j + \rho N_j$ are $p\sigma - iq_j \rho \sim |\sigma| + |\rho|$. To order $(|\sigma| + |\rho|)^{1/s}$,
therefore, the problem decouples, reducing to the computation of eigenvectors and eigenvalues of the block diagonal matrix

\[
(5.81) \quad \text{diag}\left\{ i \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_\sigma - iq_j \rho & 0 & 0 & \cdots & 0 \end{pmatrix} \right\},
\]

from which results (5.54)-(5.57) follow as before; see Section 2.2.4, *Splitting of a block-Jordan block.* (Note that the simple eigenvalue case \( s = 1 \) follows as a special case of the Jordan block computation.) Finally, the uniform condition (5.58) is a consequence of (5.55), orthonormality of bases \( \{r_j\} \), and continuity of the Taylor expansions (5.75). □

**Remark 5.18** [Z.3]. Fixing \( j \), consider a single cycle of roots \( \pi_k^j := \varepsilon^k i(\rho_\sigma - iq_j \rho)^{1/s} \) in (5.56), \( j = 1, \ldots, s, \varepsilon := 1^{1/s}, \text{ sgn } p = \text{ sgn } q_j \), on the set \( \rho \geq 0, \rho, \sigma \in \mathbb{R} \) of interest. By direct evaluation at \( \sigma = 0, \rho = 1/|q_j| \), for \( \rho > 0 \), the roots \( \pi_k^j \) split into \( s_+ \) unstable modes (Re \( \pi > 0 \)) and \( s_- \) stable modes (Re \( \pi < 0 \)), where

\[
(5.82) \quad (s_+, s_-) = \begin{cases} 
(r, r) & \text{for } s = 2r, \\
(r + 1, r) & \text{for } s = 2r + 1 \text{ and } p > 0, \\
(r, r + 1) & \text{for } s = 2r + 1 \text{ and } p < 0.
\end{cases}
\]

(At \( \sigma = 0, \rho = 1/|q_j| \), we have simply \( \pi_k^j = \varepsilon^k(-1)^{r+1} \text{ sgn } q_k^{1/s} \) which in the critical case \( s = 2r + 1 \) becomes \( \varepsilon^k(-1)^{r+2} \text{ sgn } q_k = \varepsilon^k(-1)^r \text{ sgn } p \).)

**Remark 5.19** [Z.3]. Defining \( w_j \in \mathbb{R}^{s} \) by \( w_j := (1, \alpha_j, \alpha_j^2, \ldots, \alpha_j^{s-1})^t, 1 \leq j \leq p \leq s \), with \( |\alpha_j| \leq C_1 \), we find that \( |w_1 \wedge \cdots \wedge w_p| \sim \Pi_{j<k}|\alpha_j - \alpha_k| \), with constants depending only on \( C_1 \), or equivalently

\[
(5.83) \quad \left| \det \begin{pmatrix} \alpha_1^{q_1} & \cdots & \alpha_p^{q_1} \\ \vdots & \ddots & \vdots \\ \alpha_1^{q_p} & \cdots & \alpha_p^{q_p} \end{pmatrix} \right| \leq C \Pi_{j<k}|\alpha_j - \alpha_k|,
\]

for each minor \( 0 \leq q_1 < \cdots < q_p \leq s-1 \) of matrix \( (w_1, \ldots, w_p) \in \mathbb{R}^{s \times p} \), with equality for the Vandermonde minor \( q_r := r-1 \). For, considering the determinants in (5.83) as complex polynomials \( P(\alpha_1, \ldots, \alpha_p) \), and noting that \( P = 0 \) whenever \( \alpha_j = \alpha_k \), we may use successive applications of the Remainder Theorem to conclude that \( \Pi_{j<k}(\alpha_j - \alpha_k) \) divides \( P \).

**Remark 5.20.** The direct calculations of Lemma 5.14 verifies Lemma 3.3 in the case that (H5) holds.

**Remark 5.21.** With Lemma 5.14, Corollary 5.11 follows, also, by the Kreiss symmetrizer argument of [GMWZ.2].
Lemma 5.22\[Z.3]\. Under the hypotheses of Theorem 5.5, for \(\lambda \in \Gamma^\pm\) (defined in (5.36)) and \(\rho := |(\xi, \lambda)|, \theta_1 > 0\), and \(\theta > 0\) sufficiently small, there exists a choice of bases \(\{\varphi_j^\pm\}, \{\varphi_j^\mp\}, \{\psi_j^\pm\}, \{\psi_j^\mp\}\) of solutions of the variable-coefficient eigenvalue problem such that, at \(z = 0\), the wedge products

\[
(\varphi_1') \wedge \cdots \wedge (\varphi_n') \pm (z)
\]

and determinants

\[
\det \begin{pmatrix} \Phi & \Psi \\ \Phi' & \Psi' \end{pmatrix} \pm (z)
\]

and

\[
\det \begin{pmatrix} \tilde{\Phi} & \tilde{\Psi} \\ \tilde{\Phi}' & \tilde{\Psi}' \end{pmatrix} \pm (z)
\]

are bounded in modulus (measured, in the case of wedge products, with respect to coordinates in some fixed basis) uniformly above and below, and, moreover, there hold bounds:

\[
(5.84) \quad \begin{pmatrix} \varphi_j^\pm \\ \varphi_j' \end{pmatrix} = \gamma_{21, \varphi_j^\pm} \left[ e^{\mu_j^+ x} \left( \frac{\nu_j^\pm}{\mu_j^\pm} v_j^\pm_0 \right) + O(e^{-\theta_1 |x|}) \right], \quad x_1 \gtrless 0,
\]

\[
(5.85) \quad \begin{pmatrix} \tilde{\varphi}_j^\pm \\ \tilde{\varphi}_j' \end{pmatrix} = \gamma_{21, \tilde{\varphi}_j^\pm} \left[ e^{-\mu_j^- x} \left( \frac{\nu_j^-}{-\mu_j^-} \tilde{v}_j^\pm_0 \right) + O(e^{-\theta_1 |x|}) \right], \quad x_1 \gtrless 0,
\]

and

\[
(5.86) \quad \begin{pmatrix} \psi_j^\pm \\ \psi_j' \end{pmatrix} = \gamma_{21, \psi_j^\pm} \left[ e^{\nu_j^+ x} \left( \frac{\nu_j^\pm}{\nu_j^\pm} v_j^\pm_0 \right) + O(e^{-\theta_1 |x|}) \right], \quad x_1 \gtrless 0,
\]

\[
(5.87) \quad \begin{pmatrix} \psi_j^\pm \\ \psi_j' \end{pmatrix} = \gamma_{21, \psi_j^\pm} \left[ e^{-\nu_j^- x} \left( \frac{\nu_j^-}{-\nu_j^-} \tilde{v}_j^\pm_0 \right) + O(e^{-\theta_1 |x|}) \right], \quad x_1 \gtrless 0,
\]

where \(|v_j^\pm|, |\tilde{v}_j^\pm|\) are uniformly bounded above and below, and:

(i) For fast and intermediate-slow modes, or super-slow modes for which \(\text{Im} \lambda\) is bounded distance \(\theta_1 \epsilon\) away from any associated branch singularities \(\eta_h(\tilde{\xi})\), \(v_j^\pm\) are uniformly transverse to all other \(v_k^\pm\), while, for super-slow modes for which \(\text{Im} \lambda\) is within \(\theta_1 \epsilon\) of an (necessarily unique) associated branch singularity \(\eta_l(\tilde{\xi})\), \(v_j^\pm\) are
as described in Lemma 5.14, with \( \sigma := \rho^{-1}(\text{Im } \lambda - \eta_h(\tilde{\xi})) \); moreover, the “group vectors” associated with each cluster \( v^+_h \) are uniformly transverse to each other and to all other modes. The dual vectors \( \tilde{v}^+_j \) satisfy identical bounds.

(ii) The decay/growth rates \( \mu_j^\pm / \nu_j^\pm \) satisfy

\[
|\text{Re} \; \mu_j^\pm|, |\text{Re} \; \nu_j^\pm| \sim 1,
\]

for fast modes,

\[
|\text{Re} \; \mu_j^\pm|, |\text{Re} \; \nu_j^\pm| \sim \rho
\]

for intermediate-slow modes, and

\[
|\text{Re} \; \mu_j^\pm|, |\text{Re} \; \nu_j^\pm| \sim \rho^2
\]

for super-slow modes; moreover,

\[
|\mu_j^\pm|, |\nu_j^\pm| = O(\rho)
\]

for both intermediate- and super-slow modes.

(iii) The factors \( \gamma_{21,\varphi_j^\pm}, \gamma_{21,\tilde{\varphi}_j^\pm}, \gamma_{21,\psi_j^\pm}, \text{ and } \gamma_{21,\tilde{\psi}_j^\pm} \) satisfy

\[
\gamma_{21,\beta} \sim 1, \quad \beta = \varphi_j^\pm, \psi_j^\pm, \tilde{\varphi}_j^\pm, \tilde{\psi}_j^\pm,
\]

for fast and intermediate-slow modes, and for super-slow modes for which \( \text{Im } \lambda \) is bounded distance \( \theta_1 \epsilon \) away from any associated branch singularities \( \eta_h(\tilde{\xi}), \) and

\[
\gamma_{21,\beta} \sim (|\sigma| + |\rho|)^{-t_\beta}, \quad \beta = \varphi_j^\pm, \psi_j^\pm, \tilde{\varphi}_j^\pm, \tilde{\psi}_j^\pm,
\]

for super-slow modes for which \( \text{Im } \lambda \) is within \( \theta_1 \epsilon \) of an (necessarily unique) associated branch singularity \( \eta_h(\tilde{\xi}), \) with

\[
(t_{\varphi_j^\pm}, t_{\psi_j^\pm}, t_{\tilde{\varphi}_j^\pm}, t_{\tilde{\psi}_j^\pm}) :=
\]

\[
\left\{
\begin{array}{ll}
\left( \frac{r-1}{4r}, \frac{3r-1}{4r}, \frac{3r-1}{4r}, \frac{r-1}{4r} \right) & \text{for } s = 2r, \\
\left( \frac{r}{2(2r+1)}, \frac{3r+1}{2(2r+1)}, \frac{3r}{2(2r+1)}, \frac{r-1}{2(2r+1)} \right) & \text{for } s = 2r + 1 \text{ and } p \geq 0, \\
\left( \frac{r-1}{2(2r+1)}, \frac{3r+1}{2(2r+1)}, \frac{3r}{2(2r+1)}, \frac{r}{2(2r+1)} \right) & \text{for } s = 2r + 1 \text{ and } p \leq 0;
\end{array}
\right.
\]

here, \( s := K_h^\pm \) denotes the order of the associated branch singularity \( \eta_h(\tilde{\xi}), \) and \( \sigma := \rho^{-1}(\text{Im } \lambda - \eta_h(\tilde{\xi})), \) as above.

**Proof.** Rescaling \( v_j^\pm \rightarrow \gamma_{21, v_j^\pm} v_j^\pm \) in (5.44), for any choice of \( \gamma_{21, v_j^\pm}, \) we obtain immediately relations (5.87)–(5.93) and statements (i)–(ii) from the bounds
(2.20)–(2.21) guaranteed by the conjugation Lemma together with Lemma 5.14 and discussion above; recall the relation $\mu \sim -\rho \hat{\alpha}$ in (5.48).

Using the results of Remarks 5.18–5.19, we readily obtain bounds (5.96)–(5.97) from the bounds on wedge product (5.84) and determinants (5.85)–(5.86), or equivalently (bounds (2.20)–(2.21); see also Exercise 4.20, [Z.3]) the limiting, constant-coefficient versions

\begin{equation}
\left( \frac{\overline{\phi}_1}{\phi_1} \right) \wedge \ldots \wedge \left( \frac{\overline{\phi}_n}{\phi_n} \right) \pm \left( z, \right), \tag{5.98}
\end{equation}

\begin{equation}
\det \left( \frac{\overline{\Phi}}{\Phi} \frac{\overline{\Psi}}{\Psi} \right) \pm \left( z, \right), \tag{5.99}
\end{equation}

and

\begin{equation}
\det \left( \frac{\overline{\Phi}'}{\Phi'} \frac{\overline{\Psi}'}{\Psi'} \right) \pm \left( z, \right), \tag{5.100}
\end{equation}

be bounded above and below at $z = 0$: more precisely, from the corresponding requirements on the restriction to each $s := K^{\pm}_h$-dimensional subspace of solutions corresponding to a single branch singularity $\eta^+_h(\cdot)$ (clearly sufficient, by group transversality, statement (i)).

For example, in the representative case $s = 2r$, we find from Remark 5.18(c) that there are $r$ decaying modes $\phi_j^\pm$ and $r$ growing modes $\psi_j^\pm$, which at $z = 0$ are given by rescalings $\gamma_{21,\nu_{j,k}}^\pm \nu_{j,k}^\pm$ of the vectors $\nu_{j,k}^\pm$ associated with the branch cycle in question. Choosing common scalings $\gamma_{21,\nu_{j,k}}^\pm := \gamma_\nu$ for all decaying modes $\nu_{j,k}^\pm$ and $\gamma_{21,\nu_{j,k}}^\pm := \gamma_\psi$ for all growing modes $\nu_{j,k}^\pm$, and applying the result of Remark 5.19, we find that $(\overline{\nu}_1 \wedge \ldots \wedge \overline{\nu}_n)_{\pm}(0) \sim 1$ and $\det \left( \overline{\Phi} \overline{\Psi} \right)_{\pm}(0) \sim 1$ are equivalent to

$$\gamma_\nu^r (|\sigma| + |\rho|)^{1/s} \sim 1$$

and

$$\gamma_\nu^r \gamma_\psi^r (|\sigma| + |\rho|)^{1/s} \sim 1,$$

respectively, from which we immediately obtain the stated bounds on $t_{\nu_j^\pm}$ and $t_{\psi_j^\pm}$. The bounds on $t_{\nu_j^\pm}$ and $t_{\psi_j^\pm}$ then follow by duality,

\begin{equation}
(\overline{\Phi}, \overline{\Psi}) = [(\Phi, \Psi)^{-1} S_{\xi}^{-1}]^* \tag{5.101}
\end{equation}

$$= S_{\xi}^{-1*}(\Phi, \Psi)^{-1*}.$$
$S^\xi \sim I$ as defined in (4.25), which, restricted to the $s \times s$ cycle in question, yields $\gamma\varphi \sim \chi^{-1}(|\sigma| + |\rho|)^{(s-1)/s}$ and $\gamma\psi \sim \xi^{-1}(|\sigma| + |\rho|)^{(s-1)/s}$, where $\gamma\varphi$, $\gamma\psi$ denote common scalings for the dual growing, decaying modes associated with the branch cycle in question. The computations in other cases are analogous.

**Example.** In the “generic” case of a square-root singularity $s = 2$, calculation (5.101) becomes simply

$$(\varphi, \psi) := \tilde{S}^{-1*}(\varphi, \psi)^{-1*} = (1/2)\tilde{S}^{-1*}
\begin{pmatrix}
1 & \pi^{1/2} \\
\pi^{-1/2} & -1
\end{pmatrix},$$

where

$$(\varphi, \psi) := \begin{pmatrix}
1 \\
\pi^{1/2} \\
\pi^{-1/2} \\
-1
\end{pmatrix}$$

(recall: chosen so that $\det (\varphi, \psi) = 2 \sim 1$) and $\tilde{S}$ denotes the (undetermined) restriction of $S^\xi$ to the two-dimensional total eigenspace associated with the branching eigenvalue. Note that we do not require knowledge of $\tilde{S}$ in order to determine the orders of $\varphi$ and $\psi$. This simple case is the one relevant to gas dynamics, for which glancing involves only the pair of strictly hyperbolic acoustic modes; see Remark 4.14, [Z.3].

**Remark.** The requirements $|\phi_1 \wedge \cdots \wedge \phi_n|_\pm(0) \sim 1$ are needed in order that the bases chosen in Lemma 5.14 be consistent with those defined in previous sections, in particular that they induce the same Evans function. (Note: since we do not require individual analyticity of our basis elements, we are free to choose $C^\infty$ rescalings $\gamma_{1, \beta}$ so as to exactly match the Evans function defined in Section 3). The requirements $\det (\Phi, \Psi)_\pm(0), \det (\tilde{\Phi}, \tilde{\Psi})_\pm(0) \sim 1$ are only a convenient normalization, to simplify certain calculations later on.

From now on, we will assume that the bases $\{\varphi_j^\pm\}, \{\tilde{\varphi}_j^\pm\}, \{\psi_j^\pm\}, \{\tilde{\psi}_j^\pm\}$ have been chosen as described in Lemma 5.22. For later use, we record the useful relations:

$$(t\varphi_j + t\tilde{\varphi}_j)^\pm, (t\psi_j + t\tilde{\psi}_j)^\pm \equiv (1 - 1/s),$$

$$(t\psi_j - t\varphi_j)^\pm, (t\tilde{\varphi}_j - t\tilde{\psi}_j)^\pm \equiv 1/2,$$

$$(t\varphi_j + t\tilde{\psi}_j)^\pm, (t\psi_j + t\tilde{\varphi}_j)^\pm \equiv 1/2 - 1/s,$$

$$t\varphi_j^\pm, t\tilde{\varphi}_j^\pm \leq (1/4)(1 - 1/s)$$

between exponents $t_\beta$ in (5.97); these may be verified case-by-case.

**Refined derivative bounds.** Similarly as in the one-dimensional case, for slow, dual modes the derivative bounds (5.88), (5.90) can be considerably sharpened, provided that we appropriately initialize our bases at $\rho = 0$, and this observation is significant in the Lax and overcompressive case.
Due to the special, conservative structure of the underlying evolution equations, the adjoint eigenvalue equation

\[(B^{11*} \bar{w}')' = -A^{1*} \bar{w}'\]

at \(\rho = 0\) admits an \(n\)-dimensional subspace of constant solutions; this is equivalent to the familiar fact that integral quantities are conserved under time evolution for systems of conservation laws. Thus, at \(\rho = 0\), we may choose, by appropriate change of coordinates if necessary, that slow-decaying dual modes \(e^{\pm j} \varphi_j\) and slow-growing dual modes \(e^{\pm j} \psi_j\) be identically constant, or equivalently that not only fast-decaying modes \(e^{\pm j} \varphi_j\), but also fast-growing modes \(e^{\pm j} \psi_j\) be solutions of the linearized first-order traveling wave ODE

\[B^{11} w' = A^{1} w.\]

Note that this does not interfere with our previous choice in Lemma 3.4, nor does it interfere with the specifications of Lemma 5.22, since these concern only the choice of limiting solutions \(w^{\pm j}\) of the asymptotic, constant coefficient equations at \(x_1 \to \pm \infty\), and not the particular representatives \(w_j^{\pm}\) that approach them (which, in the case of slow modes, are specified only up to the addition of an arbitrary fast-decaying mode).

**Lemma 5.23 [Z.3]**. With the above choice of bases at \(\rho = 0\), slow modes \(\tilde{\varphi}_k\), \(\tilde{\psi}_k\), for \(\lambda \in \Gamma^{\xi}\) and \(\rho := |(\tilde{\xi}, \lambda)|\) sufficiently small, satisfy

\[(\partial/\partial y_1) \tilde{\varphi}_k \leq C\rho |\tilde{\varphi}_k|, \quad |(\partial/\partial y_1) \tilde{\psi}_k | \leq C\rho |\tilde{\psi}_k|.\]

**Proof.** Recall that slow dual modes \(\tilde{\varphi}_k\) or \(\tilde{\psi}_k\), by our choice of basis, are identically constant for \(\rho = 0\), i.e \((\partial/\partial y_1) \tilde{\varphi}_k|_{\rho=0}\), \((\partial/\partial y_1) \tilde{\psi}_k|_{\rho=0}\) \(\equiv 0\), making (5.105) plausible). Indeed, the manifold of all slow dual modes, both decaying and growing, is exactly the manifold of constant functions. Now, consider the evolution of this manifold along the curve \((\tilde{\xi}, \lambda)(\tilde{\xi}_0, \tau_0, \rho)\) defined in (5.43), \(\lambda_0 =: i\tau_0\). The conglomerate slow manifold is \(C^1\) in \(\rho\), even though the separate decaying and growing manifolds may be only \(C^0\); for, it is spectrally separated from all fast modes. Thus, the subspace

\[\text{Span}\{(w, w')^t : w \in \text{slow manifold}\}\]

lies for each \(y_1\) within angle \(C\rho\) of the space \{(w, 0)^t\} of constant functions, where \(\text{"}w\text{"} := \partial/\partial y_1\). But, this is exactly the statement (5.105). \(\square\)
Bounds (5.105) are to be compared with the bounds
\[
|\partial/\partial y_1 \tilde{\psi}_k^\pm| \leq C \rho |\tilde{\psi}_k^\pm| + C \gamma_{21, \tilde{\psi}_k^\pm} e^{-\theta |y_1|},
\]
\[
|\partial/\partial y_1 \tilde{\varphi}_k^\pm| \leq C \rho |\tilde{\varphi}_k^\pm| + C \gamma_{21, \tilde{\varphi}_k^\pm} e^{-\theta |y_1|},
\]
resulting from (5.88), (5.90), and (5.94).

**Low-frequency Evans function bounds.** The results of Section 5.1 suggest that the zeroes \(\lambda_*(\tilde{\xi})\) of the Evans function satisfy
\[
\text{Re } \lambda_*(\tilde{\xi}) \leq -\theta |\tilde{\xi}|, \quad |\text{Im } \lambda_*(\tilde{\xi})| \leq C |\tilde{\xi}|; \quad \theta, C > 0,
\]
for uniformly inviscid stable shocks, i.e. they move linearly into the stable complex half-plane with respect to \(|\tilde{\xi}|\), and
\[
\lambda_*(\tilde{\xi}) = \lambda_{s_j} := i \tau_j (\tilde{\xi}) - \beta_j (\tilde{\xi}) |\tilde{\xi}|^2 + o(|\tilde{\xi}|^2), \quad j = 1, \ldots, \ell,
\]
\(\tau_j\) real, distinct, \(\beta_j\) real, \(> 0\), for weakly inviscid stable shocks satisfying the refined dynamical stability condition, where \(i \tau_j = \lambda_j (\tilde{\xi})\) are the roots of \(\Delta (\tilde{\xi}, \cdot)\) (recall, strong refined dynamical stability requires that they be simple, Definition 1.21). The following result quantifies these observations via appropriate polar coordinate computations centered around the refined dynamical stability condition.

**Lemma 5.24[Z.3].** Under the hypotheses of Theorem 5.5, for \(\lambda \in \partial \Lambda \tilde{\xi}\) and \(\rho := |(\tilde{\xi}, \lambda)|\) sufficiently small, there holds
\[
\text{Re } \lambda_*(\tilde{\xi}) \leq \theta |\tilde{\xi}|; \quad |\text{Im } \lambda_*(\tilde{\xi})| \leq C |\tilde{\xi}|; \quad \theta, C > 0,
\]
(5.107)

for uniformly inviscid stable (U.I.S) shocks, and
(5.108)

\[
\lambda_*(\tilde{\xi}) = \lambda_{s_j} := i \tau_j (\tilde{\xi}) - \beta_j (\tilde{\xi}) |\tilde{\xi}|^2 + o(|\tilde{\xi}|^2), \quad j = 1, \ldots, \ell,
\]
\(\tau_j\) real, distinct, \(\beta_j\) real, \(> 0\), for weakly inviscid stable shocks satisfying the refined dynamical stability condition, where \(i \tau_j = \lambda_j (\tilde{\xi})\) are the roots of \(\Delta (\tilde{\xi}, \cdot)\) (recall, strong refined dynamical stability requires that they be simple, Definition 1.21). The following result quantifies these observations via appropriate polar coordinate computations centered around the refined dynamical stability condition.

**Lemma 5.24[Z.3].** Under the hypotheses of Theorem 5.5, for \(\lambda \in \partial \Lambda \tilde{\xi}\) and \(\rho := |(\tilde{\xi}, \lambda)|\) sufficiently small, there holds
(5.109)

\[
|D(\tilde{\xi}, \lambda)|^{-1} \leq C \rho^{-\ell}
\]
for uniformly inviscid stable (U.I.S) shocks, and
(5.110)

\[
|D(\tilde{\xi}, \lambda)|^{-1} \leq C \rho^{-\ell+1} \left[ \min_j |\text{Im } (\lambda) - i \tau_j (\tilde{\xi})| + |\text{Re } (\lambda) + \beta_j (\tilde{\xi}) |\tilde{\xi}|^2| + o(|\tilde{\xi}|^2) \right]^{-1}
\]
only ready, where \(\tau_j, \beta_j\) are as in (5.108), and \(\Lambda \tilde{\xi}\) is as in (2.62).

**Proof.** Define the curves
(5.111)

\[
(\tilde{\xi}, \lambda)(\rho, \tilde{\xi}_0, \lambda_0) := (\rho \tilde{\xi}_0, \rho \lambda_0 - \theta (\rho |\lambda_0|^2 - \theta (\rho |\tilde{\xi}_0|^2)^2)^2,
\]
for \(\tilde{\xi}_0 \in \mathbb{R}^{d-1}\), \(\text{Re } \lambda_0 = 0, |\tilde{\xi}_0| + |\lambda_0| = 1\), and set
(5.112)

\[
D(\rho, \tilde{\xi}_0, \lambda_0) := D(\tilde{\xi}(\rho, \tilde{\xi}_0, \lambda_0), \lambda(\rho, \tilde{\xi}_0, \lambda_0)).
\]

Inspection of the proof of Lemma 3.3 (either in [MeZ.2] or the later calculations in Section 5.3.2) show that continuity of slow decaying modes holds not only along
rays, but also along the curved paths \((\tilde{\xi}, \lambda)(\rho, \tilde{\xi}_0, \lambda_0)\) defined in (5.111), provided \(\theta > 0\) is taken sufficiently small. Thus, we obtain by a calculation identical to that in the proof of Theorem 3.5, for \(\theta > 0\) sufficiently small, that

\[
\bar{D}(\tilde{\xi}_0, \lambda_0, 0) = \rho^\ell \Delta(\tilde{\xi}_0, \lambda_0) + o(\rho^\ell).
\]

In the U.I.S. case, \(\bar{\Delta}(\tilde{\xi}_0, \lambda_0) \neq 0\), hence, for \(\rho\) sufficiently small,

\[
|\bar{D}(\tilde{\xi}_0, \lambda_0, \rho)|^{-1} \leq C|\bar{\Delta}(\tilde{\xi}_0, \lambda_0)|^{-1} \rho^{-\ell} \leq C_2(|\bar{\xi}| + |\lambda|)^{-\ell}
\]
as claimed.

It remains to verify (5.110) for \(\lambda_0\) in the vicinity of a point \(i\tau_j(\tilde{\xi}_0)\) such that \(\bar{\Delta}(\tilde{\xi}_0, i\tau_j(\tilde{\xi}_0)) = 0\), where, by assumption, \(\bar{\Delta}\) and \(D\) are analytic, with \(\bar{\Delta}_\lambda \neq 0\). Introducing again the function

\[
g_{\tilde{\xi}, \rho}(\lambda) := \rho^{-\ell} D(\rho \tilde{\xi}, \rho \lambda)
\]
used in Section 5.1, we thus have

\[
\bar{D}(\tilde{\xi}_0, \lambda_0, \rho) = \rho^\ell g_{\tilde{\xi}, \rho}(\lambda_0 - \theta \rho)
\]

\[
= \rho^\ell [g_{\tilde{\xi}, 0}(i\tau_j) + g_{\tilde{\xi}, 0}(i\tau_j) \rho + \bar{\Delta}_\lambda(\tilde{\xi}_0, \lambda_0, \rho) = 0, \quad \tilde{\Delta}_\lambda(\tilde{\xi}_0, \lambda_0, \rho) \neq 0, \quad \text{and} \quad \bar{\Delta}_\lambda \neq 0,
\]
together with definitions \(\beta_j := (g_\rho / g_\lambda)|\tilde{\xi}_0|^{-2}\) and \(\lambda(\tilde{\xi}_0, \lambda_0, \rho) := (\lambda_0 - \theta \rho)\rho\), we obtain (factoring out the term \(g_\lambda\))

\[
|\bar{D}(\tilde{\xi}_0, \lambda_0, \rho)| \geq C^{-1}|\rho^\ell||\bar{\Delta}_\lambda||\lambda_0 - \theta \rho - i\tau_j + \beta_j|\tilde{\xi}_0|^2 \rho + \Theta|
\]

\[
= C^{-1}|\rho^\ell - 1||\bar{\Delta}_\lambda||\lambda - i\tau_j(\tilde{\xi}) + \beta_j||\tilde{\xi}|^2 + \rho \Theta|
\]

where

\[
\Theta = O(|\lambda_0 - i\tau_j|^2 + \rho^2).
\]

The result then follows by the observation that \(|\tilde{\xi}_0|/|\tau_j(\tilde{\xi}_0)|\) is bounded from zero (recall, \(\Delta(0, 1) \neq 0\)), whence \(\rho = O(\tilde{\xi})\). □

**Scattering coefficients.** It remains to estimate the behavior of coefficients \(M_{jk}^\pm, d_{jk}^\pm\) defined in Corollary 2.26, as \(\rho \to 0, \rho := |(\xi, \lambda)|\). Consider coefficients \(M_{jk}^\pm\). Expanding (2.101) using Cramer’s rule, and setting \(z = 0\), we obtain

\[
M_{jk}^\pm = D^{-1}C_{jk}^\pm,
\]
where

\[(5.120) \quad C^+ := (I, 0) \left( \frac{\Phi^+}{\Phi^\ell} \quad \frac{\Phi^-}{\Phi^\ell} \right)^{\text{adj}} \left( \frac{\Psi^-}{\Psi^\ell} \right)_{z=0} \]

and a symmetric formula holds for \( C^- \). Here, \( P^{\text{adj}} \) denotes the adjugate matrix of a matrix \( P \), i.e. the transposed matrix of minors. As the adjugate is polynomial in the entries of the original matrix, it is evident that, at least away from branch singularities \( \eta_j^\pm(\tilde{\xi}) \), \( |C^\pm| \) is uniformly bounded and therefore

\[(5.121) \quad |M_{jk}^\pm| \leq C_1 |D^{-1}| \leq C_2 \rho^{-\ell} \begin{cases} 1, & \text{U.I.S.} \\ \left[ \min_j \rho^{-1}(|\text{Im} \ (\lambda) - \tau_j(\tilde{\xi})| + \rho) \right]^{-1}, & \text{otherwise} \end{cases}
\]

by Lemma 5.24, where \( C_1, C_2 > 0 \) are uniform constants.

However, the crude bound (5.121) hides considerable cancellation, a fact that will be crucial in our analysis. Again focusing on the curve \((\tilde{\xi}, \lambda)(\tilde{\xi}_0, \lambda_0, \rho)\) defined in (5.43), let us relabel the \( \{ \varphi^\pm \} \) so that, at \( \rho = 0 \),

\[(5.122) \quad \varphi_j^+ \equiv \varphi_j^- = \partial \bar{U}^\delta / \partial \delta_j, \quad j = 1, \ldots, \ell.
\]

(Note: as observed in Chapter 3, the fast decaying modes, among them \( \{ \partial \bar{U}^\delta / \partial \delta_j \} \), can be chosen independent of \((\tilde{\xi}_0, \lambda_0)\) at \( \rho = 0 \).

In the case that \( \tilde{\Delta}(\tilde{\xi}_0, \lambda_0) = 0 \) or equivalently \( (\partial / \partial \rho)^\ell \tilde{D}(\xi_0, \lambda_0, \rho) = 0 \), there is an additional dependency among

\[ (\partial / \partial \rho)\varphi_1^\pm, \ldots, (\partial / \partial \rho)\varphi_\ell^\pm \quad \text{and} \quad \varphi_1^\pm, \ldots, \varphi_n^\pm, \]

so that we can arrange either

\[(5.123) \quad \varphi_{\ell+1}^+ = \varphi_{\ell+1}^- \]

or else (after \( C^1 \) change in coordinates)

\[(5.124) \quad (\partial / \partial \rho)\varphi_1^+ = (\partial / \partial \rho)\varphi_1^-
\]

modulo slow decaying modes. The corresponding functions \( \varphi_{\ell+1}, (\partial / \partial \rho)\varphi_1 \), respectively, have an interpretation in the one-dimensional case as “effective eigenfunctions,” see [ZH], Section 5–6 for further discussion.

By assumption (1.42), in fact only case (5.124) can occur. For, review of the calculation giving \( (\partial / \partial \rho)^\ell \tilde{D}(\tilde{\xi}_0, \lambda, \rho) = \Delta(\tilde{\xi}_0, \lambda) \) reveals that a dependency of form (5.123) implies dependence of rows involving \( r_j^\pm \) in \( \Delta \). With convention (5.124), we have the sharpened bounds:
Lemma 5.25[Z.3]. Under the hypotheses of Theorem 5.5, for $\lambda \in \Gamma^\xi$ and $\rho := |(\tilde{\xi}, \lambda)|$ sufficiently small, there holds
\begin{equation}
|M^\pm_{jk}|, |d^\pm_{jk}| \leq C \gamma_{22,\beta} \tilde{\gamma}_1, \quad \beta = M^\pm_{jk}, d^\pm_{jk},
\end{equation}
where
\begin{equation}
\gamma_{22,M^\pm_{jk} \gamma_{21,\varphi^\pm_{jk} \gamma_{21,\psi^\pm_{jk}}}} \leq \left[1 + \sum_j (|\sigma_j^+| + |\rho|)^{-\frac{1}{2}(1-1/K^+_j)} \right] \left[1 + \sum_j (|\sigma_j^-| + |\rho|)^{-\frac{1}{2}(1-1/K^-_j)} \right],
\end{equation}
\begin{equation}
(\gamma_{22,d_{jk} \gamma_{21,\varphi} \gamma_{21,\psi}})^\pm \leq 1 + \sum_j (|\sigma_j^+| + |\rho|)^{(1-1/K^+_j)},
\end{equation}
with $\gamma_{21,\beta}$ as defined in (5.96)–(5.97), $\sigma^\pm_j := \rho^{-1}(\Im \lambda - \eta^\pm_j(\tilde{\xi}))$, $\eta_j(\cdot)$ and $K^\pm_j$ as in (H6), and, for uniformly inviscid stable (U.I.S) shocks,
\begin{equation}
\tilde{\gamma}_1 := \begin{cases} \rho^{-1} & \text{for } j = 1, \cdots, \ell, \\ 1 & \text{otherwise,} \end{cases}
\end{equation}
while, for weakly inviscid stable (W.I.S) shocks,
\begin{equation}
\tilde{\gamma}_1 := \begin{cases} \left( \min_j |\Im(\lambda) - i\tau_j(\tilde{\xi})| + \rho^2 \right)^{-1} & \text{for } j = 1, \\ \rho^{-1} & \text{for } j = 2, \cdots, \ell, \\ 1 & \text{otherwise,} \end{cases}
\end{equation}
$\tau_j(\cdot)$ as in (5.108). (Here, as above, U.I.S. and W.I.S. denote “uniform inviscid stable” and “weak inviscid stable,” as defined, respectively, in (1.32), Section 1.4, and (1.28), Section 1.3; the classification of Lax, overcompressive, and undercompressive types is given in Section 1.2).

That is, apart from the factor induced by branch singularities, blowup in $M_{jk}$ occurs to order $\rho^{-1}|D^{-1}|$ rather than $|D^{-1}|$, and, more importantly only in fast-decaying modes.

Proof. Formula (5.120) may be rewritten as
\begin{equation}
C^+_{jk} = \det \left( \begin{array}{c} \varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^+, \varphi_j^+, \cdots, \varphi_n^+, \Phi^- \\ \varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_j^-, \varphi_j^+, \cdots, \varphi_n^+, \Phi^- \end{array} \right)_{|z=0},
\end{equation}
from which we easily obtain the desired cancellation in $M^+ = C^+ D^{-1}$. For example, in the U.I.S. case, we obtain for $j > 1$ that, along path $(\tilde{\xi}, \lambda)(\tilde{\xi}_0, \lambda_0, \rho)$, we have, away from branch singularities $\eta_j^\pm(\tilde{\xi})$:
\begin{equation}
C^+_{jk} = \det \left( \begin{array}{c} \varphi_1^+ + \rho \varphi_1^+, \cdots, \varphi_n^+ + \rho \varphi_n^+ \\ \cdots, \cdots, \varphi_n^+ + \rho \varphi_n^+ \end{array} \right) = \mathcal{O}(\rho^\ell) \leq C|D|,
\end{equation}
yielding \(|M_{jk}| = |C_{jk}||D|^{-1} \leq C\) as claimed, by elimination of \(\ell\) zero-order terms, using linear dependency among fast modes at \(\rho = 0\). Note, similarly as in the proof of Lemma 5.24, that we require in this calculation only that fast modes be \(C^1\) in \(\rho\), while slow modes may be only continuous. For this situation, \(\gamma_{22} \sim 1\) suffices.

In the vicinity of one or more branch singularities, we must also take account of blowup in associated slow modes, as quantified by the factors \(\gamma_{21,\beta}\) in Lemma 5.22. Branch singularities not involving \(\phi_j^+\) or \(\psi_k^-\) clearly do not affect the calculation, since we have chosen a normalization for which complete cycles of branching (super-slow) modes \(\phi_j^+\) have wedge products of order one; thus, \(\gamma_{22} \sim 1\) again suffices for this case. In the vicinity of a branch singularity involving either or both of the modes \(\phi_j^+\) or \(\psi_k^-\), however, there arises an additional blowup that must be accounted for.

Since this blowup involves only slow modes, it enters \((5.131)\) simply as an additional multiplicative factor corresponding to the modulus of their wedge products at \(z = 0\); as in the proof of Lemma 5.22, these can be estimated by the wedge products of the corresponding modes for the limiting, constant coefficient equations at \(x_1 = \pm \infty\), which at \(z = 0\) are exactly the vectors \(v_j, v_k^j\) described in Lemma 5.14. Since complete cycles have order one wedge product, as noted above, we need only compute the blowup in branch the cycles directly involving \(\phi_j^+ / \psi_k^-\), since these are modified in expression \((5.131)\), respectively by deleting one decaying mode, \(\phi_j^+\), and augmenting one dual, growing mode, \(\psi_k^-\).

Using the result of Remark 5.19, and the scaling \(\gamma_{21,\beta}\) given in Lemma 5.22 for the modulus of individual modes, we find that the deleted cycle has wedge product of order

\[
\pi^{(\tilde{r}-1)(\tilde{r}-2)/2} \pi^{-(\tilde{r}-1)(\tilde{r}-1)/2} = \pi^{-(\tilde{r}-1)/2} = \gamma_{21,\phi_j^+},
\]

where \(\tilde{r}\) denotes the number of decaying modes in the original cycle: \(\tilde{r} = r\) for \(s := K^+_h = 2r\) or \(s = 2r + 1\) and \(p > 0\), otherwise \(\tilde{r} = r + 1\), and \(\pi := (|\sigma| + |\rho|)^{1/s}\) as in the notation of Lemma 5.22. Here, \(s := K^+_h\) is the multiplicity of the nearby branch singularity \(\eta^+_{h}\) associated with mode \(\phi_j^+\), and \(\gamma = 21, \phi_j^+\) is as defined in \((5.97)\).

Similarly, the augmented cycle involving \(\psi_k^-\) yields a wedge product of order

\[
\pi^{(\tilde{r}+1)(\tilde{r})/2} \pi^{-(\tilde{r})(\tilde{r}-1)/2} \pi^{-(2s-\tilde{r}-2)/2} = \pi^{-(3\tilde{r}-2s+1)/2} = \gamma_{21,\psi_k^-},
\]

where \(\gamma_{21,\psi_k^-}\) is as defined in \((5.96)-(5.97)\). The asserted bound on \(\gamma_{22}\) then follows by the observation (see \((5.102)\)) that

\[
t_{\phi_j^+}, t_{\psi_k^-} \leq (1/4)(1 - 1/K^+_h)
\]

in \((5.97)\) \((K^+_h := s =: 2r\) in the notation of formulae \((5.97)\)). This completes the estimation of \(|M_{jk}|\) in the case \(x_1 \geq y_1\); the case \(x_1 \leq y_1\) may be treated by a symmetric argument.
The bounds on $|d_{jk}^\pm|$ follow similarly, using (2.100) in place of (2.99). In these cases, however, there is the new possibility that the augmented mode $\psi_k$ and the deleted mode $\varphi_j$ may belong to the same cycle, since both are associated with the same spatial infinity. In this case, the total number of modes remains fixed, with a growth mode replacing a decay mode, and the resulting blowup in the associated wedge product is simply

$$(5.134) \quad (\gamma_{21,\psi_j}/\gamma_{21,\varphi_j})^\pm.$$  

The asserted bound on $(\gamma_{22}\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k})^\pm$ then follows from relation

$$\begin{aligned}
(t_{\psi_j} - t_{\varphi_j} + t_{\varphi_j} + t_{\tilde{\psi}_k})^\pm &= (t_{\psi_j} - t_{\varphi_j} + t_{\varphi_j} + t_{\tilde{\psi}_j})^\pm \\
&= 1 - 1/K_h^\pm,
\end{aligned}$$

again obtained by inspection of formulae (5.97) (see (5.102)), where $K_h^\pm$ as usual denotes the order of the associated branch singularity. If on the other hand the augmented and deleted modes belong to different cycles, then the computation reduces essentially to that of the previous case, yielding

$$\gamma_{22} = (\gamma_{21,\varphi_j}\gamma_{21,\tilde{\psi}_k})^\pm,$$

from which the result follows as before (indeed, we obtain a slightly better bound, see (5.102)). \qed

**Remark 5.26.** Note that frequency blowup in term $M_{jk}\varphi_j^+\tilde{\psi}_k^-$ splits evenly between coefficient $M_{jk}$ and modes $\varphi_j^+\tilde{\psi}_k^-$, see (5.132)–(5.133).

**Lax and Overcompressive Case.** For Lax and overcompressive shocks, we can say a bit more. First, observe that duality relation

$$(5.135) \quad \det \left( \begin{array}{c} \tilde{\Psi}^- \\ \tilde{\Psi}' \\ \tilde{\Phi}^- \\ \tilde{\Phi}' \end{array} \right)^* S \left( \begin{array}{c} \Phi^+ \\ \Phi'^+ \\ \Phi^- \\ \Phi'^- \end{array} \right) = \det \left( \begin{array}{c} \tilde{\Psi}^- \\ \tilde{\Psi}' \\ \Phi^+ \\ \Phi'^+ \end{array} \right) S \left( \begin{array}{c} \Phi^+ \\ \Phi'^+ \\ \Phi^- \\ \Phi'^- \end{array} \right) \begin{array}{c} 0 \\ I \end{array} \right)$$

yields that $D(\tilde{\xi},\lambda)$ and the dual Evans function

$$(5.136) \quad \tilde{D} := \det \left( \begin{array}{c} \tilde{\Psi}^- \\ \tilde{\Psi}' \\ \tilde{\Phi}^- \\ \tilde{\Phi}' \end{array} \right)|_{x_1=0}.$$
vanish to the same order as \((\tilde{\xi}, \lambda) \to (0, 0)\), since the determinants of \(\begin{pmatrix} \tilde{\Psi}^- & \tilde{\Phi}^- \\ \tilde{\Psi}^- & \tilde{\Phi}^- \end{pmatrix} \), \(S, \) and \(\begin{pmatrix} \Phi^+ & \Psi^+ \\ \Phi^+ & \Psi^+ \end{pmatrix} \) are by construction bounded above and below. Indeed, subspaces \(\ker \begin{pmatrix} \Phi^+ & \Phi^- \\ \Phi^+ & \Phi^- \end{pmatrix} \) and \(\ker \begin{pmatrix} \tilde{\Phi}^+ & \tilde{\Phi}^- \\ \tilde{\Phi}^+ & \tilde{\Phi}^- \end{pmatrix} \) can be seen to have the same dimension, \(\dim \ker \begin{pmatrix} \tilde{\Psi}^- & \tilde{\Phi}^- \\ \tilde{\Psi}^- & \tilde{\Phi}^- \end{pmatrix}^* S \begin{pmatrix} \Phi^+ \\ \Phi^+ \end{pmatrix} \), which at \(\rho = 0\) is equal to \(\ell\). This computation partially recovers the results of [ZH], section 6, without the assumption of smoothness in \(\{\phi_j^\pm\}, \{\psi_j^\pm\}\). The “contracted” Evans function

\[
\det \begin{pmatrix} \tilde{\Psi}^- & \tilde{\Phi}^- \\ \tilde{\Psi}^- & \tilde{\Phi}^- \end{pmatrix}^* S \begin{pmatrix} \Phi^+ \\ \Phi^+ \end{pmatrix}
\]

is discussed further in [BSZ.1].

It follows that there is an \(\ell\)-fold intersection between

\[
\text{Span} \left( \begin{pmatrix} \tilde{\Psi}^-_j \\ \tilde{\Phi}^-_j \end{pmatrix} \right) \quad \text{and} \quad \text{Span} \left( \begin{pmatrix} \tilde{\Psi}^+_j \\ \tilde{\Phi}^+_j \end{pmatrix} \right)
\]

We now recall the important observation of Lemma 4.28. Recall, Section 4.5.2, that we have fixed a choice of bases such that, at \(\rho = 0\), both slow-decaying dual modes \(\tilde{\phi}_j^\pm\) and slow-growing dual modes \(\tilde{\psi}_j^\pm\) are identically constant, or equivalently that not only fast-decaying modes \(\phi_j^\pm\), but also fast-growing modes \(\psi_j^\pm\) are solutions of the linearized traveling wave ODE (5.104). With this choice of basis, we have for Lax and overcompressive shocks that that the only bounded solutions of the adjoint eigenvalue equation at \(\rho = 0\) are constant solutions, or, equivalently, fast-decaying modes \(\psi_j\) at one infinity are fast-growing at the other, and fast-decaying \(\tilde{\psi}_j\) at one infinity are fast-growing at the other. Combining the above two results, we have:

**Lemma 5.27.** Under the hypotheses of Theorem 5.5, with \(|(\tilde{\xi}, \lambda)| \) sufficiently small, for Lax and overcompressive shocks, with appropriate basis at \(\rho = 0\) (i.e. slow dual modes taken identically constant), there holds

\[
|M_{jk}|, |d_{jk}| \leq C\gamma_{22}
\]

if \(\tilde{\psi}_k\) is a fast mode, and

\[
|M_{jk}|, |d_{jk}| \leq C\gamma_{22}\rho
\]

if, additionally, \(\phi_j\) is a slow mode, where \(\gamma_{22}\) is as defined in (5.127)–(5.126).

**Proof.** Transversality, \(\gamma \neq 0\), follows from (D2), so that the results of Lemma 4.28 hold. Consider first, the more difficult case of uniform inviscid stability, \(\Delta(\xi_0, \lambda) \neq 0\).
We first establish the bound (5.137). By Lemma 5.25, we need only consider
\( j = 1, \ldots, \ell \), for which \( \phi_j = \phi_j^+ \). By Lemma 4.28, all fast-growing modes \( \psi_k^+ \), \( \psi_k^- \)
lie in the fast-decaying manifolds \( \text{Span}(\varphi_1^+, \ldots, \varphi_{\ell}^-) \), \( \text{Span}(\varphi_1^+, \ldots, \varphi_{\ell}^+) \) at \( -\infty \), \( +\infty \) respectively. It follows that in the right hand side of (5.130), there is at \( \rho = 0 \)
a linear dependency between columns

\[
\varphi_1^+, \ldots, \varphi_{j-1}^+, \psi_k^+, \varphi_j^+, \varphi_{\ell}^+ \text{ and } \varphi_j^- = \varphi_j^+
\]

i.e. an \( \ell \)-fold dependency among columns

\[
\varphi_1^+, \ldots, \psi_k^-, \ldots, \varphi_{\ell}^+ \text{ and } \varphi_1^-, \ldots, \varphi_{\ell}^-
\]

all of which, as fast modes, are \( C^1 \) in \( \rho \). It follows as in the proof of Lemma 5.25
that \( |C_{jk}| \leq C\gamma_{22}\rho^\ell \) for \( \rho \) near zero, giving bound (5.137) for \( M_{jk} \). If \( \varphi_j^+ \)
is a slow mode, on the other hand, then the same argument shows that there is a linear
dependency in columns

\[
\varphi_1^+, \ldots, \psi_k^-, \ldots, \varphi_{\ell}^+ \text{ and } \varphi_1^-, \ldots, \varphi_{\ell}^-
\]

and an \((\ell + 1)\)-fold dependency in (5.140), since the omitted slow mode \( \varphi_j^+ \) plays
no role in either linear dependence; thus, we obtain the bound (5.138), instead.

Analogous calculations yield the result for \( d_{jk}^\pm \) as well.

Proof of Proposition 5.10. The proof of Proposition 5.10 is now straightforward. Collecting the results of Lemmas 5.22–5.23, we have the spatial decay bounds

\[
|\varphi_j(x_1)| \leq C\gamma_{21, \varphi_j} e^{-\theta|x_1|},
\]

\[
|\varphi_j'(x_1)| \leq C\gamma_{21, \varphi_j} e^{-\theta|x_1|},
\]

for fast forward modes,

\[
|\varphi_j(x_1)| \leq C\gamma_{21, \varphi_j} e^{-\theta_0^2|x_1|},
\]

\[
|\varphi_j'(x_1)| \leq C\gamma_{21, \varphi_j} (\rho e^{-\theta_0^2|x_1|} + e^{-\theta|x_1|}),
\]
for slow forward modes
\begin{equation}
|\tilde{\psi}_k(y_1)| \leq C\gamma_{21,\tilde{\psi}_k} e^{-\theta|y_1|}, \\
|\tilde{\psi}_k'(y_1)| \leq C\gamma_{21,\tilde{\psi}_k} e^{-\theta|y_1|},
\end{equation}
\label{5.144}

for fast dual modes, and
\begin{equation}
|\tilde{\psi}_k(y_1)| \leq C\gamma_{21,\tilde{\psi}_k} e^{-\rho^2\theta|y_1|}, \\
|\tilde{\psi}_k'(y_1)| \leq C\gamma_{21,\tilde{\psi}_k} (\rho e^{-\theta\rho^2|y_1|} + \alpha e^{-\theta|y_1|}),
\end{equation}
\label{5.145}

for slow dual modes, where \(\gamma_{21,\beta}\) are as defined in (5.96)–(5.97) and
\[\alpha = \begin{cases} 
0 & \text{for Lax and overcompressive shocks}, \\
1 & \text{for undercompressive shocks}.
\end{cases}\]

Likewise, the results of Lemmas 5.25 and 5.27 give the frequency growth bounds
\begin{equation}
|M_{jk}^\pm|, |d_{jk}^\pm| \leq C\gamma_{22}
\end{equation}
\label{5.146}

for slow modes \(\varphi_j^\pm\), and
\begin{equation}
|M_{jk}^\pm|, |d_{jk}^\pm| \leq C\rho^{-1}\gamma_{22}\gamma_1
\end{equation}
\label{5.147}

for fast modes \(\varphi_j^\pm\), where \(\gamma_1\) is as defined in (5.37) and \(\gamma_{22}\) as in (5.127)–(5.126).

In the critical case that \(\varphi_j^\pm\) is slow and \(\tilde{\psi}_k^\pm\) is fast, (5.146) can be sharpened to
\begin{equation}
|M_{jk}^\pm|, |d_{jk}^\pm| \leq C\gamma_{22}(\rho + \alpha),
\end{equation}
\label{5.148}

\(\alpha\) as above.

Combining these bounds, we may estimate the various terms arising in the decompositions of Corollary 2.26 as:
\[|\phi_j M_{jk} \tilde{\psi}_k|, |\phi_j d_{jk} \tilde{\psi}_k| \leq C\gamma_{21,\varphi_j} \gamma_{21,\tilde{\psi}_k} \gamma_{22,\gamma_1} \rho^{-1} e^{-\theta|x_1|} e^{-\theta\rho^2|y_1|}\]
and
\[|(\partial/\partial y_1) \phi_j M_{jk} \tilde{\psi}_k|, |(\partial/\partial y_1) \phi_j d_{jk} \tilde{\psi}_k| \leq C\gamma_{21,\varphi_j} \gamma_{21,\tilde{\psi}_k} \gamma_{22,\gamma_1} \rho^{-1} e^{-\theta|x_1|} (\rho e^{-\theta\rho^2|y_1|} + \alpha e^{-\theta|y_1|})\]

for fast modes \(\varphi_j\):
\[|\phi_j M_{jk} \tilde{\psi}_k|, |\phi_j d_{jk} \tilde{\psi}_k| \leq C\gamma_{21,\varphi_j} \gamma_{21,\tilde{\psi}_k} \gamma_{22} e^{-\theta\rho^2|x_1|} e^{-\theta\rho^2|y_1|}\]
and

\[ |(\partial / \partial y_1) \phi_j M_{jk} \tilde{\psi}_k|, |(\partial / \partial y_1) \phi_j d_{jk} \tilde{\psi}_k| \leq C_{21, \varphi_j} \gamma_{21, \tilde{\psi}_k}^{22}e^{-\theta \rho^2|x_1|}(\rho e^{-\theta \rho^2|y_1|} + \alpha e^{-\theta |y_1|}) \]

for slow modes \( \varphi_j \);

\[ |\varphi_j \tilde{\varphi}_j| \pm \leq C(\gamma_{21, \varphi_j} \gamma_{21, \tilde{\varphi}_j}) e^{-\theta |x_1 - y_1|} \]

and

\[ |(\partial / \partial y_1) \varphi_j \tilde{\varphi}_j| \pm \leq C(\gamma_{21, \varphi_j} \gamma_{21, \tilde{\varphi}_j}) e^{-\theta |x_1 - y_1|} \]

for fast modes \( \varphi_j \);

\[ |\varphi_j \tilde{\varphi}_j| \pm \leq C(\gamma_{21, \varphi_j} \gamma_{21, \tilde{\varphi}_j}) e^{-\theta \rho^2|x_1 - y_1|} \]

and

\[ |(\partial / \partial y_1) \varphi_j \tilde{\varphi}_j| \pm \leq C(\gamma_{21, \varphi_j} \gamma_{21, \tilde{\varphi}_j}) \rho e^{-\theta \rho^2|x_1 - y_1|} \]

for slow modes \( \varphi_j \);

\[ |\psi_j \tilde{\psi}_j| \pm \leq C(\gamma_{21, \psi_j} \gamma_{21, \tilde{\psi}_j}) e^{-\theta |x_1 - y_1|} \]

and

\[ |(\partial / \partial y_1) \psi_j \tilde{\psi}_j| \pm \leq C(\gamma_{21, \psi_j} \gamma_{21, \tilde{\psi}_j}) e^{-\theta |x_1 - y_1|} \]

for fast modes \( \psi_j \); and

\[ |\psi_j \tilde{\psi}_j| \pm \leq C(\gamma_{21, \psi_j} \gamma_{21, \tilde{\psi}_j}) e^{-\theta \rho^2|x_1 - y_1|} \]

and

\[ |(\partial / \partial y_1) \psi_j \tilde{\psi}_j| \pm \leq C(\gamma_{21, \psi_j} \gamma_{21, \tilde{\psi}_j}) \rho e^{-\theta \rho^2|x_1 - y_1|} \]

for slow modes \( \psi_j \). Using bounds (5.127)–(5.126), and recalling (see (5.102)) that

\[ (t_{\varphi_j} + t_{\tilde{\varphi}_j}) \pm, (t_{\psi_j} + t_{\tilde{\psi}_j}) \pm \equiv 1 - 1/K_j^\pm, \]

we thus obtain the result by checking term by term. \( \square \)
5.3.2. High-frequency bounds.

Modifying the auxiliary energy estimate and one-dimensional resolvent estimates already derived, we obtain in straightforward fashion the following high-frequency resolvent bounds.

**Proposition 5.28.** Under the hypotheses of Theorem 5.5, for $|\tilde{\xi}|$ sufficiently large, and some $C$, $\theta > 0$,

$$e^{\theta t}f|_{\tilde{H}^1(x_1, \tilde{\xi})} \leq Ce^{-\theta t}|f|_{\tilde{H}^1(x_1, \tilde{\xi})}$$

for $|f|_{\tilde{H}^1(x_1, \tilde{\xi})} := (1 + |\tilde{\xi}|)|f|_{\tilde{H}^1(x_1)}$.

**Proof.** Carrying out a Fourier-transformed version of the auxiliary energy estimate of Propositions 4.15 and 5.9 in the simpler, linearized setting, we obtain

$$(d/dt)\mathcal{E}(W(t)) \leq -\theta \mathcal{E}(t) + C|W|^{2}_{L^2(x_1)},$$

$\theta > 0$, where

$$\mathcal{E}(W) := M(1 + |\tilde{\xi}|^2)(W, \alpha \tilde{A}^0 W) + \langle \alpha K(\partial x_1, i\tilde{\xi})W, W \rangle + M\langle \partial x_1 W, \alpha \hat{A}^0 \partial x_1 W \rangle,$$

$M, C > 0$ sufficiently large constants, $K(\partial x)$ a first-order differential operator as defined in (5.19)–(5.20), or, more generally, an analogous first-order pseudodifferential operator as discussed in the proof of Proposition 5.9 (see also [Z.4]). Observing that $|W|^{2}_{L^2} \leq C|\tilde{\xi}|^{-2}\mathcal{E}(W)$ can be absorbed in $\theta\mathcal{E}(W)$, we obtain

$$(d/dt)\mathcal{E}(W(t)) \leq - (\theta/2)\mathcal{E}(t),$$

from which the result follows by the fact that $\mathcal{E}(W)^{1/2}$ is equivalent to the $\tilde{H}^1(x_1, \tilde{\xi})$ norm of $W$. \qed

**Proposition 5.29.** Under the hypotheses of Theorem 5.5, for $|\tilde{\xi}| \leq R$ and $|\lambda|$ sufficiently large, $\text{Re } \lambda \geq -\eta$ is contained in the resolvent set $\rho(\tilde{L}_\xi)$ for some uniform $\eta > 0$ sufficiently small, with resolvent kernel $G_{\tilde{\xi}, \lambda}(x, y)$ satisfying the same decomposition (4.175)–(4.187) as in the one-dimensional case, but with dissipation coefficients $\eta_j^s$ replaced by multi-dimensional versions $\eta_j(\tilde{\xi}, x)^*$, exponentially decaying to asymptotic values

$$\text{Re } \sigma_j(\tilde{\xi}, \pm \infty)^* > 0.$$  

**Proof.** For $|\tilde{\xi}|$ bounded and $|\lambda|$ sufficiently large, we may treat $\tilde{\xi}$ terms as first-order perturbations in the one-dimensional analysis in the proof of Proposition 4.33, order $|\lambda|^{-1}$ after the rescaling of (4.189)–(4.193). These affect only the bounds on the hyperbolic block, modifying the form of $\eta_j^s$ in (4.202). We do not require the precise dependence of the resulting $\eta_j(\tilde{\xi}, x)^*$ on $\tilde{\xi}$, but only the bound (5.151), which follows as in the one-dimensional case by (2.55) together with the fact that $-\text{Re } \sigma_j(\tilde{\xi}, \pm \infty)$ corresponds to the zero-order term in the expansion of hyperbolic modes of symbol $-A\tilde{\xi} - B\tilde{\xi}$. \qed
5.3.3 Bounds on the solution operator.

We are now ready to establish the claimed bounds on the linearized solution operator. Define “low-” and “high-frequency” parts of the linearized solution operator \( S(t) \)

\[
S_1(t) := \frac{1}{(2\pi)^d} \int_{|\xi| \leq r} \oint_{\Gamma \tilde{\xi}} e^{\lambda t + i \tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi}
\]

and

\[
S_2(t) := e^{Lt} - S_1(t),
\]

\( \Gamma \tilde{\xi} \) as in (5.36) and \( r > 0 \) sufficiently small.

**Proof of Proposition 5.7.** We estimate \( S_1 \) and \( S_2 \) in turn, for simplicity restricting to the case of a uniformly stable Lax or overcompressive shock, \( \gamma_1 = 1 \).

**\( S_1 \) bounds.** Let \( \hat{u}(x_1, \tilde{\xi}, \lambda) \) denote the solution of \( (L_{\tilde{\xi}} - \lambda) \hat{u} = \hat{f} \), where \( \hat{f}(x_1, \tilde{\xi}) \) denotes Fourier transform of \( f \), and

\[
u(x, t) := S_1(t)f = \frac{1}{(2\pi)^d} \int_{|\xi| \leq r} \oint_{\Gamma \tilde{\xi}} e^{\lambda t + i \tilde{\xi} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda d\tilde{\xi}.
\]

Bounding \( |\hat{f}|_{L^\infty(\tilde{\xi}, L^1(x_1))} \leq |f|_{L^1(x_1, \tilde{\xi})} = |f|_1 \) using Hausdorff-Young’s inequality, and appealing to the \( L^1 \to L^p \) resolvent estimates of Corollary 5.11, we may thus bound

\[
|\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^p(x_1)} \leq |f|_1 b(\tilde{\xi}, \lambda),
\]

where \( b := C \gamma_2 \rho^{-1} \).

**\( L^2 \) bounds.** Using in turn Parseval’s identity, Fubini’s Theorem, the triangle inequality, and our \( L^1 \to L^2 \) resolvent bounds, we may estimate

\[
|\nu|_{L^2(x_1, \tilde{\xi})}(t) = \frac{1}{(2\pi)^d} \int_{x_1} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\tilde{\lambda} \in \Gamma(\tilde{\xi})} e^{\lambda t + i \tilde{\lambda} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda \right| d\tilde{\xi} dx_1 \right)^{1/2} \\
= \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\tilde{\lambda} \in \Gamma(\tilde{\xi})} e^{\lambda t + i \tilde{\lambda} \cdot \tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda \right| d\tilde{\xi} \right)^{1/2} \\
\leq \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\tilde{\lambda} \in \Gamma(\tilde{\xi})} |e^{\lambda t + i \tilde{\lambda} \cdot \tilde{x}}| \hat{f}(x_1, \tilde{\xi}) d\lambda \right| d\tilde{\xi} \right)^{1/2} \\
\leq |f|_1 \left( \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\tilde{\lambda} \in \Gamma(\tilde{\xi})} e^{\lambda t \tilde{\lambda} \cdot \tilde{x}} b(\tilde{\xi}, \lambda) d\lambda \right| d\tilde{\xi} \right)^{1/2} ,
\]

from which we readily obtain the claimed bound on \( |S_1(t)f|_{L^2(x)} \) using the bounds on \( b \). Specifically, parametrizing \( \Gamma(\tilde{\xi}) \) by

\[
\lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R},
\]
and observing that in nonpolar coordinates
\[ \rho^{-1} \gamma_2 \leq \left( |k| + |\tilde{\xi}| \right)^{-1} \left( 1 + \sum_{j \geq 1} \left( \frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{\frac{1}{\epsilon} - 1} \right) \]
(5.155)
\[ \leq \left( |k| + |\tilde{\xi}| \right)^{-1} \left( 1 + \sum_{j \geq 1} \left( \frac{|k - \tau_j(\tilde{\xi})|}{\rho} \right)^{\epsilon - 1} \right), \]
where \( \epsilon := \frac{1}{\max_j s_j} \) (0 < \( \epsilon \) < 1 chosen arbitrarily if there are no singularities), we obtain a contribution bounded by
(5.156)
\[ C|f| \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \int_{-\infty}^{+\infty} e^{-\theta(k^2 + |\tilde{\xi}|^2)t} (\rho)^{-1} \gamma_2 dk \right|^2 d\tilde{\xi} \right)^{1/2} \]
\[ \leq C|f|_1 \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left( e^{-2\theta|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \right) \left| \int_{-\infty}^{+\infty} e^{-\theta|k|^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon - 1} dk \right|^2 d\tilde{\xi} \right)^{1/2} \]
\[ + C \sum_{j \geq 1} |f|_1 \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left( e^{-2\theta|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \right) \left| \int_{-\infty}^{+\infty} e^{-\theta|k|^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon - 1} dk \right|^2 d\tilde{\xi} \right)^{1/2} \]
\[ \leq C|f|_1 \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left( e^{-2\theta|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-2\epsilon} \right) \left| \int_{-\infty}^{+\infty} e^{-\theta|k|^2 t} |k - \tau_j(\tilde{\xi})|^{\epsilon - 1} dk \right|^2 d\tilde{\xi} \right)^{1/2} \]
\[ \leq C|f|_1 t^{-(d-1)/4} \]
as claimed. Derivative bounds follow similarly.

\( L^\infty \) bounds. Similarly, using Hausdorff–Young’s inequality, we may estimate
\[ |u|_{L^\infty(x_1,\tilde{\xi})}(t) \leq \sup_{x_1} \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\lambda \in \Gamma(\tilde{\xi})} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \right| d\tilde{\xi} \]
(5.157)
\[ \leq \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\lambda \in \Gamma(\tilde{\xi})} |e^{\lambda t}| |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^\infty(x_1)} d\lambda d\tilde{\xi} \]
\[ \leq |f|_1 \frac{1}{(2\pi)^d} \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \oint_{\lambda \in \Gamma(\tilde{\xi})} e^{\Re \lambda t} b(\tilde{\xi}, \lambda) d\lambda d\tilde{\xi} \right| \]
to obtain the claimed bound on \(|S_1(t) f|_{L^\infty}\). Parametrizing \( \Gamma(\tilde{\xi}) \) again by
\[ \lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R}, \]
we obtain a contribution bounded by
(5.158)
\[ C|f|_1 \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} \left| \int_{-\infty}^{+\infty} e^{-\theta(k^2 + |\tilde{\xi}|^2)t} (\rho)^{-1} \gamma_2 dkd\tilde{\xi} \right|^2 \right)^{1/2} \]
\[ \leq C|f|_1 \left( \int_{\tilde{\xi} \in \mathbb{R}^{d-1}} e^{-\theta|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-\epsilon} \left| \int_{-\infty}^{+\infty} e^{-\theta|k|^2 t} |k|^{\epsilon - 1} dk \right|^2 d\tilde{\xi} \right)^{1/2} \]
\[ \leq C|f|_1 t^{-(d-1)/2} \]
as claimed. Derivative bounds follow similarly.

**General** $2 \leq p \leq \infty$. Finally, the general case follows by interpolation between $L^2$ and $L^\infty$ norms.

**$S_2$ bounds.** Using

$$S(t) = \frac{1}{(2\pi)^{d-1}} \int e^{\xi \cdot \tilde{x}} e^{L\xi^t} d\tilde{\xi},$$

we may split $S_2(t)$ into two parts,

$$S_2^I(t) := \frac{1}{(2\pi)^{d-1}} \int_{|\xi| \geq R} e^{\xi \cdot \tilde{x}} e^{L\xi^t} d\tilde{\xi},$$

$R > 0$ sufficiently large, and

$$S_2^{II}(t) := \frac{1}{(2\pi)^{d-1}} \int_{|\xi| \leq R} e^{\xi \cdot \tilde{x}} e^{L\xi^t} d\tilde{\xi} - S_1(t)$$

$$= \frac{1}{(2\pi)^{d-1}} \int_{|\xi| \leq R} \text{P.V.} \int_{\gamma_0 + i\infty}^{\gamma_0 - i\infty} e^{i\xi \cdot \tilde{x} + \lambda t} (L_\xi - \lambda)^{-1} d\lambda d\tilde{\xi}$$

$$- \frac{1}{(2\pi)^{d-1}} \int_{|\xi| \leq R} \int_{1}^{\infty} e^{i\xi \cdot \tilde{x} + \lambda t} (L_\xi - \lambda)^{-1} d\lambda d\tilde{\xi}.$$

The first part, $S_2^I(t)$ satisfies the claimed bounds by Parseval’s identity together with (5.149), together with the observation that $\partial_\tilde{x}$ commutes with $L_\xi$ and thus all $S_j^k$, by invariance of coefficients with respect to $\tilde{x}$. The second part, $S_2^{II}(t)$, satisfies the claimed bounds by Parseval’s identity together with calculations like those of the corresponding one-dimensional high- and intermediate-frequency estimates in the proof of Proposition 4.39 and the $H^1(x_1) \rightarrow H^1(x_1)$ estimates of Lemma 4.14, together with the same observation that $\partial_\tilde{x}$ commutes all $S_j^k$.

This completes the proof, and the article. □

**APPENDICES A. AUXILIARY CALCULATIONS.**

**A.1. Applications to example systems.**

In this appendix, we present simple conditions for (A1)–(A3) and (H1)–(H2), and use them to verify the hypotheses for various interesting examples from gas dynamics and MHD. Consider a system (1.1) satisfying conditions (1.5) and (1.6). (Note: (1.6) is implied by (A3) under the class of transformations considered, since $\text{Re} \overline{\partial_\xi} > 0 \Rightarrow \text{Re} \sigma(A^0)^{-1} \overline{b} = (\partial W/\partial U)b(\partial U/\partial W) > 0$.)

**Definition 6.1.** Following [Kaw], we define functions $\eta(G), q^i(G) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ to be a convex entropy, entropy flux ensemble for (1.1), (1.6) if: (i) $d^2 \eta > 0$ ($\eta$ convex), (ii) $dq_i^j = dq^j_i$, and (iii) $d^2 \eta B^{jk}$ is symmetric (hence positive semidefinite, by (1.6)).
Proposition 6.2 [KSh]. A necessary and sufficient condition that a system (1.1), (1.6) can be put in symmetric form (1.7) by a change of coordinates $U \rightarrow W(U)$ for $U$ in a convex set $\mathcal{U}$ is existence of a convex entropy, entropy flux ensemble $\eta$, $q^j$ defined on $U$, with $W = d\eta(U)$ (known as an “entropy variable”).

Proof. ($\Leftrightarrow$) The change of coordinates $U \rightarrow W := d\eta(U)$ is invertible on any convex domain, since its Jacobian $d^2\eta$ is symmetric positive definite, by Definition 6.1 (i), and thus we can rewrite (1.1) as (1.7), as claimed, where $\tilde{F}^j(W) := F^j(U(W))$ and $\tilde{B}^{jk}(W) := B^{jk}(U(W)) \partial U/\partial W = B^{jk} d^2\eta^{-1}$. Symmetry of $d\tilde{F}^j = dF^j d^2\eta^{-1}$ and $\tilde{B}^{jk} = B^{jk} d^2\eta^{-1}$ follow from symmetry of $d^2\eta dF^j$ and $d^2\eta B^{jk}$, the first a well-known consequence of Definition 6.1 (ii) [G, Bo, Se.3], the second just Definition 6.1 (iii). Finally, block structure and uniform ellipticity of $\tilde{B}^{jk}$ follow by symmetry of $B^{jk}$, the fact that the $w^l$ rows must vanish, by the corresponding property of left factor $B^{jk}$, and the fact that $\Re \sigma \sum_{j,k} b^{jk} \xi_j \xi_k \geq \theta |\xi|^2$, since $B^{jk}$ is similar to $B^{jk}$.

($\Rightarrow$) Symmetry of $\tilde{A}^0 = dW(U)$ implies that $d\eta(U) := W(u)$ is exact on any simply connected domain, while positive definiteness implies that $\eta$ is convex. Likewise, symmetry of $d(d\eta dF^j)$ (following by symmetry of $d^2\eta dF^j$, a consequence of symmetry of $dF^j d^2\eta^{-1}$, plus the reverse calculation to that cited above) implies that $dq^j(U) := d\eta dF^j$ is exact. Finally, symmetry of $d^2\eta B^{jk}$ follows by symmetry of $B^{jk} d^2\eta^{-1}$. □

Proposition 6.3 [GMWZ.4]. Necessary and sufficient conditions that there exist coordinate change $W(U)$ and matrix multiplier $S(W)$ taking (1.1) to form (1.11) satisfying (A1)–(A3) are: (i) there exist symmetric, positive definite (right) symmetrizers $Q(U)$ such that $(A^j Q)_{\pm}$ and $(A^j Q)_{11}$ are symmetric, and $B^{jk}Q = \text{block-diag} \{0, \beta^{jk}\}$ with $\beta^{jk}$ satisfying uniform ellipticity condition (1.6), and (ii) there exists $w^{II} : U \rightarrow \mathbb{R}^r$ such that $B^{jk}(U)U_{x_k} = (0, b^{jk}(U)w^{II}(U)_{x_k})$.

If there exist such transformations, then $S$ must be lower block-triangular and $B^{jk}(\partial U/\partial W)$ block-diagonal, with $S(U) = (\partial U/\partial W)^{tr} Q^{-1}$, and

$$W(U) := (w^l(U), w^{II}(U))$$

for $w^l$ invertible and and $w^{II}$ satisfying (ii). Moreover, we may always choose $W$ with $(\partial W/\partial U)$ lower block-triangular, in which case $\tilde{A}^0 = S(\partial U/\partial W)$ is block-diagonal as well. (Indeed, this structure follows already from block structure of $B^{jk}$ and positive symmetric definiteness of $\tilde{A}^0$ alone, without any further considerations; conversely, $W(U) := (w^l(U), w^{II}(U))$ and $S = \tilde{A}^0(\partial U/\partial W)$ for symmetric positive definite block-diagonal $\tilde{A}^0$ is sufficient to imply block structure of $B^{jk}$.)

Proof. ($\Rightarrow$) Define $W(U) = (w^l, w^{II})(U)$, where $w^l$, $w^{II}$ are as described in the hypotheses. Then, $\partial W/\partial U$ is lower block triangular, of form

$$
\begin{pmatrix}
\frac{dw^l}{\partial w^{II}/\partial u^l} & 0 \\
\frac{\partial w^{II}/\partial u^l}{\partial w^{II}/\partial u^{II}}
\end{pmatrix},
$$
with $B_{22}^{jk} = \hat{b}^{jk}(\partial w^I/\partial u^I)$ by assumption (ii), hence (taking, e.g., $j = k = 1$) is invertible, by assumption (1.6). Moreover, $B^{jk} := B^{jk}(\partial U/\partial W)$ by assumption (ii) is block-diagonal of form block-diag $\{0, \hat{b}^{jk}\}$. Thus, choosing a lower block-triangular multiplier

$$S(W) := P(W)(\partial W/\partial U), \quad P = \text{block-diag}\{P_1, P_2\},$$

for any symmetric positive definite $P_j$, we obtain form (1.11), with $\hat{A}^0 = P$ symmetric positive definite and $\tilde{B}^{jk} = \text{block-diag}\{0, \tilde{b}^{jk}\}$ in the required block-diagonal form. Indeed, only multipliers of this form will suffice, since $(\partial W/\partial U)B^{jk}(\partial U/\partial W)$ is also of block-diagonal form, by lower triangularity of $(\partial W/\partial U)$, so that $P$ must be lower triangular as well to preserve this property, but also $P = \hat{A}^0$ is required to be symmetric. Writing the block-diagonal $\tilde{B}^{jk}$ as $B^{jk}Q\left(Q^{-1}(\partial U/\partial W)\right)$, where $B^{jk}Q$ is block-diagonal, we find similarly that $Q^{-1}(\partial U/\partial W)$ must be upper block-triangular to preserve block structure. Therefore, $S(U) := (\partial U/\partial W)^{tr}Q^{-1}$ is lower block-triangular, and so is its product with the lower block-triangular $\partial U/\partial W$, so that $\hat{A}^0 = S(\partial U/\partial W) = (\partial U/\partial W)^{tr}Q^{-1}(\partial U/\partial W)$ is both lower block-triangular and symmetric positive definite, hence block-diagonal as well. Further, expanding

$$S(U)B^{jk}(\partial U/\partial W) = (\partial U/\partial W)^{tr}Q^{-1}(B^{jk}Q)Q^{-1}(\partial U/\partial W)$$

$$= S(B^{jk}Q)S^{tr} = \text{block-diag}\{0, S_{22}(B^{jk}Q)_{22}S^{tr}_{22}\},$$

we find that $\tilde{B}^{jk}$ satisfies ellipticity condition (1.6) as well. Likewise, lower block-triangularity of $S$ yields $SA^j(\partial U/\partial W)_{11} = S_{11}(A^jQ)_{11}S^{tr}_{11}$, verifying symmetry of $SA^j(\partial U/\partial W)_{11}$, and $SA^j(\partial U/\partial W)_{\pm} = (S(A^jQ)S^{tr})_{\pm}$ and $SB^{jk}(\partial U/\partial W)_{\pm} = (S(B^{jk}Q)S^{tr})_{\pm}$, verifying symmetry of $SA^j(\partial U/\partial W)_{\pm}$ and $SB^{jk}(\partial U/\partial W)_{\pm}$.

($\Leftarrow$) By nonsingularity of $(\partial U/\partial W)$, and vanishing of the first block-row of $SB^{jk}(\partial U/\partial W)$, we find that the first block-row of $SB^{jk}$ must vanish, hence $S$ must be lower block-triangular by vanishing of the first block-row of $B^{jk}$ together with nonsingularity of $B_{22}^{jk}$. It follows by nonsingularity of $S$ that $S_{11}$ and $S_{22}$ are nonsingular, whence the product of lower block-triangular $S$ with vanishing first block-row $B^{jk}(\partial U/\partial W)$ can be block-diagonal only if $B^{jk}(\partial U/\partial W)$ is already block-diagonal, implying condition (ii). From the assumption that $\hat{A}^0 = S(\partial U/\partial W)$ is symmetric positive definite, we find that

$$Q := S^{-1}(\partial U/\partial W)^{tr} = (\partial U/\partial W)(\hat{A}^0)^{-1}(\partial U/\partial W)^{tr} = (\partial U/\partial W)S^{-1,tr}$$

is symmetric positive definite as well. Factoring $A^jQ = S^{-1}\left(SA^j(\partial U/\partial W)\right)S^{-1,tr}$, $B^{jk}Q = S^{-1}\left(SB^{jk}(\partial U/\partial W)\right)S^{-1,tr}$, we find, again by lower block-triangularity of $S$, that right symmetrizer $Q$ preserves all properties of $\hat{A}^j$ and $\tilde{B}^{jk}$, thus verifying (i). Once existence of such $Q$ is verified, it follows from the analysis in the first part that we can take $W$ of form $(w^I(u^I), w^I(U))$ as claimed. □
Remark 6.4. The above construction may be recognized as a generalization of the Kawashima normal form (N), described, e.g., in [GMWZ4]; indeed, at the endstates (more generally, wherever symmetry of \(A^j, B^{jk}\) are enforced), the two constructions coincide. From the proof of Proposition 6.3, we find, more generally, that, assuming any subset of the properties asserted on \(A^jQ, B^{jk}Q\), these properties will be inherited by the system (1.11), independent of the others.

Corollary 6.5. Suppose it is known that system (1.1) admits right symmetrizers \(Q_\pm\) at \(U_\pm\) as described above, for example, if there exist local convex entropies \(\eta_\pm(U)\) near these states. Then, necessary and sufficient conditions for existence of a transformed system (1.11) satisfying (A1)–(A3) are (ii) of Proposition 6.3, together with the existence of symmetric positive definite matrices \(P_1\) and \(P_2\) such that \(P_1\) simultaneously symmetrizes \(\alpha_\pm := (\partial w^I/\partial u^I)A_\pm^I(\partial w^I/\partial u^I)^{-1}, P_1\alpha_\pm\) symmetric, and \(P_2\) simultaneously stabilizes \(\beta^{jk} := (\partial w^{I\prime}/\partial u^{I\prime})B^{jk}_{22} = (\partial w^{I\prime}/\partial u^{I\prime})^{-1}, W\) as defined in (ii) above, in the sense that \(\beta^{jk}\) satisfy a uniform ellipticity condition (1.6).

Proof. Necessity follows from Proposition 6.3, block-diag \(\{P_1, P_2\} = \tilde{A}^0\). Sufficiency follows by a partition of unity argument, interpolating the symmetric positive definite matrices \(P_j\) guaranteed at each point \(U\), taking care at the endstates \(U_\pm\) to choose the special \(P_j\) (guaranteed by the existence of symmetrizers \(Q_\pm\)) that also symmetrize \(A^j\). □

Corollary 6.5 suggests a strategy for verifying (A1)–(A3), namely to make the complete coordinate change \(U \to W(U), S(U) = (\partial W/\partial U), with W = (u^I, w^{I\prime}(U))\) for simplicity, then look directly for simultaneous symmetrizer/stabilizers \(P_1/P_2\). This works extremely well for the Navier–Stokes equations of compressible gas dynamics, which appear as

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= \mu \Delta u + (\lambda + \mu)\nabla \text{div}u, \\
(\rho(e + \frac{1}{2}u^2))_t + \text{div}(\rho(e + \frac{1}{2}u^2)u + pu) &= \text{div}(\tau \cdot u) + \kappa \Delta T,
\end{align*}
\]

where \(\rho > 0\) denotes density, \(u \in \mathbb{R}^d\) fluid velocity, \(T > 0\) temperature, \(e = e(\rho, T)\) internal energy, and \(p = p(\rho, T)\) pressure. Here, \(\tau := \lambda \text{div}(u)I + 2\mu Du\), where \(Du_{jk} = \frac{1}{2}(u_{xj}^j + u_{xk}^k)\) is the deformation tensor, \(\lambda(\rho, T) > 0\) and \(\mu(\rho, T) > 0\) are viscosity coefficients, and \(\kappa(\rho, T) > 0\) is the coefficient of thermal conductivity.

For these equations, \(U = (\rho, \rho u, E = e + |u|^2/2)\) and \(W = (\rho, u, T)\), and \(\alpha^j, \beta^{jk}\) are symmetric, with \(\sum_j \alpha^j \rho x_j = u \cdot \nabla \rho\) (scalar convection) and

\[
\sum_{jk}(\beta^{jk}(u, T)x_j)x_j = \rho^{-1}(\mu \Delta u + (\mu + \lambda)\nabla \text{div}u, \kappa T_\varepsilon \Delta T)
\]
(see, e.g., [GMWZ.4]), whence suitable \( P_j \) exist if and only if \( T_e > 0 \), in which case they may be simply taken as the identity. Here, \( E \) represents total energy density. Likewise, MHD may be treated similarly, with \( W = (\rho, u, T, B) \), \( B \) the magnetic field. Note that this indeed allows the case of van der Waals equation of state.

Alternatively, we may start with the Kawashima form (N) at the endstates, and try to deduce directly an appropriate modification preserving block structure along the profile. For example, examining form (N) for Navier–Stokes equations, for which (see, e.g., [KS], or equation (2.13) of [GMWZ.4])

\[
\tilde{A}_0^j = \begin{pmatrix}
\frac{p}{\rho}/\rho & 0 & 0 \\
0 & \rho I_3 & 0 \\
0 & 0 & \rho e_T/T
\end{pmatrix},
\]

\[
\sum_j \xi_j \tilde{A}^j = \begin{pmatrix}
\left(\frac{p}{\rho}/\rho\right)(u \cdot \xi) & \frac{p}{\rho} \xi & 0 \\
\rho(u \cdot \xi) I_3 & \rho e_T/T(u \cdot \xi) & p_T \xi^t \\
0 & \rho e_T/T(u \cdot \xi) & 0
\end{pmatrix},
\]

and

\[
\sum_{j,k} \tilde{B}^{jk} = \begin{pmatrix}
0 & 0 & 0 \\
\mu|\xi|^2 I_3 + (\mu + \lambda)\xi^t \xi & 0 & 0 \\
0 & T^{-1}|\xi|^2 & 0
\end{pmatrix},
\]

we find that we may simply multiply the first row by \( \chi(U) p_1^{-1} + (1 - \chi(U)) \) to obtain form (1.11) satisfying (A1)–(A3), where \( \chi \) is a smooth cutoff function supported on the set where the Navier–Stokes equations support a convex entropy and equal to unity at \( U_{\pm} \). A similar procedure applies to MHD.

By direct calculation, we find that the eigenvalues of \((\tilde{A}_0^s)^{-1} \sum_j \tilde{A}^j \xi_j = \sum_j A^j \xi_j\) for gas dynamics are of constant multiplicity with respect to both \( U \) and \( \xi \); see Appendix C. For MHD on the other hand, they are of constant multiplicity (under appropriate nondegeneracy conditions) only with respect to \( U \) (see [Je], pp. 122-127), and this limits the applications for the moment to one dimension. In both cases, \( A_s = (u \cdot \xi) \) is scalar, hence constant multiplicity. Thus, to complete the verification of (A1)–(A3), (H0)–(H2), it remains only to check that \( \det (A^1 - s) \) does not vanish at the endstates \( U_{\pm} \), and that \( A^1 + s = u^1 - s \) does not vanish along the profile, both purely one-dimensional consideration associated with the traveling-wave ODE.

We conclude by examining further various example systems in one dimension, expressed for simplicity in Lagrangian coordinates. For ease of notation, we drop the superscripts \( j \) and \( jk \) from \( A^j \), \( A^j_s \) and \( B^{jk} \).

**Example 6.6.** (standard gas dynamics) The general Navier–Stokes equations of compressible gas dynamics, written in Lagrangian coordinates, appear as

\[
v_t - u_x = 0, \tag{6.2}
\]

\[
u_t + p_x = ((\nu/v)u_x)_x, \\
(e + u^2/2)_t + (pu)_x = ((\kappa/v)T_x + (\mu/v)uu_x)_x,
\]
where $v > 0$ denotes specific volume, $u$ velocity, $e$ internal energy, $T = T(v, e)$ temperature, $p = p(v, e)$ pressure, and $\mu > 0$ and $\kappa > 0$ are coefficients of viscosity and heat conduction, respectively.

Note, as described in [MeP], that the “incomplete equation of state” $p = p(v, e)$ is sufficient to close the compressible Euler equations, consisting of the associated hyperbolic system (1.4), whereas the compressible Navier–Stokes equations require also the relation of temperature to $v$ and $e$. Accordingly, following [MeP], consider the “complete equation of state”

\begin{equation}
(6.3) \quad e = e(v, s)
\end{equation}

expressing internal energy in terms of $v$ and entropy $s$, defining temperature and pressure through the fundamental thermodynamic relation (combining first and second laws)

\begin{equation}
(6.4) \quad de = T ds - pdv,
\end{equation}

under the standard constraints of positive temperature,

\begin{equation}
(6.5) \quad T = e_s > 0,
\end{equation}

and global thermodynamic stability (precluding van der Waals type equation of state),

\begin{equation}
(6.6) \quad e(v, s) \text{ convex}.
\end{equation}

Note that we do not require positivity of $p = e_v$ or $p_e$, which may fail under certain interesting circumstances (for example, in water, near the transition from liquid to ice).

Under assumption (6.5), we may invert relation (6.3) to obtain $s = s(v, e)$ as a function of the primary variables $v$ and $e$. A straightforward calculation then gives

**Lemma 6.7.** For $T = e_s$ positive, $e(v, s)$ is convex if and only if $-s(v, e)$ is convex, if and only if $-s(v, E - u^2/2)$ is convex with respect to $(v, u, E)$, where $E := e + u^2/2$.

An immediate consequence of Lemma 6.7 is that

\begin{equation}
(6.7) \quad (\partial/\partial E)T(v, E - u^2/2) = T_e = (e_s)_e = (1/s_e)_e = -s_e e / s_e^2 > 0,
\end{equation}

whence (6.2) has the structural form (1.1)–(1.5), where the lower triangular matrix

\[ b_2 = \begin{pmatrix} 1/v & 0 \\ * & (\partial/\partial E)T \end{pmatrix} \]

evidently has real, positive spectrum $1/v$, $(\partial/\partial E)T$. Likewise, it is easily checked that genuine coupling, (A2), holds.
Appealing to relation (6.4), we find that

\[(6.8) \quad \eta(v, u, E) := -s(v, u, E - u^2/2), \quad q(v, u, E) := 0\]

satisfy conditions (ii)-(iii) of Definition 6.1. (Here, it is helpful to take \(u = 0\) by Galilean invariance; for details, see, e.g., Section 9 of [LZe].) By Lemma 6.7, condition (i) is satisfied as well, whence \(\eta, q\) is a convex entropy, entropy flux pair for (6.2). Applying Proposition 6.2, we thus find that (A1)–(A3) are satisfied for (6.2) written in terms of the entropy variable

\[(6.9) \quad d\eta = (-s_v, us_e, -s_e) = (e_v/e_s, u/e_s, -1/e_s) = T^{-1}(-p, u, -1).\]

Straightforward calculation then gives \(A_s = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0\) as in the isentropic case, verifying (H1) and completing the verification of the hypotheses under assumptions (6.5) and (6.6). Note: zero-speed profiles, \(s = 0\) do not exist. For, stationary solutions of (6.2) are easily seen to be constant in \(u, p, T\), and therefore \(e, \mu, v\) must be constant as well.

As discussed above, the assumption of global thermodynamic stability may be relaxed to \(T_c > 0\) together with thermodynamic stability at endstates \(U_\pm\), thus accommodating models for van der Waals gas dynamics. Again, stationary profiles cannot occur, since \(U = (v, u, e)\) must remain constant so long as it stays sufficiently near \(U_\pm\), and thus remains so forever. This completes the verification of (A1)–(A3), (H0)–(H2) in the general case; condition (H3) holds for extreme, Lax-type profiles, but not necessarily in general.

**Example 6.8.** (MHD) Next, consider the equations of MHD:

\[(6.10) \quad v_t - u_{1x} = 0,\]

\[u_{1t} + (p + (1/2\mu_0)(B_2^2 + B_3^2))_x = ((\nu/v)u_{1x})_x,\]

\[u_{2t} - ((1/\mu_0)B_1B_3)_x = ((\nu/v)u_{2x})_x,\]

\[u_{3t} - ((1/\mu_0)B_1^*B_3)_x = ((\nu/v)u_{3x})_x,\]

\[(vB_2)_t - (B_1^*u_2)_x = ((1/\sigma\mu_0v)B_{2x})_x,\]

\[(vB_3)_t - (B_1^*u_3)_x = ((1/\sigma\mu_0v)B_{3x})_x,\]

\[e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2\mu_0)v(B_2^2 + B_3^2)\]

\[+ [(p + (1/2\mu_0)(B_2^2 + B_3^2))u_1 - (1/\mu_0)B_1^*(B_2u_2 + B_3u_3)]_x\]

\[= ((\nu/v)u_1u_{1x} + (\mu/v)(u_2u_{2x} + u_3u_{3x}) +\]

\[(\kappa/v)T_x + (1/\sigma\mu_0^2v)(B_2B_{2x} + B_3B_{3x})]_x,\]

where \(v\) denotes specific volume, \(u = (u_1, u_2, u_3)\) velocity, \(p = P(v, e)\) pressure, \(B = (B_1^*, B_2, B_3)\) magnetic induction, \(B_1^*\) constant, \(e \) internal energy, \(T = T(v, e)\).
temperature, and \( \mu > 0 \) and \( \nu > 0 \) the two coefficients of viscosity, \( \kappa > 0 \) the coefficient of heat conduction, \( \mu_0 > 0 \) the magnetic permeability, and \( \sigma > 0 \) the electrical resistivity.

Calculating similarly as in the Navier–Stokes case, under the same assumptions (6.5) and (6.6), we find again that \( \eta := -s, \quad q := 0 \) is an entropy, entropy flux pair for system (6.10), \( \eta, q \) now considered as functions of the conserved quantities

\[
(v, u_1, u_2, u_3, vB_2, vB_3, E),
\]

where \( E := e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2\mu_0)v(B_2^2 + B_3^2) \) denotes total energy; for details, see again Section 9 of [LZe]. Likewise, we obtain by straightforward calculation that \( \Lambda_* = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0 \), verifying (H1) and completing the verification of the hypotheses under assumptions (6.5) and (6.6), plus the additional assumption (possibly superfluous, as in the Navier–Stokes case) that speed \( s \) be nonzero.

**Example 6.9.** (MHD with infinite resistivity/permeability) An interesting variation of (6.10) that is of interest in certain astrophysical parameter regimes is the limit in which either electrical resistivity \( \sigma \), magnetic permeability \( \mu_0 \), or both, go to infinity, in which case the right-hand sides of the fifth and sixth equations of (6.10) go to zero and there is a three-dimensional set of hyperbolic modes \( (v, vB_2, vB_3) \) instead of the usual one. Nonetheless, \( \Lambda_* = d\tilde{F}_{11} = d\tilde{F}_{12}b_2^{-1}b_1 \equiv 0 \) in this case as well; that is, though now vectorial, hyperbolic modes still experience passive, scalar convection, and so (H1) remains valid whenever \( s \neq 0 \). (Note: in Lagrangian coordinates, zero speed corresponds to “particle”, or fluid velocity.) Reduction to symmetric form may be achieved by the same entropy, entropy flux pair as in the standard case of Example 6.8, under assumptions (6.5) and (6.6).

**Example 6.10.** (multi-species gas dynamics or MHD) Another simple example for which the hyperbolic modes are vectorial is the case of miscible, multi-species flow, neglecting species diffusion, in either gas- or magnetohydrodynamics. In this case, the hyperbolic modes consist of \( k \) copies of the hyperbolic modes for a single species, where \( k \) is the number of total species, with a single, scalar convection rate \( \Lambda_* \equiv 0 \), corresponding to passive convection by the common fluid velocity.

**A.2. Structure of viscous profiles.**

In this appendix, we prove the results cited in the introduction concerning structure of viscous profiles.

**Proof of Lemma 1.6.** Differentiating (1.16) and rearranging, we may write (1.16)–(1.17) in the alternative form

\[
\begin{pmatrix}
  u'^I \\
  u'^{II}
\end{pmatrix}' = \begin{pmatrix}
  dF_{11}^I & dF_{12}^I \\
  b_1^{11} & b_2^{11}
\end{pmatrix}^{-1} \begin{pmatrix}
  0 \\
  f^{II} - f^{II}
\end{pmatrix}.
\]
Linearizing (6.11) about $U_\pm$, we obtain

$$
(6.12) \quad \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}' = \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 0 \\
 F_{21}^1 & F_{22}^1 \end{array} \right)_{\mid (U_\pm)} \begin{pmatrix} u^I \\ u^{II} \end{pmatrix},
$$

or, setting

$$
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)_{\mid (U_\pm)} \begin{pmatrix} u^I \\ u^{II} \end{pmatrix},
$$

the pair of equations

$$
z_1' = 0
$$

and

$$
(6.13) \quad z_2' = \left( \begin{array}{cc} F_{21}^1 & F_{22}^1 \\
 b_{21}^1 & b_{22}^1 \end{array} \right) \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 \\
 I_r \end{array} \right)_{\mid (U_\pm)} z_2,
$$

the latter of which evidently describes the linearized ODE on manifold (1.16).

Observing that

$$
\det \left( \begin{array}{cc} F_{21}^1 & F_{22}^1 \\
 b_{21}^1 & b_{22}^1 \end{array} \right) \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 \\
 I_r \end{array} \right)_{\mid (U_\pm)} =
$$

by (H2) and (H1)(i), we find that the coefficient matrix of (6.13) has no zero eigenvalues. On the other hand, it can have no nonzero purely imaginary eigenvalues $i\xi$, since otherwise

$$
\left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right) \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 \\
 u^{II} \end{array} \right) = i\xi \begin{pmatrix} 0 \\ u^{II} \end{pmatrix},
$$

and thus

$$
\left[ -i\xi \begin{pmatrix} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{pmatrix} - \xi^2 \begin{pmatrix} 0 & 0 \\
 b_{11}^1 & b_{12}^1 \end{pmatrix} \right] \left( \begin{array}{cc} F_{11}^1 & F_{12}^1 \\
 b_{11}^1 & b_{12}^1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 \\
 u^{II} \end{array} \right) = \left( \begin{array}{cc} 0 \\
 0 \end{array} \right)
$$

for $\xi \neq 0 \in \mathbb{R}$, in violation of (2.55), Corollary 2.19. Thus, we find that $U_\pm$ are hyperbolic rest points, from which (2.18) follows. □

**Proof of Lemma 1.7.** We follow the notation of Sections 1.2 and 3.1. Equating by consistent splitting/continuous extension to $\lambda = 0$ the dimensions of the stable subspace $S^+$ of the limiting coefficient matrix $A_+(\lambda)$ appearing in eigenvalue equation (2.17) at $\lambda = 0$ and at $\lambda \to +\infty$, we obtain

$$
\dim \mathcal{U}(A_+) + d_+ = \dim \mathcal{U}(A_{*+}) + r,
$$
or

\[(n - i_+) + d_+ = \dim \mathcal{U}(A_{*+}) + r,\]

and similarly at \(x \to -\infty\), where the lefthand side follows from Lemma 4.20 and the righthand side from the proof of Lemma 2.21. Rearranging, we obtain (1.19). If there exists a connecting profile, then we have by (H1)(i) that \(\dim S(A_{*})\) and \(\dim \mathcal{U}(A_{*})\) are constant along the profile and sum to \(n - r\). Thus, \(\dim S(A_{*-}) + \dim \mathcal{U}(A_{*+}) = n - r\), and we obtain \(n - i = r - d\) by summing relations (1.19). □

### A.3. Asymptotic ODE estimates.

For completeness, we prove in this appendix the asymptotic ODE estimates cited in Section 2.2. We first recall the gap lemma of [GZ,KS,ZH].

**Lemma 6.11 (the gap lemma).** Consider (2.17), under assumption (h0), with \(\theta < \theta\). If \(V^{-}(\lambda)\) is an eigenvector of \(A_{-}\) with eigenvalue \(\mu(\lambda)\), both analytic in \(\lambda\), then there exists a solution of (2.17) of form

\[W(\lambda, x) = V(x, \lambda) e^{\mu(\lambda)x},\]

where \(V\) is \(C^1\) in \(x\) and locally analytic in \(\lambda\) and for each \(j = 0, 1, \ldots\) satisfies

\[V(x, \lambda) = V^{-}(\lambda) + O(e^{-\theta|x|}|V^{-}(\lambda)|), \quad x < 0,\]

for all \(\theta < \theta\). Moreover, if \(\Re \mu(\lambda) > \Re \bar{\mu}(\lambda) - \theta\) for all (other) eigenvalues \(\bar{\mu}\) of \(A_{-}\), then also \(W\) is uniquely determined by (6.14), and (6.14) holds for \(\bar{\theta} = \theta\).

**Proof.** Setting \(W(x) = e^{\mu x} V(x)\), we may rewrite \(W' = A W\) as

\[V' = (A_{-} - \mu I) V + \theta V, \quad \theta := (A_{-} - A_{-}) = O(e^{-\theta|x|}),\]

and seek a solution \(V(x, \lambda) \to V^{-}(x)\) as \(x \to \infty\). Choose \(\bar{\theta} < \theta_1 < \theta\) such that there is a spectral gap \(\Re (\sigma A_{-} - (\mu + \theta_1)) > 0\) between \(\sigma A_{-}\) and \(\mu + \theta_1\). Then, fixing a base point \(\lambda_0\), we can define on some neighborhood of \(\lambda_0\) to the complementary \(A_{-}\)-invariant projections \(P(\lambda)\) and \(Q(\lambda)\) where \(P\) projects onto the direct sum of all eigenspaces of \(A_{-}\) with eigenvalues \(\bar{\mu}\) satisfying \(\Re (\bar{\mu}) < \Re (\mu + \theta_1)\) and \(Q\) projects onto the direct sum of the remaining eigenspaces, with eigenvalues satisfying \(\Re (\bar{\mu}) > \Re (\mu) + \theta_1\). By basic matrix perturbation theory (eg. [Kat]) it follows that \(P\) and \(Q\) are analytic in a neighborhood of \(\lambda_0\), with

\[|e^{(A_{-} - \mu I)x} P| \leq C(e^{\theta_1 x}), \quad x > 0, \quad |e^{(A_{-} - \mu I)x} Q| \leq C(e^{\theta_1 x}), \quad x < 0.\]

It follows that, for \(M > 0\) sufficiently large, the map \(T\) defined by

\[TV(x) = V^{-} + \int_{-\infty}^{x} e^{(A_{-} - \mu I)(x-y)} P \theta(y) V(y) dy - \int_{x}^{-M} e^{(A_{-} - \mu I)(x-y)} Q \theta(y) V(y) dy\]

(6.17)
is a contraction on $L^\infty(-\infty,-M]$. For, applying (6.16), we have

\begin{equation}
|TV_1 - TV_2| \leq C|V_1 - V_2| \left( \int_{-\infty}^x e^{\theta_1(x-y)}e^{\theta y}dy + \int_x^{-M} e^{\theta_1(x-y)}e^{\theta y}dy \right)
\leq C_1|V_1 - V_2| \left( e^{\theta_1 x}e^{(\theta-\theta_1)y}|_{-\infty}^x + e^{\theta_1 x}e^{(\theta-\theta_1)y}|_{x}^{-M} \right)
\leq C_2|V_1 - V_2|\infty e^{-\theta M} < \frac{1}{2}|V_1 - V_2|\infty.
\end{equation}

By iteration, we thus obtain a solution $V \in L^\infty(-\infty,-M]$ of $V = TV$ with $V \leq C_3|V^-|$; since $T$ clearly preserves analyticity $V(\lambda,x)$ is analytic in $\lambda$ as the uniform limit of analytic iterates (starting with $V_0 = 0$). Differentiation shows that $V$ is a bounded solution of $V = TV$ if and only if it is a bounded solution of (6.15) (exercise). Further, taking $V_1 = V, V_2 = 0$ in (6.18), we obtain from the second to last inequality that

\begin{equation}
|V - V^-| = |T(V) - T(0)| \leq C_2 e^{\theta x}|V| \leq C_4 e^{\theta x}|V^-|,
\end{equation}

giving (6.14). Analyticity, and the bounds (6.14), extend to $x < 0$ by standard analytic dependence for the initial value problem at $x = -M$. Finally, if $\text{Re } (\mu(\lambda)) > \text{Re } (\bar{\mu}(\lambda)) - \frac{\theta}{2}$ for all other eigenvalues, then $P = I, Q = 0$, and $V = TV$ must hold for any $V$ satisfying (6.14), by Duhamel’s principle. Further, the only term appearing in (6.18) is the first integral, giving (6.19) with $\theta = \theta$. □

**Proof of Lemma 2.5.** Substituting $W = P_+Z$ into (2.17), equating to (2.19), and rearranging, we obtain the defining equation

\begin{equation}
P'_+ = \kappa_+P_+ - P_+\kappa, \quad P_+ \rightarrow I \text{ as } x \rightarrow +\infty.
\end{equation}

Viewed as a vector equation, this has the form $P'_+ = AP_+$, where $A$ approaches exponentially as $x \rightarrow +\infty$ to its limit $A_+$, defined by $A_+P := \kappa_+P - P_+\kappa_+$. The limiting operator $A_+$ evidently has analytic eigenvalue, eigenvector pair $\mu \equiv 0, P_+ \equiv I$, whence the result follows by Lemma 6.11 for $j = k = 0$. The $x$-derivative bounds $0 < k \leq K + 1$ then follow from the ODE and its first $K$ derivatives, and the $\lambda$-derivative bounds from standard interior estimates for analytic functions. A symmetric argument gives the result for $P_-$. □

**Proof of Proposition 2.13.** Setting $\Phi_2 := \psi_1\psi_2^{-1}, \psi_1 \in \mathbb{C}^{(N-k)\times k}, \psi_2 \in \mathbb{C}^{k\times k}$, where $(\psi_1^t, \psi_2^t)^t \in \mathbb{C}^{N\times k}$ satisfies (2.29), we find after a brief calculation that $\Phi_2$ satisfies

\begin{equation}
\Phi'_2 = (M_1\Phi_2 - \Phi_2M_2) + \delta Q(\Phi_2),
\end{equation}

where $Q$ is the quadratic matrix polynomial $Q(\Phi) := \Theta_{12} + \Theta_{11}\Phi - \Phi\Theta_{22} + \Phi\Theta_{21}\Phi$. Viewed as a vector equation, this has the form

\begin{equation}
\Phi'_2 = M\Phi_2 + \delta Q(\Phi_2),
\end{equation}

\text{STABILITY OF LARGE-AMPLITUDE SHOCK WAVES} 183
with linear operator $M\Phi := M_1\Phi - \Phi M_2$. Note that a basis of solutions of the
decoupled equation $\Phi' = M\Phi$ may be obtained as the tensor product $\Phi = \phi \hat{\phi}^*$ of
bases of solutions of $\phi' = M_1\phi$ and $\hat{\phi}' = -M_2^*\hat{\phi}$, whence we obtain from (4.257)
the bound

$$(6.23) \quad |F^{y-x}| \leq |\hat{F}^{y-x}| |\hat{F}^{y-x}| = |\hat{F}^{y-x}| (|F^{y-x}|^{-1}) \leq Ce^{-\eta|x-y|}$$

for $x > y$, where $F$ denotes the flow of the full decoupled matrix-valued equation, $F_j$
the flow of $\phi' = M_j\phi$ and $\hat{F}_j$ the adjoint flow associated with equation $\hat{\phi}' = -M_j^*\hat{\phi}$.
(Recall the standard duality relation $\hat{F}_j = F_1j$).

That is, $F$ is uniformly exponentially decaying in the forward direction. Thus,
assuming only that $\Phi_2$ is bounded at $-\infty$, we obtain by Duhamel’s principle the
integral fixed-point equation

$$(6.24) \quad \Phi_2(x) = T\Phi_2(x) := \delta \int_{-\infty}^x F^{y-x}Q(\Phi_2)(y) \, dy.$$ 

Using (6.23), we find that $T$ is a contraction of order $O(\delta/\eta)$, hence (6.24) deter-
mines a unique solution for $\delta/\eta$ sufficiently small, which, moreover, is order $\delta/\eta$ as
claimed. The bound on $\partial_x F_j$ then follows by differentiation of (6.24), using the
fact that $\partial_x F^{y-x} \leq C F^{y-x}$, since the coefficients of the decoupled (linear) flow
are bounded. A symmetric argument establishes existence of $\Phi_1$. \qed

**Remarks 6.12.** 1. Note that the above stable/unstable manifold construc-
tion requires only boundedness of coefficients at infinity, and not any special (e.g.,
asymptotically constant or periodic) structure. This reflects the local nature of
large frequency/short time estimates in the applications.

2. Equation (6.21) could alternatively be derived from the point of view of
Lemma 2.5, as defining a change of coordinates

$$P := \begin{pmatrix} I & \Phi_2 \\ \Phi_1 & I \end{pmatrix}$$

conjugating (2.29) to diagonal form. The result of Proposition 2.13 shows that
there is a unique such transformation that is bounded on the whole line.

3. If desired, one can obtain a full, and explicit Neumann series expansion in
powers of $\delta/\eta$ from the fixed-point equation (6.24), using our explicit description
of the flow $F$ as the tensor product $\Phi = \phi \hat{\phi}^*$ of bases of solutions of $\phi' = M_1\phi$ and
$\hat{\phi}' = -M_2^*\hat{\phi}$. For our purposes, it is only the existence and not the precise form of
the expansion that is important.

**Proof of Corollary 2.14.** Similarly as in the proof of Proposition 2.13 just
above, we may obtain a fixed-point representation

$$F_j^{y-x} = T F_j^{y-x} := \hat{F}_1^{y-x} + \delta \int_y^x \hat{F}_1^{y-z}(\Theta_{11} + \Theta_{12} \Phi_2)(z, \epsilon) F_j^{z-x} \, dz,$$

where $T$ is a contraction of order $O(\delta/\eta)$, yielding the result. \qed
Lemma 6.13. Let $d_1 \in \mathbb{C}^{m_1 \times m_1}$ and $d_2 \in \mathbb{C}^{m_2 \times m_2}$ have norm bounded by $C_1$ and respective spectra separated by $1/C_2 > 0$. Then, the matrix commutator equation

$$d_1 X - X d_2 = F,$$

$X \in \mathbb{C}^{m_1 \times m_2}$, is soluble for all $F \in \mathbb{C}^{m_1 \times m_2}$, with $|X| \leq C(C_1, C_2)|F|$.

**Proof.** Consider (6.25) as a matrix equation $DX = F$ where $X$ and $F$ are vectorial representations of the $m_1 \times m_2$ dimensional quantities $X$ and $F$, and $D$ is the matrix representation of the linear operator (commutator) corresponding to the lefthand side. It is readily seen that $\sigma(D)$ is just the difference $\sigma(d_1) - \sigma(d_2)$ between the spectra of $d_1$ and $d_2$, with associated eigenvectors of form $r_1 l_2^*$, where $r_1$ are right eigenvectors associated with $d_1$ and $l_2$ are left eigenvectors associated with $d_2$. (Here as elsewhere, $^*$ denotes adjoint, or conjugate transpose of a matrix or vector). By assumption, therefore, the spectrum of $D$ has modulus uniformly bounded below by $1/C_2$, whence the result follows. \(\square\)

**Proof of Proposition 2.15.** Substituting $W = T\tilde{W}$ into (2.43) and rearranging, we obtain

$$\tilde{W}' = (T^{-1}AT - T^{-1}T')\tilde{W}$$

$$= (T^{-1}AT - \varepsilon T^{-1}T_y)\tilde{W},$$

yielding the defining relation

$$T^{-1}AT - \varepsilon T^{-1}T_y = D$$

for $T$.

By (h2), there exists a uniformly well-conditioned family of matrices $T_0(x)$ such that $T_0^{-1}A_0 T_0 = D_0, D_0$ as in (2.42); moreover, these may be chosen with the same regularity in $y$ as $A_0$ (i.e., the full regularity of $A$). Expanding

$$T^{-1}(y, \varepsilon) = \left( I - \left( \sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j \right) + \left( \sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j \right)^2 - \ldots \right) T_0^{-1}$$

by Neumann series, and matching terms of like order $\varepsilon^j$, we obtain a hierarchy of systems of linear equations of form:

$$D_0 T_0^{-1}T_j - T_0^{-1}T_j D_0 - F_j = D_j,$$

where $F_j$ depends only on $A_k$ for $0 \leq k \leq j$ and $T_k$, $(d/dy)T_k$ for $0 \leq k \leq j - 1$.

On off-diagonal blocks $(k, l)$, (6.28) reduces to

$$d_l [T_0^{-1}T_j]^{(k, l)} - [T_0^{-1}T_j]^{(k, l)} d_k = F_j^{(k, l)},$$
uniquely determining \([T_0^{-1}T_j]^{(k,l)}\), by Lemma 6.13. On diagonal blocks \((k,k)\), we are free to set \([T_0^{-1}T_j]^{(k,k)} = 0\), whereupon (6.28) reduces to

\[
[D_j]^{(k,k)} = -[F_j]^{(k,k)},
\]

determining the remaining unknown \([D_j]^{(k,k)}\).

Thus, we may solve for \(D_j, T_j\) at each successive stage in a well-conditioned way. Moreover, it is clear that the regularity of \(D_j, T_j\) is as claimed, since we lose one order of regularity at each stage, through the dependence of \(F_j\) on derivatives of lower order \(T_k, k < j\).

\[\square\]

A.4. Expansion of the one-dimensional Fourier symbol

In this appendix, we record the Taylor expansions about \(\xi = 0\) and \(\xi = \infty\) of the Fourier symbol \(P(i\xi) = -i\xi A^1(x) - \xi^2 B^{11}(x)\) of the one-dimensional frozen, constant-coefficient operator at a given \(x\). Henceforth, we suppress the superscripts for \(A\) and \(B\).

**Low frequency expansion.** Under assumption (P0), straightforward matrix perturbation theory [Kat] yields that the second-order expansion at \(i\xi = 0\) of \(P(i\xi)\) with respect to \(i\xi\) is just

\[
P = -i\xi A - \xi^2 R \text{diag} \{\beta_1, \ldots, \beta_n\} L,
\]

\[
\text{diag} \{\beta_1, \ldots, \beta_n\} := LBR,
\]

where \(L, R\) are composed of left, right eigenvectors of \(A\), as described in the introduction.

**High frequency expansion.** We next consider the expansion of \(P\) at infinity with respect to \((i\xi)^{-1}\). Recall that

\[
B := \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_\ast = A_{11} - A_{12} b_1 b_2^{-1}.
\]

\[\text{Claim.}\] To zeroth order, "hyperbolic" dispersion relations

\[
\lambda(i\xi) \in \sigma(-\xi^2 B - i\xi A), \quad \lambda \sim \xi,
\]

are given by

\[
\lambda_j(\xi) = -i\xi a_j^* - \eta_j^* + \ldots, \quad j = 1, \ldots n - r,
\]

with corresponding right and left eigenvectors

\[
R_j^\ast(i\xi) = \begin{pmatrix} r_j^* \\ -b_2^{-1} b_1 r_j^* \end{pmatrix} + \ldots \quad \text{and} \quad L_j^\ast(i\xi) = \begin{pmatrix} l_j^* \\ 0 \end{pmatrix} + \ldots,
\]
where

\begin{equation}
(6.35) \quad \eta_j^* := l_j^* D^* r_j^*, \quad D_* := A_{12} b_2^{-1} \left[A_{21} - A_{22} b_2 b_1 + b_2^{-1} b_1 A_* \right],
\end{equation}

and $a_j^*$, $r_j^*$, $l_j^*$ are eigenvalues and associated right and left eigenvectors of $A_*$. The remaining $r$ “parabolic” relations $\lambda \sim \xi^2$ are given to second order by

\begin{equation}
(6.36) \quad \lambda_{n-r+j}(\xi) = -\xi^2 \gamma_j + \ldots, \quad j = 1, \ldots, r,
\end{equation}

with corresponding right and left eigenvectors

\begin{equation}
(6.37) \quad R_{n-r+j}(i\xi) = \begin{pmatrix} 0 \\ s_j \end{pmatrix} + \ldots \quad \text{and} \quad L_{n-r+j}(i\xi) = \begin{pmatrix} \gamma_j^{-1} b_{tr} t_j \\ t_j \end{pmatrix},
\end{equation}

where $\gamma_j$, $s_j$, $t_j$ are the eigenvalues and right and left eigenvectors of $b_2$.

**Proof.** Expressing $P(i\xi) = -\xi^2 (B + \epsilon A)$, $\epsilon := (i\xi)^{-1}$ and expanding around $\epsilon = 0$ by standard matrix perturbation theory, we obtain the results after rearrangement; for details, see the dual calculation carried out in Appendix A2, [MaZ.1] in the relaxation case. \(\square\)

**Remark 6.14.** Note that we have without calculating that $\text{Re} \, \sigma \eta_j^* > 0$ as $x \to \pm \infty$, by the spectral bounds of Lemma 2.18.

**APPENDIX B. A WEAK VERSION OF MÉTIVIER’S THEOREM.**

In this appendix, we present a simplified version of the results of Métivier [Mé.1, FM, Mé.5] on inviscid stability of small-amplitude shock waves, establishing sharp uniform lower bounds on the Lopatinski determinant but not the associated uniform $L^2$ bounds on the solution operator. This gives the flavor of Métivier’s analysis, while avoiding many of the technical difficulties; at the same time, it illustrates in the simpler, inviscid setting some of the issues arising in the verification of strong spectral stability in the viscous small-amplitude case [PZ, FreS.2]. This material, presented first in [Z.2], was developed in part together with H. Freistühler; for an extension to the overcompressive case, see [FreZ.2]. Thanks also to K. Jenssen for helpful comments and corrections.

Consider an inviscid system

\begin{equation}
(7.1) \quad U_t + \sum_j F^j(U) x_j,
\end{equation}

and a one-parameter family of stationary Lax $p$-shocks (defined in (1.23)) $(U_+^\varepsilon, U_-^\varepsilon)$ with normal $e_1$ and amplitude $|U_+^\varepsilon - U_-^\varepsilon| := \varepsilon$, satisfying the Rankine–Hugoniot condition

\[ [F^1(U_+^\varepsilon)] = 0, \]
with $U_\pm$ converging in the small-amplitude limit $\varepsilon \to 0$ to some base point $U_0$. Setting $A^j := dF^j(U)$, and denoting by $\bar{a}_j$, $\bar{r}_j$, $\bar{l}_j$ the eigenvalues and associated right and left eigenvectors of $A^1$, we make the assumptions:

(h0) $F^j \in C^2$ (regularity).

(h1) There exists $A^0(\cdot)$, symmetric positive definite and smoothly depending on $U$, such that $A^0A^j(U)$ is symmetric for all $1 \leq j \leq d$ in a neighborhood of $U_0$ (simultaneous symmetrizability, $\Rightarrow$ hyperbolicity).

(h2) $\bar{a}_p = 0$ is a simple eigenvalue of $dF^1(U_0)$ with $\nabla \bar{a}_p \cdot \bar{r}_p(U_0) \neq 0$ (strict hyperbolicity and genuine nonlinearity of the principal characteristic field $a_p$ in the normal spatial direction $x_1$).

(h3) $\bar{r}_p$ is not an eigenvector of $A^{\xi} := \sum_{j \neq 1} \xi_j A^j$ for $\xi \in \mathbb{R}^{d-1} \neq 0$ (sufficient for extreme shocks, $p = 1$ or $n$); more generally,

$$
\langle \bar{r}_p, A^0A^{\xi}\tilde{\Pi}(A^1 - \bar{a}_p)^{-1}\Xi A^{\xi}\bar{r}_p \rangle \neq 0
$$

for all $\xi \in \mathbb{R}^{d-1} \neq 0$, where $\tilde{\Pi}$ denotes the eigenprojection complementary to the eigenprojection of $A^1$ onto Range ($\bar{r}_p$) (genuine hyperbolic coupling [Mé.1–4,FreZ.2]).

Assumptions (h0)–(h2) are standard conditions satisfied by many physical systems. Assumption (h3) is a technical condition that will emerge through the analysis. It ensures that the system does not decouple in the $p$th field, in particular excluding the (non-uniformly stable) scalar case.

Propositions 1.10 and 1.12 and Remark 1.14.1, show that the described scenario of a converging family of Lax $p$-shocks is typical under (h0)–(h2). As pointed out by Métivier, condition (h3) has the following easily checkable characterization (proof deferred to the end of this section), which shows also that (h3) is generically satisfied in two dimensions or for extreme shocks in higher dimensions, in both cases corresponding to nonsingularity of the associated Hessian ($1 \times 1$ for two dimensions; semidefinite for extreme shocks, as a consequence of hyperbolicity). Conditions (h0)–(h3) are generically satisfied for extreme shocks of gas- and magnetohydrodynamics near points $U_0$ of thermodynamic stability.

Lemma 7.1 [Mé.1]. Under assumptions (h0)–(h2), condition (h3) is equivalent to strict concavity (resp. convexity) of eigenvalue $a_p(U, \xi)$ of $A^\xi := \sum_j \xi_j A^j$ with respect to $\xi$ at the base point $U = U_0$, $\xi = (\xi_1, \xi) = (1, 0)$.

Then, we have the following two stability results, respectively concerning the one- and multi-dimensional case.

Proposition 7.2 [M.4]. (One-dimensional stability) In dimension $d = 1$, hypotheses (h0)–(h2) are sufficient to give uniform stability for $\varepsilon$ sufficiently small; moreover, we have

$$
|\Delta(0, \lambda)| \geq C^{-1}\varepsilon|\lambda|,
$$

with $C$ a positive constant.
for some $C > 0$, uniformly as $\varepsilon \to 0$.

**Proof.** Recalling (1.29), we have

$$|\Delta(0, \lambda)| := |\det (\bar{r}_1^-, \ldots, \bar{r}_{p-1}^-, \lambda[U], \bar{r}_{p+1}^+, \ldots, \bar{r}_n^+)|.$$  

Without loss of generality choosing $\bar{r}_j(U)$ continuously in $U$, $|\bar{r}_j| \equiv 1$, we have

$$\bar{r}_j^\pm = \bar{r}_j(U_0) + o(1)$$

as $\varepsilon \to 0$. At the same time,

$$[U] = \varepsilon(\bar{r}_p(U_0) + o(1)),$$

by the Lax structure theorem, Theorem 1.10. Combining, we have

$$|\Delta(0, \lambda)| = \varepsilon|\lambda| |\det (\bar{r}_1, \ldots, \bar{r}_n)(U_0) + o(1)|,$$

giving the result. \(\square\)

**Proposition 7.3 [Mé.1]. (Multidimensional stability).** Assuming (h0)–(h3), we have uniform stability for $\varepsilon := |U_+ - U_-|$ sufficiently small; moreover

$$|\Delta(\xi, \lambda)| \geq C^{-1}\varepsilon^{3/2} |(\xi, \lambda)|,$$

for some uniform $C > 0$.

**Proof.** Without loss of generality take $d = 2$ and (by the change of coordinates $A^j \to (A^0)^{1/2}A^j(A^0)^{-1/2}$) $A^1, A^2$ symmetric, $A^0 = I$. For simplicity, restrict to the extreme shock case $p = 1$; the intermediate case goes similarly [Mé.1]. Then, up to a well-conditioned rescaling, we may rewrite (1.29) as

$$\Delta = l_1^+ \cdot (\lambda[U] + i\xi_2[F^2(U)]),$$

where $l_1^+(\xi, \lambda)$ denotes the unique stable left eigenvector for $\text{Re} \lambda > 0$ of

$$A_+(\xi, \lambda) := (\lambda + i\xi_2A_2^2)(A_1^1)^{-1},$$

extended by continuity to $\text{Re} \lambda = 0$. We use form (7.7) to simplify the computations.

**Observation 1.** By appropriate choice of coordinates, we may arrange that

$$A_1^1 = \begin{pmatrix} -\hat{\xi} & 0 \\ 0 & A_1 \end{pmatrix},$$
where $\tilde{A}^{1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric, positive definite and bounded uniformly above and below, $\hat{\varepsilon} \sim \varepsilon$ by the Lax structure theorem and genuine nonlinearity, condition (h2);

$$A^{2}_{+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \tilde{A}^{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\tilde{A}^{2} \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric; and

$$\lambda[U] + i\xi_{2}[F^{2}(U)] = \varepsilon \begin{pmatrix} \lambda \\ i\xi_{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} (c + \mathcal{O}(\varepsilon)),$$

c \neq 0. (Here, $\mathcal{O}(\varepsilon)$ is matrix-valued, for brevity of exposition.)

**Proof of Observation.** First, take coordinates such that $A^{1}$ is block-diagonal and $A^{2}$ symmetric. Changing to a moving coordinate frame $x^{'}_{2} = x_{2} - ct$ in the transverse direction, we may arrange that $(A^{2}_{+})_{11} = 0$, i.e., $\langle \tilde{r}_{1}^{+}, A^{2}_{+} \tilde{r}_{1}^{+} \rangle = 0$, hence by a change of dependent variables in the second $((n-1) \times (n-1))$ block of $A^{1}$ that $A^{2}_{+} \tilde{r}_{1}^{+}$ lies in the second coordinate direction. Since $A^{2}_{+} \tilde{r}_{1}^{+} \neq 0$, by (h3), we may arrange by a rescaling of $x_{2}$ that $|A^{2}_{+} \tilde{r}_{1}^{+}| = 1$. This verifies (7.8)–(7.9). Relation (7.10) then follows by the Lax structure theorem and Taylor expansion about $U_{+}$ of $F^{2}(U)$.

From here on, we drop the hat, and write $\varepsilon$ for $\hat{\varepsilon}$.

**Observation 2.** The matrix $A_{+}(\tilde{\xi}, \lambda) : = (\lambda + i\xi_{2}A^{2}_{+})(A^{1}_{+})^{-1}$ then has the form

$$\begin{pmatrix} -\lambda/\varepsilon & i\xi_{2}/a & 0 \\ -i\xi_{2}/\varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (|\xi_{2}| + |\lambda|) \begin{pmatrix} 0 & \mathcal{O}(\varepsilon) & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix},$$

$a > 0$, or, “balancing” using transformation

$$T := \begin{pmatrix} \varepsilon^{-1/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix},$$

$$TA_{+}T^{-1} = \begin{pmatrix} -\lambda/\varepsilon & i\xi_{2}/a\varepsilon^{1/2} & 0 \\ -i\xi_{2}/\varepsilon^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (|\xi_{2}| + |\lambda|) \begin{pmatrix} 0 & \mathcal{O}(\varepsilon^{1/2}) & \mathcal{O}(\varepsilon^{-1/2}) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix},$$

$$= \varepsilon^{-1/2} \begin{pmatrix} -\hat{\lambda} & i\xi_{2}/a & w^{Tr} \\ -i\xi_{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (|\xi_{2}| + \varepsilon^{1/2}|\hat{\lambda}|) \begin{pmatrix} 0 & \mathcal{O}(\varepsilon^{1/2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}.$$
where $\hat{\lambda} := \lambda \varepsilon^{-1/2}$, $|w| = \mathcal{O}(1)$.

**Proof of Observation 2.** We have from (7.8) that

\[
(A_+^1)^{-1} = \begin{pmatrix}
-\varepsilon^{-1} & 0 \\
0 & (\hat{A}^1)^{-1}
\end{pmatrix},
\]

where

\[
(\hat{A}^1)^{-1} = \begin{pmatrix}
a^{-1} & * \\
* & *
\end{pmatrix}
\]

is symmetric positive definite, whence the diagonal entry $a^{-1}$ is symmetric positive definite as well. The result then follows by direct computations, combining (7.9) with (7.14). $\square$

Without loss of generality fixing $\xi_2 \equiv 1$ (the case $\xi_2 = 0$ may be obtained by continuity, and has anyway been treated already in the one-dimensional case), we consider the resulting matrix perturbation problem (7.12).

**Case (i) ($|\lambda| \gg \varepsilon^{1/2}$).** First we treat the straightforward case $|\lambda| \gg 1$. In this case, $-\hat{\lambda}$ dominates all entries in the rescaled perturbation problem (7.12), from which we find that the associated left eigendirection of $TA_+T^{-1}$ is $(1, 0, \ldots, 0)^T + o(1)$, and thus the left eigenvector $l_+^1$ of $A_+$ is (rescaled to order one)

\[
(7.15) \quad \varepsilon^{1/2}(1, 0, \ldots, 0)^T + o(1) T \sim (1, 0, \ldots, 0)^T + o(\varepsilon^{1/2}).
\]

Computing $\Delta = l_+^1 \cdot (\lambda[U] + i\xi_2[F^2(U)])$ following (7.7), we obtain from (7.10) and (7.15) that $|\Delta| \sim \varepsilon \lambda \geq C^{-1}\varepsilon^{3/2}$, as asserted.

**Case (ii) ($|\lambda| \leq C\varepsilon^{1/2}$).** Next, we treat the critical case that $|\hat{\lambda}|$ is uniformly bounded. Noting that the upper lefthand corner

\[
(7.16) \quad \alpha := \begin{pmatrix}
-\lambda & i\xi_2/a \\
i\xi_2 & 0
\end{pmatrix} = \begin{pmatrix}
-\hat{\lambda} & i/a \\
-i & 0
\end{pmatrix}
\]

of block-upper triangular matrix

\[
A = \begin{pmatrix}
-\lambda & i\xi_2/a & w^T \\
-i\xi_2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
-\hat{\lambda} & i/a & w^T \\
-i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

is uniformly nonsingular, with determinant $-\xi_2^2/a = -1/a < 0$, we have by standard matrix perturbation theory that the stable left eigenvector $L_+^1$ of

\[
TA_+T^{-1} = \varepsilon^{-1/2}A + \mathcal{O}(1)
\]
is an $O(\varepsilon^{1/2})$ perturbation of the left stable eigenvector $\mathbf{L} = (\ell^\text{Tr}, w^\text{Tr})^\text{Tr}$ of $A$, where

$$\ell := (1, i/a\mu)^\text{Tr}$$

is the left stable eigenvector of $\alpha$ and

$$\mu := -\hat{\lambda} - \sqrt{\hat{\lambda}^2 + 4/a}$$

is the associated stable eigenvalue. Scaling to order one length, we thus have

$$l_1^+ = \varepsilon^{1/2} T \mathbf{L}_1^+.$$  

Computing $\Delta = l_1^+ \cdot (\lambda[U] + i\xi_2[F^2(U)])$ following (7.7), we obtain from (7.10), (7.17), and the above developments that

$$\Delta = \varepsilon^{1/2}(\mathbf{L}_1^+ + O(\varepsilon^{1/2})) \cdot T \varepsilon \begin{pmatrix} \lambda \\ i\xi_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} (c + O(\varepsilon))$$

$$= \varepsilon^{3/2} \ell \cdot \begin{pmatrix} \hat{\lambda} \\ i \end{pmatrix} (1 + O(\varepsilon^{1/2}))$$

$$= \varepsilon^{3/2} \mu(1 + O(\varepsilon^{1/2})),$$

yielding $|\Delta| \sim \varepsilon^{3/2}$ as claimed.

This completes the proof. $\square$

**Remarks.** 1. The upper bound in case (ii) shows that (7.6) is sharp.

2. [Mé.1] The matrix perturbation problem (7.11) may be recognized as essentially that arising from the canonical $2 \times 2$ example

$$F^1(u, v) := \begin{pmatrix} u^2/2 \\ v \end{pmatrix}, \quad F^2(u, v) := \begin{pmatrix} v \\ u \end{pmatrix},$$

$U_\pm = (\mp \varepsilon, 0)$, for which the system at $U_+$ is the $2 \times 2$ wave-type equation

$$A^1 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the resulting matrix perturbation problem agrees with the determining $2 \times 2$ upper-lefthand block of (7.11). That is, whereas one-dimensional behavior is (at least at small amplitudes) essentially scalar, Proposition 7.2, multi-dimensional
behavior involves in an essential way the coupling between different hyperbolic modes (precisely two for each fixed transverse direction $\tilde{\xi}$).

3. In the intermediate shock case $1 < p < n$, the strengthened version (7.2) of condition (h3) may be seen to correspond (see development of (7.14)) to nonvanishing of $a$ in the $2 \times 2$ blocks analogous to $\alpha$ in (7.16) (one block arising at $+\infty$ and one at $-\infty$).

**Proof of Lemma 7.1.** Without loss of generality take $d = 2$ and (by the change of coordinates $A^j = (A^0)^{1/2} A^j (A^0)^{-1/2}$) $A^1, A^2$ symmetric, $A^0 = I$. Let

$$a_p(\tilde{\xi}) = a_p^0 + a_p^1 \tilde{\xi} + a_p^2 \tilde{\xi}^2$$

and

$$r_p(\tilde{\xi}) = r_p^0 + r_p^1 \tilde{\xi} + \ldots,$$

denote the eigenvalues and eigenvectors of

$$A(\tilde{\xi}) := (A^1 + \tilde{\xi} A^2),$$

analytic by simplicity of $a_p$, (h2), without loss of generality $|r_p^0| = 1$. Expanding the defining relations

$$(A(\tilde{\xi}) - a_p(\tilde{\xi}))r_p(\tilde{\xi}) = 0,$$

and matching terms of like order in $\tilde{\xi}$, we obtain

**(0th order):**

$$(A^1 - a_p^0) r_p^0 = 0,$$

**(1st order):**

$$(A^1 - a_p^0) r_p^1 + (A^2 - a_p^1) r_p^0 = 0,$$

**(2nd order):**

$$(A^1 - a_p^0) r_p^2 + (A^2 - a_p^1) r_p^1 + (-a_p^2) r_p^0 = 0.$$

Taking the inner product of $r_p^0$ with (7.23), we obtain

$$(7.25) \quad 0 = \langle r_p^0, (A^2 - a_p^1) r_p^0 \rangle,$$

and thus

$$(7.26) \quad a_p^1 = \langle r_p^0, A^2 r_p^0 \rangle.$$
Next, we may solve modulo span \((r^0_p)\) for \(r^1_p\) in (7.23) by taking the inner product with \(r^0_k\) for \(k \neq p\), to obtain

\[
\langle r^0_k, r^1_p \rangle = -\frac{\langle r^0_k, (A^2 - a^1_p)r^0_p \rangle}{a^0_k - a^0_p},
\]

hence

\[
r^1_p = -\sum_{k \neq p} \frac{\langle r^0_k, (A^2 - a^1_p)r^0_p \rangle}{a^0_k - a^0_p} r^0_k
\]

is a solution. Finally, taking the inner product of \(r^0_p\) with (7.24), and using (7.28) plus symmetry of \(A^2\), we obtain expansion

\[
a^2_p = -\sum_{k \neq p} \frac{\langle r^0_k, (A^2 - a^1_p)r^0_p \rangle}{a^0_k - a^0_p} (A^2 - a^1_p)r^0_k
\]

\[
= -\sum_{k \neq p} \frac{\langle r^0_k, (A^2 - a^1_p)r^0_p \rangle^2}{a^0_k - a^0_p}.
\]

Observing that \(r^0_p\), by (7.25) is an eigenvector of \(A^2\) if and only if

\[
(A^2 - a^1_p)r^0_p = 0 \mod (\text{span}_{k \neq p}(r^0_k)),
\]

we have the result in the extreme case \(p = 1\) or \(n\), for which \(a^0_k - a^0_p\) for \(k \neq p\) has a definite sign. More generally, observing that \(\langle A^2 - a^1_p \rangle r^0_p = \bar{\Pi}A^2 r^0_p\), we may rewrite (7.28) as

\[
r^1_p = (A^1 - a^0_p)^{-1}\bar{\Pi}A^2 r^0_p
\]

to obtain

\[
a^2_p = \langle r^0_p, (A^2 - a^1_p)r^1_p \rangle
\]
\[
= \langle \bar{\Pi}A^2 r^0_p, r^1_p \rangle
\]
\[
= \langle \bar{\Pi}A^2 r^0_p, (A^1 - a^0_p)^{-1}\bar{\Pi}A^2 r^0_p \rangle
\]

in place of (7.29). Thus, \(a^2_p \neq 0\) is equivalent to

\[
\langle r^0_p, A^2\bar{\Pi}(A^1 - a^0_p)^{-1}\bar{\Pi}A^2 r^0_p \rangle \neq 0,
\]

as claimed. \(\square\)
APPENDIX C. EVALUATION OF THE LOPATINSKI CONDITION FOR GAS DYNAMICS.\textsuperscript{31}

(Previously submitted.)

REFERENCES


\textsuperscript{31}Contributed by K. Jenssen and G. Lyng.


Lord Rayleigh (J.W. Strutt), On the stability, or instability, of certain fluid motions, II, Scientific Papers, 3 (1887) 2–23, Cambridge University Press.


Lord Rayleigh (J.W. Strutt), On the question of the stability of the flow of fluids, Scientific Papers, 3 (1892a) 575–584, Cambridge University Press.


