STABILITY AND DYNAMICS OF VISCOUS SHOCK WAVES

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Abstract. We examine from a classical dynamical systems point of view stability, dynamics, and bifurcation of viscous shock waves and related solutions of nonlinear pde. The central object of our investigations is the Evans function: its meaning, numerical approximation, and behavior in various asymptotic limits.

1. Introduction and motivating examples. The object of this course is to study, using techniques related to the Evans function, the stability and dynamics of traveling-wave solutions $u(x,t) = \bar u(x-st)$ of various systems of viscous conservation laws arising in continuum mechanics:

$$u_t + f(u)_x = (B(u)u_x)_x. \quad u \in \mathbb{R}^n. \tag{1.1}$$

Our emphasis is on concrete applications: in particular, numerical verification of stability/bifurcation conditions for viscous shock waves. The techniques we shall describe should have application also to other problems involving stability of traveling waves, particularly for large systems, or systems with no spectral gap between decaying and neutral modes of the linearized operator about the wave. We begin by introducing some equations from Biology and Physics possessing traveling wave solutions.

Reaction–diffusion equation. Consider a non-conservative problem:

$$u_t + g(u) = u_{xx}, \qquad u = u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{1.2}$$

where:

$$g(u) = dG(u) = (1/2)(u^2-1)u \quad \text{and} \quad G(u) = (1/8)(u^2-1)^2. \tag{1.3}$$

This equation is instructive to study, as it combines the effects of: (i) spatial diffusion, exemplified by the heat equation $u_t = u_{xx}$, with (ii) reaction, exemplified by the gradient flow $u_t = -dG(u)$ for which G represents a potential (chemical, for example) of the state associated with the value u (for example, densities of various chemical constituents).

Equation (1.2) models propagation of electro-chemical signals along a nerve axon, pattern-formation in chemical systems, and population dynamics. It also serves as a model for phase-transitional phenomena in material science, where u is an "order parameter" and G a free energy potential.

Burgers equation. The following scalar viscous conservation law:

$$u_t + f(u)_x = u_{xx}, \qquad u = u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{1.4}$$

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where:

$$f(u) = u^2/2, \tag{1.5}$$

serves as a simple model for gas dynamics, traffic flow, or shallow-water waves, with u representing the density of some conserved quantity and f its flux through a fixed point x.

Isentropic gas dynamics. The equations of an isentropic compressible gas, take the form in Lagrangian coordinates (that is, coordinates in which x represents initial particle location, as opposed to Eulerian coordinates, in which x represents fixed location):

$$v_t - u_x = 0, \qquad u_t + p(u)_x = \left(\frac{u_x}{v}\right)_x, \tag{1.6}$$

where u represents specific volume (one over density), v velocity, and p pressure. System (1.6) models ideal gas dynamics when $p' < 0$. In the case $p' \gtrless 0$, it models phase-transitional gas dynamics or elasticity; thus generalizing both of the previous two examples.

Here, we consider mainly a γ-law pressure, modeling an ideal isentropic polytropic gas:

$$p(v) = av^{-\gamma}, \tag{1.7}$$

where $a > 0$ and $\gamma > 1$ are constants that characterize the gas.

Isentropic MHD. In Lagrangian coordinates, the equations for planar compressible isentropic magnetohydrodynamics take the form:

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + (p + (1/2\mu_0)(B_2^2 + B_3^2))_x = ((1/v)u_{1x})_x, \\ u_{2t} - ((1/\mu_0)B_1^* B_2)_x = ((\tilde{\mu}/v)u_{2x})_x, \\ u_{3t} - ((1/\mu_0)B_1^* B_3)_x = ((\tilde{\mu}/v)u_{3x})_x, \\ (vB_2)_t - (B_1^* u_2)_x = ((1/\sigma\mu_0 v)B_{2x})_x, \\ (vB_3)_t - (B_1^* u_3)_x = ((1/\sigma\mu_0 v)B_{3x})_x, \end{cases} \tag{1.8}$$

where v denotes specific volume, $u = (u_1, u_2, u_3)$ velocity, $p = p(v)$ pressure, $B = (B_1^*, B_2, B_3)$ magnetic induction, B_1^* constant (by the divergence-free condition $\nabla_x \cdot B \equiv 0$), $\tilde{\mu} = \frac{\mu}{2\mu+\eta}$ where $\mu > 0$ and $\eta > 0$ are the respective coefficients of Newtonian and dynamic viscosity, $\mu_0 > 0$ the magnetic permeability, and $\sigma > 0$ the electrical resistivity [A, C, J].

We take again a γ-law pressure (1.7). Though we do not specify η, we have in mind mainly the value $\eta = -2\mu/3$ typically prescribed for (nonmagnetic) gas dynamics [Ba],[1] or $\tilde{\mu} = 3/4$. An interesting, somewhat simpler, subcase is that of infinite electrical resistivity $\sigma = \infty$, in which the last two equations of (1.8) are replaced by:

$$(vB_2)_t - (B_1^* u_2)_x = 0, \qquad (vB_3)_t - (B_1^* u_3)_x = 0, \tag{1.9}$$

[1]Predicted by Chapman-Enskog expansion of Boltzmann's equation.

and only the velocity variables (u_1, u_2, u_3) experience parabolic smoothing, through viscosity. Here, we are assuming a planar multidimensional flow, i.e., one that depends only on a single space-direction x. In the simplest, *parallel case* when $u_2 = u_3 = B_2 = B_3 \equiv 0$, the equations reduce to those of isentropic gas dynamics (1.6).

2. Traveling and standing waves. We shall study *traveling waves* to (1.1), which are solutions of the form:

$$u(x,t) = \bar{u}(x - st), \tag{2.1}$$

where s is a constant *speed* of the *profile* $\bar{u}(\cdot)$ connecting *endstates*:

$$\lim_{z \to \pm\infty} \bar{u}(z) = u_\pm. \tag{2.2}$$

Other types of profiles, such as waves that are oscillatory at infinity, are also interesting but beyond our scope. If $s = 0$ the wave (2.1) is called a *standing wave* of the associated evolution equation. Note that a traveling wave may always be converted to a standing wave by the change of coordinates $x \mapsto x - st$, corresponding to frame moving with the speed s.

Traveling waves have practical importance, as the simplest solutions that maintain their form while still reflecting the spatial dynamics of the problem; in particular, except in the trivial case $u \equiv \text{const}$, they depend nontrivially on the variable x. We derive below nontrivial traveling-wave solutions for each of the models just described, choosing parameters carefully in order to have explicit solutions.

Generally, one must approximate profiles numerically, since for arbitrary choices of f, g, p, the profile equations are not explicitly soluble. However, the properties of the solutions remain similar, in particular, *profiles decay exponentially as* $x \to \pm\infty$, which is the principle property we will need in what follows. Numerical approximation of profiles is another interesting problem, but one that has already been well studied [Be].

Reaction–diffusion equation. Observing that

$$\partial_t \bar{u}(x - st) = -s\bar{u}', \quad \partial_x \bar{u}(x - st) = \bar{u}', \quad \partial_x^2 \bar{u}(x - st) = \bar{u}'', \tag{2.3}$$

we obtain for a solution (2.1) the *profile equation*: $-s\bar{u}' + dG(\bar{u}) = \bar{u}''$. Taking for simplicity $s = 0$, we find that it becomes a *nonlinear oscillator*, which is a Hamiltonian equation:

$$dG(\bar{u}) = \bar{u}'', \tag{2.4}$$

that may be reduced to first order. Specifically, multiplying by $\bar{u}'$ on both sides, we obtain:

$$dG(\bar{u})\bar{u}' = \bar{u}'\bar{u}'' \quad \text{where} \quad G(\bar{u})' = [(\bar{u}')^2/2]'.$$

Integrating from $-\infty$ to x then yields: $(\bar u')^2 = 2[G(\bar u) - G(u_-)]$. Thanks to (2.2), we have $\bar u'(\pm\infty) = 0$, hence, recalling (2.4), we have $dG(u_\pm) = 0$, where $u_\pm = \pm 1, 0$. As $u = 0$ is a nonlinear center admitting no orbital connections, we have: $u_\pm = \pm 1, \quad G(u_\pm) = 0$ and

$$\bar u' = \pm\sqrt{2G(\bar u)} = \pm\sqrt{(1/4)(\bar u^2 - 1)^2} = \pm(1/2)(\bar u^2 - 1).$$

Choosing the sign $+$, we obtain finally, the profile solution (unique up to translation in x):

$$\bar u(x) = -\tanh(x/2). \tag{2.5}$$

Burgers equation. Using again (2.3), we obtain as the equation for a Burgers profile:

$$-s\bar u' + f(\bar u)' = \bar u''. \tag{2.6}$$

Integrating from $-\infty$ to x reduces (2.6) to a first-order equation:

$$\bar u' = (f(\bar u) - s\bar u) - (f(u_-) - su_-).$$

For definiteness taking again $s = 0$, $u_- = 1$, we obtain $\bar u' = (1/2)(1 - \bar u^2)$ as in the reaction–diffusion case, and thus the same tanh solution (2.5), again unique up to translation.

Gas dynamics. Similarly, we find for (1.6) the profile equations:

$$-s\bar v' - \bar u' = 0 \qquad -s\bar u' + p(\bar v)' = (\bar v^{-1}\bar u')'. \tag{2.7}$$

Substituting the first equation into the second, and integrating from $-\infty$ to x, we obtain:

$$\bar v' = (-s)^{-1}\bar v[(p(\bar v) + s^2\bar v) - (p(v_-) + s^2 v_-)]. \tag{2.8}$$

Choosing $s = -1$, $v_- = 1$, $a = v_+$, taking a isothermal, or constant-temperature, pressure $p(v) = av^{-1}$, and setting $v_m := \frac{v_- + v_+}{2}$, $\delta := v_- - v_+$, and $\bar v := v_m + (\delta/2)\bar\theta$, we obtain, finally, $\bar\theta' = (\delta/2)(\bar\theta^2 - 1)$, hence $\bar\theta(x) = -\tanh((\delta/2)x)$, giving a solution [Tay]:

$$\bar v := v_m - (\delta/2)\tanh((\delta/2)x) \tag{2.9}$$

similar to the Burgers profile (2.5), with $\bar u = \bar v + C$, C an arbitrary constant.

MHD. In the "parallel case" the equations of isentropic MHD reduce to those of isentropic gas dynamics, and so we have the same family of traveling waves already studied. However, from the standpoint of behavior, there is a major difference that we wish to point out. Namely, taking $v_- = 1$,

$u_{1,-} = 0$, $s = -1$, and for simplicity taking $\sigma = \infty$, but *without making the parallel assumptions*, we obtain the more general profile equation:

$$\begin{aligned} v' &= v(v-1) + v(p - p_-) + \frac{1}{2\mu_0 v}((B_- + B_1^* w)^2 - v^2 B_-^2), \\ \tilde{\mu} w' &= vw - \frac{B_1^*}{\mu_0}\big(B_-(1-v) + B_1^* w\big), \end{aligned} \tag{2.10}$$

where $w = (u_2, u_3)$ denotes transverse velocity components, and $B \equiv v^{-1}(B_- + B_1^* w)$.

The point we wish to emphasize is that even in the case of a parallel traveling-wave profile, for which $B_{2,\pm} = B_{3,\pm} = u_{2,\pm} = u_{3,\pm} = 0$, there may exist other traveling-wave profiles connecting the same endstates, but which are not of parallel type. Indeed, this depends precisely on the type of the rest points at $\pm\infty$, i.e., the number of stable (negative real part) and unstable (positive real part) eigenvalues of the coefficient of the linearized equations about these equilibria, which are readily determined by the observation that v and w equations decouple in the parallel case [BHZ].

3. Spectral stability and the eigenvalue equations. A fundamental issue in physical applications is *nonlinear* (time-evolutionary) *stability* of traveling-wave solutions $\bar{u}$ in (2.1). The question is whether or not a perturbed solution $\tilde{u} = \bar{u} + u$, with perturbation u sufficiently small at $t = 0$, approaches as $t \to +\infty$ to some translate $\bar{u}(x - st - x_0)$ of the original wave $\bar{u}(\cdot)$.

Observe that a perturbation $u(x,t) = \bar{u}(x - x_0 - st) - \bar{u}(x - st)$ is a traveling wave itself, hence does not decay. Thus, approach to a translate "orbital stability" is all that we can expect. Unstable solutions are impermanent, since they disappear under small perturbations, and typically play a secondary role if any in behavior: for example, as transition states lying on separatrices between basins of attraction of neighboring stable solutions, or as background organizing centers from which other solutions bifurcate.

Let us briefly recall the study of asymptotic stability of an equilibrium $\bar{U} \in \mathbb{R}^n$ of an autonomous ODE in $\mathbb{R}^n$:

$$U_t = \mathcal{F}(U), \qquad \mathcal{F}(\bar{U}) = 0. \tag{3.1}$$

We first linearize about $\bar{U}$, to obtain:

$$U_t = LU := d\mathcal{F}(\bar{U})U,$$

where the operator L has constant coefficients, since $\bar{U}$ is constant and $\mathcal{F}$ is autonomous. The formal justification for this step is that if perturbation $U = \tilde{U} - \bar{U}$ is small, then terms of second order are much smaller than terms of first order in the perturbation, and so can be ignored. If the solutions of (3.1) decay as $t \to \infty$, we say that $\bar{U}$ is *linearly* (asymptotically) *stable.*

Linear stability is nearly but not completely necessary for *nonlinear stability*, defined as decay as $t \to \infty$ of perturbations $U = \tilde{U} - \bar{U}$ that are sufficiently small at $t = 0$, for a solution $\tilde{U}$ of (3.1). A necessary condition for linear stability is *spectral stability*, or nonexistence of eigenvalues λ of L with $\Re\lambda \geq 0$. For, if there are $W \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ such that $LW = \lambda W$ with $\Re\lambda \geq 0$, then $U(t) = e^{\lambda t}W$ is a *growing mode* of (3.1), or a solution that does not decay.

These three notions of stability are linked in $\mathbb{R}^n$ by *Lyapunov's Theorem*, asserting that linearized and spectral stability are equivalent, and imply nonlinear exponential stability. One generally studies the simplest condition of spectral stability which corresponds to nonexistence of solutions W, λ with $\Re\lambda \geq 0$ of the eigenvalue equation:

$$LW = \lambda W. \tag{3.2}$$

This may be studied by Nyquist diagram, Routh–Hurwitz stability criterion [Hol], or other methods based on the characteristic polynomial $p(\lambda) = \det(\lambda \mathrm{Id} - L)$ associated with L.

We will pursue a similar program in the case of interest for PDEs, linearizing and studying the associated eigenvalue equations with the aid of the *Evans function*, which is an infinite-dimensional generalization of the characteristic polynomial. As a first step, we derive here the linearized and eigenvalue equations for the 4 described models, and then put them in a common framework as (complex) systems of first-order ODE:

$$W' = A(\lambda, x)W, \quad W \in \mathbb{C}^n,\ A \in \mathbb{C}^{n\times n}. \tag{3.3}$$

Reaction–diffusion equation. Let $\tilde{u}$ and $\bar{u}$ be two solutions of (1.2). The variable $u = \tilde{u} - \bar{u}$ satisfies the *nonlinear perturbation equation*:

$$u_t + (g(\bar{u} + u) - g(\bar{u})) = u_{xx}. \tag{3.4}$$

Taylor expanding $(g(\bar{u}+u)-g(\bar{u})) = dg(\bar{u})u+O(u^2)$ and dropping nonlinear terms of higher order, we obtain the linearized equation:

$$u_t + dg(\bar{u})u = u_{xx} \tag{3.5}$$

where $dg(\bar{u}) = d^2G(\bar{u}) = (1/2)(3\bar{u}^2 - 1)$, $dg(\bar{u}) = d^2G(\bar{u}) = (1/2)(3\bar{u}^2 - 1)$, and $\bar{u}(x) = -\tanh(x/2)$. Seeking *normal modes* $u(x,t) = e^{\lambda t}w(x)$, we obtain the *eigenvalue equation*:

$$\lambda w + dg(\bar{u})w = w''. \tag{3.6}$$

Observe now a complication that (3.4), by translational-invariance of the original equation (1.2), admits stationary solutions $u(x,t) \equiv \bar{u}(x+c)-\bar{u}(x)$ which do not decay. Likewise, the linearized equation (3.5) admits the stationary solutions $u(x,t) = c\bar{u}'(x)$ and $\bar{u}(x+c) - \bar{u}(x) = c\bar{u}'(x) + O(c^2)$, hence the two (nonlinear vs. linear) solutions approach each other as $c \to 0$. Also, as remarked above, we cannot expect in terms of stability more than approach to a translation for (3.4) and a multiple $c\bar{u}'$ of $\bar{u}'$ for (3.5). As spectral stability, therefore, we require that *there exist no solution $w \in L^2$, $\lambda \in \mathbb{C}$ of the eigenvalue equation* (3.6) *with* $\Re\lambda \geq 0$ *except for the "translational solution" $w = c\bar{u}'$, $\lambda = 0$.*

Adjoining first derivatives, (3.6) may be written in the form (3.3) as:

$$\begin{pmatrix} w \\ w' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda + dg(\bar{u}) & 0 \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix}, \tag{3.7}$$

where $dg(\bar{u}) = (1/2)(3\bar{u}^2 - 1)$ and $\bar{u} = -\tanh(x/2)$.

Burgers equation. We obtain the perturbation equation:

$$u_t + (f(\bar{u} + u) - f(\bar{u}))_x = u_{xx}.$$

Expanding $f(\bar{u} + u) - f(\bar{u}) = df(\bar{u})u + O(u^2)$ and discarding the nonlinear part, we get the linearized equation $u_t + (df(\bar{u})u)_x = u_{xx}$, and:

$$\lambda w + (df(\bar{u})w)' = w'', \qquad \text{where } df(\bar{u}) = \bar{u}, \quad \bar{u}(x) = -\tanh(x/2), \tag{3.8}$$

which is the eigenvalue equation. A standard trick is to integrate (3.8), substitute $\breve{w}(x) = \int_{-\infty}^{x} w(z)\,dz$, and recover the *integrated eigenvalue equation*:

$$\lambda\breve{w} + df(\bar{u})\breve{w}' = \breve{w}''. \tag{3.9}$$

LEMMA 3.1 ([ZH]). *The eigenvalues of* (3.8) *and* (3.9) *for* $\Re\lambda \geq 0$ *coincide, except at* $\lambda = 0$.

The advantage of (3.9) vs. (3.8) is that there is no bounded solution $\breve{w}$ for $\lambda = 0$, that is, we have removed the eigenvalue $\lambda = 0$ by integration. As spectral stability for Burgers equation, we hence require that *there is no solution $\breve{w} \in L^2$, $\lambda \in \mathbb{C}$ of* (3.9) *with* $\Re\lambda \geq 0$.

The equation (3.9) has the equivalent form (3.3) as:

$$\begin{pmatrix} \breve{w} \\ \breve{w}' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda & df(\bar{u}) \end{pmatrix} \begin{pmatrix} \breve{w} \\ \breve{w}' \end{pmatrix}, \qquad df(\bar{u}) = \bar{u} = -\tanh(x/2). \tag{3.10}$$

Gas dynamics. Reasoning as before, we obtain in the frame $\tilde{x} = x - st$:

$$\begin{aligned} v_t - sv_{\tilde{x}} - u_{\tilde{x}} &= 0 \\ u_t - su_{\tilde{x}} + \big((dp(\bar{v}) + \bar{v}^{-2}\bar{u}_{\tilde{x}})v\big)_{\tilde{x}} &= ((1/\bar{v})u_{\tilde{x}})_{\tilde{x}}, \end{aligned} \tag{3.11}$$

and the eigenvalue system:

$$\begin{aligned} \lambda v - sv' - u' &= 0 \\ \lambda u - su' + \big((dp(\bar{v}) + \bar{v}^{-2}\bar{u}')v\big)' &= ((1/\bar{v})u')'. \end{aligned} \tag{3.12}$$

Integrating, we obtain finally the integrated eigenvalue equations

$$\begin{aligned} \lambda\check{v} + \check{v}' - \check{u}' &= 0 \\ \lambda\check{u} + \check{u}' + (dp(\bar{v}) + \bar{v}^{-2}\bar{u}')\check{v}' &= (1/\bar{v})\tilde{u}''. \end{aligned} \tag{3.13}$$

As for the Burgers equation, we require as spectral stability *that there exist no solution* $\check{w} \in L^2$, $\lambda \in \mathbb{C}$ *of the system* (3.13) *with* $\Re\lambda \geq 0$.

Following [HLZ], we get that (3.13) can be put in the form (3.3) as:

$$\begin{pmatrix} \check{u} \\ \check{v} \\ \check{v}' \end{pmatrix}' = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\bar{v} & \lambda\bar{v} & f(\bar{v}) - \lambda \end{pmatrix} \begin{pmatrix} \check{u} \\ \check{v} \\ \check{v}' \end{pmatrix}, \tag{3.14}$$

where:

$$f(\bar{v}) = 2\bar{v} - (\gamma - 1)\Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)\Big(\frac{v_+}{\bar{v}}\Big)^\gamma - \Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)v_+^\gamma - 1. \tag{3.15}$$

MHD. Linearizing (1.8) about a parallel shock $(\bar{v}, \bar{u}_1, 0, 0, B_1^*, 0, 0)$, we get:

$$\begin{cases} v_t + v_x - u_{1x} = 0 \\ u_{1t} + u_{1x} - a\gamma\left(\bar{v}^{-\gamma-1}v\right)_x = \Big(\dfrac{u_{1x}}{\bar{v}} + \dfrac{\bar{u}_{1x}}{\bar{v}^2}v\Big)_x \\ u_{2t} + u_{2x} - \dfrac{1}{\mu_0}(B_1^*B_2)_x = \tilde{\mu}\left(\dfrac{u_{2x}}{\bar{v}}\right)_x \\ u_{3t} + u_{3x} - \dfrac{1}{\mu_0}(B_1^*B_3)_x = \tilde{\mu}\left(\dfrac{u_{3x}}{\bar{v}}\right)_x \\ (\bar{v}B_2)_t + (\bar{v}B_2)_x - (B_1^*u_2)_x = \left(\left(\dfrac{1}{\sigma\mu_0}\right)\dfrac{B_{2x}}{\bar{v}}\right)_x \\ (\bar{v}B_3)_t + (\bar{v}B_3)_x - (B_1^*u_3)_x = \left(\left(\dfrac{1}{\sigma\mu_0}\right)\dfrac{B_{3x}}{\bar{v}}\right)_x, \end{cases} \tag{3.16}$$

which is a decoupled system, consisting of the linearized isentropic gas dynamic equations in (v, u_1) about profile $(\bar{v}, \bar{u}_1)$, and two copies of an equation in variables $(u_j, \bar{v}B_j)$, $j = 2, 3$.

Introducing integrated variables $\tilde{v} := \int v$, $\tilde{u} := \int u_1$ and $w_j := \int u_j$, $\alpha_j := \int \bar{v}B_j$, $j = 2, 3$, we find that the integrated linearized eigenvalue equations decouple into the integrated linearized eigenvalue equations for gas dynamics (3.13) and two copies of:

$$\begin{cases} \lambda w + w' - \dfrac{B_1^*\alpha'}{\mu_0\bar{v}} = \mu\dfrac{w''}{\bar{v}} \\ \lambda\alpha + \alpha' - B_1^*w' = \dfrac{1}{\sigma\mu_0\bar{v}}\left(\dfrac{\alpha'}{\bar{v}}\right)' \end{cases} \tag{3.17}$$

in variables (w_j, α_j), $j = 2, 3$. As spectral stability, therefore, we require the pair of conditions that *there exist no solution* $\check{w} \in L^2$, $\lambda \in \mathbb{C}$ *with* $\Re\lambda \geq 0$ *of either* (3.13) *or the integrated transverse system* (3.17).

The transverse eigenvalue equations (3.17) may be written as a first-order system (3.3), indexed by the three parameters B_1^*, σ, and $\tilde{\mu}_0 := \sigma\mu_0$:

$$\begin{pmatrix} w \\ \tilde{\mu}w' \\ \alpha \\ \frac{\alpha'}{\sigma\mu_0\hat{v}} \end{pmatrix}' = \begin{pmatrix} 0 & 1/\tilde{\mu} & 0 & 0 \\ \lambda\hat{v} & \hat{v}/\tilde{\mu} & 0 & -\sigma B_1^*\hat{v} \\ 0 & 0 & 0 & \sigma\mu_0\hat{v} \\ 0 & -B_1^*\hat{v}/\tilde{\mu} & \lambda\hat{v} & \sigma\mu_0\hat{v}^2 \end{pmatrix} \begin{pmatrix} w \\ \mu w' \\ \alpha \\ \frac{\alpha'}{\sigma\mu_0\hat{v}} \end{pmatrix}. \tag{3.18}$$

For the four models of physical interest, we have found traveling-wave solutions and characterized spectral stability in terms of existence of solutions decaying as $x \to \pm\infty$ of an associated eigenvalue equation of linear ordinary differential type. However, such solution in general cannot be carried out explicitly. Following, we will study the question of existence numerically by an efficient "shooting" method based on the Evans function, and analytically by the examination of various asymptotic limits.

4. The Evans function: construction and an example. We now present the Evans function introduced in [E1]–[E4] and elaborated and generalized in [AGJ, PW, GZ, HuZ2]. Let L be a linear differential operator with asymptotically constant coefficients along a preferred spatial direction x. Assume that its eigenvalue equation $(L - \lambda \mathrm{Id})w = 0$ may be expressed as a 1st order ODE in an appropriate phase space:

$$W' = A(x,\lambda)W, \qquad \lim_{x\to\pm\infty} A(x,\lambda) = A_\pm(\lambda), \tag{4.1}$$

with A analytic in λ as a function from $\mathbb{C}$ to $C^1(\mathbb{R}, \mathbb{C}^{n\times n})$. Further, assume:

(h0) For $\Re\lambda > 0$, the dimension k of the stable subspace S_+ of $A_+(\lambda)$ and dimension $n-k$ of the unstable subspace U_- of $A_-(\lambda)$ sums to the dimension n of the entire phase space ("consistent splitting" [AGJ]) and that associated eigenprojections $\Pi_\pm$ (which must be automatically analytic on $\Re\lambda > 0$) extend analytically to $\Re\lambda = 0$.

(h1) In a neighborhood of any $\lambda_0 \in \{\Re\lambda \geq 0\}$, for fixed C, $\theta > 0$:

$$|A - A_\pm|(x,\lambda) \leq Ce^{-\theta|x|} \quad \text{for } x \gtrless 0.$$

Condition (h1) holds in the traveling-wave context whenever the underlying wave is a connection between hyperbolic rest points, defined as equilibria whose linearized equations have no center subspace.

Introduce now the complex ODE:

$$R' = \Pi' R, \quad R(\lambda_*) = R_*, \tag{4.2}$$

where $'$ denotes $d/d\lambda$, $\lambda_* \in \{\Re\lambda \geq 0\}$ is fixed, $\Pi = \Pi_\pm$, and $R = R_\pm$ with R_+ and R_- being $n\times k$ and $n\times(n-k)$ complex matrices. Let R_* be full rank and satisfy $\Pi(\lambda_*)R_* = R_*$, its columns constituting a basis for the stable (resp. unstable) subspace of A_+ (resp. A_-).

LEMMA 4.1 ([K, Z10, Z11]). *For $\Lambda \subset \{\Re\lambda \geq 0\}$ simply connected, there exists a unique global analytic solution R of* (4.2) *on Λ such that (i)* $\operatorname{rank} R \equiv \operatorname{rank} R^*$*, (ii) $\Pi R \equiv R$, (iii) $\Pi R' \equiv 0$.*

Property (iii) indicates that the Kato basis R is an optimal choice in the sense that it involves its minimal variation. From (ii) and (iii), we obtain the theoretically useful alternative formulation: $R' = [\Pi', \Pi]R$, where $[P, Q]$ denotes the commutator $PQ - QP$.

The following fundamental result is known as the *conjugation lemma*:

LEMMA 4.2 ([MeZ1, PZ]). *Assuming (h1), there exist coordinate changes $W = P^{\pm}Z_{\pm}$, $P_{\pm} = Id + \Theta^{\pm}$, defined and uniformly invertible on $x \gtrless 0$, with:*

$$|(\partial/\partial\lambda)^j \Theta^p_{\pm}| \leq C(j)e^{-\tilde\theta|x|} \quad \text{for } x \gtrless 0, \quad 0 \leq j, \tag{4.3}$$

for any $0 < \bar\theta < \theta$, converting (4.1) *to the constant-coefficient limiting systems $Z' = A_{\pm}Z$.*

Consequently, we see that on $\Re\lambda > 0$, the manifolds of solutions of (4.1) decaying as $x \to \pm\infty$ are spanned by $\{W_j^{\pm} := (PR_j)^{\pm}\}$, where $R_j^{\pm}$ are the columns of $R^{\pm}$ as in Lemma 4.1.

We now introduce the main:

DEFINITION 4.1 ([AGJ, PW, GZ, Z1, HuZ2, HSZ]). *Let $W_1^+, \dots, W_k^+$ and $W_{k+1}^-, \dots, W_n^-$ be analytically-chosen (in λ) bases of the manifolds of solutions decaying as $x \to +\infty$ and $-\infty$. The Evans function on Λ is:*

$$\begin{aligned} D(\lambda) &= \det(P^+R_1^+, \cdots, P^+R_k^+, P^-R_{k+1}^-, P^-R_n^-)|_{x=0}, \\ &= \det\begin{pmatrix} W_1^+ & \cdots & W_k^+ & W_{k+1}^- & \cdots & W_n^- \end{pmatrix}_{|x=0}. \end{aligned} \tag{4.4}$$

PROPOSITION 4.1 ([GJ1, GJ2, MaZ3]). *The Evans function is analytic in λ on $\Re\lambda \geq 0$. On $\Re\lambda > 0$, its zeros agree in location and multiplicity with eigenvalues of L.*

Note that the above is indeed "the" Evans function, and not "an" Evans function, since we have uniquely specified a choice of asymptotic bases $R_j^{\pm}$. In general, the Evans function, similarly to the background profile, *is only numerically evaluable.*

4.1. A fundamental example. Consider Burgers' equation, $u_t + (u^2)_x = u_{xx}$, and the family of stationary viscous shocks:

$$\hat u^{\epsilon}(x) := -\epsilon\tanh(\epsilon x/2), \qquad \lim_{z\to\pm\infty} \hat u^{\epsilon}(z) = \mp\epsilon \tag{4.5}$$

of amplitude $|u_+ - u_-| = 2\epsilon$. The integrated eigenvalue equation (3.9) appears as $w'' = \hat u^{\epsilon} w' + \lambda w$. It reduces by the linearized Hopf–Cole transformation $w = \operatorname{sech}(\epsilon x/2)z$ to the constant-coefficient linear oscillator $z'' = (\lambda + \epsilon^2/4)z$, yielding exact solutions:

$$w^{\pm}(x,\lambda) = \operatorname{sech}(\epsilon x/2)e^{\mp\sqrt{\epsilon^2/4+\lambda}x}$$

which decay as $x \to \pm\infty$, with asymptotic behavior:

$$\mathcal{W}^{\pm}(x,\lambda) \sim e^{\mu_\pm(\lambda)x}\mathcal{V}_\pm(\lambda),$$

where $\mu_\pm(\lambda) := \mp(\epsilon/2 + \sqrt{\epsilon^2/4+\lambda}$ and $\mathcal{V}_\pm := (1, \mu_\pm(\lambda))^T$ are the eigenvalues and eigenvectors of the limiting constant-coefficient equations at $x = \pm\infty$ for $\mathcal{W}_\pm := (w, w')^T_\pm$.

Defining an Evans function $\mathcal{D}(\lambda) = \det(\mathcal{W}^-, \mathcal{W}^+)_{|x=0}$, we may compute explicitly:

$$\mathcal{D}(\lambda) = -2\sqrt{\epsilon^2/4+\lambda}.$$

However, this is not "the" Evans function $D(\lambda) = \det(W^-, W^+)_{|x=0}$ specified in the previous sections, which is constructed, rather, from a special basis $W_\pm = c_\pm(\lambda)\mathcal{W}^\pm \sim e^{\mu_\pm x}V^\pm$, where $V_\pm = c_\pm(\lambda)\mathcal{V}^\pm$ are "Kato" eigenvectors determined uniquely (up to a constant factor independent of λ) by the property [K, GZ, HSZ] that there exist left eigenvectors $\tilde{V}^\pm$ with:

$$(\tilde{V}\cdot V)^\pm \equiv \text{constant}, \quad (\tilde{V}\cdot V')^\pm \equiv 0. \tag{4.6}$$

Computing dual eigenvectors $\tilde{\mathcal{V}}^\pm = (\lambda+\mu^2)^{-1}(\lambda, \mu_\pm)$ satisfying $(\tilde{\mathcal{V}}\cdot\mathcal{V})^\pm \equiv 1$, and setting $V^\pm = c_\pm\mathcal{V}^\pm$, $\tilde{V}^\pm = \mathcal{V}^\pm/c_\pm$, we find that (4.6) is equivalent to the complex ODE:

$$\dot{c}_\pm = -\Big(\frac{\tilde{V}\cdot\dot{V}}{\tilde{V}\cdot V}\Big)^\pm c_\pm = -\Big(\frac{\dot{\mu}}{2\mu-\epsilon}\Big)_\pm c_\pm, \tag{4.7}$$

which may be solved by exponentiation, yielding the general solution:

$$c_\pm(\lambda) = C(\epsilon^2/4+\lambda)^{-1/4}.$$

Initializing at a fixed nonzero point, say $c_\pm(1) = 1$, and noting that $D_\epsilon(\lambda) = c_-c_+\mathcal{D}_\epsilon(\lambda)$, we thus obtain the remarkable formula:

$$D_\epsilon(\lambda) \equiv -2\sqrt{\epsilon^2/4+1}. \tag{4.8}$$

That is, with the "Kato" normalization, the Evans function associated with a Burgers shock is not only stable (nonvanishing), but *identically constant.*

5. Abstract stability theorem: the Evans condition. Consider a viscous shock solution:

$$U(x,t) = \bar{U}(x-st), \qquad \lim_{z\to\pm\infty}\bar{U}(z) = U_\pm, \tag{5.1}$$

of a hyperbolic–parabolic system of conservation laws:

$$U_t + F(U)_x = (B(U)U_x)_x, \qquad x\in\mathbb{R},\ U,\ F\in\mathbb{R}^n,\ B\in\mathbb{R}^{n\times n}. \tag{5.2}$$

Profile $\bar{U}$ satisfies the traveling-wave ODE:

$$B(U)U' = F(U) - F(U_-) - s(U - U_-). \tag{5.3}$$

In particular, the condition that $U_\pm$ be rest points implies the *Rankine–Hugoniot condtions*:

$$F(U_+) - F(U_-) = s(U_+ - U_-) = 0 \tag{5.4}$$

of inviscid shock theory. Following this analogy, we define the *degree of compressivity* d as the total number of incoming hyperbolic characteristics toward the shock profile minus the dimension of the system, that is $d = \dim U(A_-) + \dim S(A_+) - n$, where $d = 1$ corresponds to a standard Lax shock, $d \le 0$ to an undercompressive shock, and $d \ge 2$ to an overcompressive shock. Denote $A(U) = DF(U)$ and let:

$$B_\pm := \lim_{z\to\pm\infty} B(z) = B(U_\pm), \qquad A_\pm := \lim_{z\to\pm\infty} A(z) = DF(U_\pm). \tag{5.5}$$

Following [TZ3, Z2, Z3], we introduce the structural assumptions:

(A1) $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, b nonsingular, where $U \in \mathbb{R}^n$, $U_1 \in \mathbb{R}^{n-r}$, $U_2 \in \mathbb{R}^r$, and $b \in \mathbb{R}^{r\times r}$. Moreover, $F_1(U)$ is *linear in* U (strong block structure).

(A2) There exists a smooth, positive definite matrix field $A^0(U)$, without loss of generality block-diagonal, such that $A^0_{11}A_{11}$ is symmetric, $A^0_{22}b$ is positive definite (not necessarily symmetric), and $(A^0A)_\pm$ is symmetric.

(A3) No eigenvector of $A_\pm$ lies in Ker$B_\pm$ (genuine coupling [Kaw, KSh]).

In all of the examples, $A_{11} = \alpha\mathrm{Id}$, corresponding with simple transport along fluid particle paths, with genuine coupling equivalent to A_{12} full rank.

To (A1)–(A3), we add the following hypotheses. Here and elsewhere, $\sigma(M)$ denotes the spectrum of a matrix or linear operator M.

(H0) $F, B \in C^k$, $k \ge 4$.

(H1) $\sigma(A_{11})$ is real, with constant multiplicity, $\sigma(A_{11}) < s$ or $\sigma(A_{11}) > s$.

(H2) $\sigma(A_\pm)$ is real, semisimple, and does not include s.

(H3) Considered as connecting orbits of (5.3), $\bar{U}$ lies in an ℓ-dimensional manifold, $\ell \ge 1$, of solutions (5.1) connecting fixed pair of endstates $U_\pm$.

Conditions (A1)–(A3) and (H0)–(H3) are a slightly strengthened version of the corresponding hypotheses of [MaZ4, Z2, Z3] for general systems with "real", or partially parabolic viscosity, the difference lying in the assumed in (A1) linearity of the U_1 equation. The class of equations satisfying our assumptions, though not complete, is sufficiently broad to include many models of physical interest, in particular the compressible

Navier–Stokes equations and the equations of compressible magnetohydrodynamics (MHD), expressed in Lagrangian coordinates, with either ideal or "real" van der Waals-type equation of state.

We remark that much more general assumptions (in particular, allowing Eulerian coordinates) are sufficient for the stability results reported here [MaZ4, Z1]. Lagrangian coordinates are important for later material on conditional stability and bifurcation, and so we make the stronger assumptions here for the sake of a common presentation.

Also, we notice that by the change of coordinates $x \mapsto x - st$, we may assume without loss of generality $s = 0$ as we shall do from now on. Linearizing (5.2) about $\bar{U}$ yields:

$$U_t = LU := -(AU)_x + (BU_x)_x, \tag{5.6}$$

$$B(x) := B(\bar{U}(x)), \quad A(x)V := dF(\bar{U}(x))V - (dB(\bar{U}(x))V)\bar{U}_x, \tag{5.7}$$

for which the generator L possesses [He, Sa, ZH] both a translational zero-eigenvalue and essential spectrum tangent at zero to the imaginary axis.

Conditions (H1)–(H2) imply that $U_\pm$ are nonhyperbolic rest points of ODE (5.3) expressed in terms of the U_2-coordinate, whence:

$$|\partial_x^r(\bar{U} - U_\pm)(x)| \le Ce^{-\eta|x|}, \qquad 0 \le r \le k+1, \tag{5.8}$$

for $x \gtrless 0$, some η, $C > 0$. Let d_2 be the dimension of the unstable manifold at $-\infty$ of the traveling-wave ODE expressed in the U_2 coordinate plus the dimension of the stable one at $+\infty$, minus the dimension r of the ODE.

Proposition 5.1 ([MaZ3]). *Under (A1)–(A3), (H0)–(H2), d_2 is equal to the degree of compressivity d, hence, if the traveling-wave connection is transversal, the set of all traveling-wave connections between $U_\pm$ form an ℓ-dimensional manifold $\{\bar{U}^\alpha\}$, $\alpha \in \mathbb{R}^\ell$, in the vicinity of $\bar{U}$, with $\ell = d$. In any case, for Lax or overcompressive shocks ($d \ge 1$), the linearized equations have a d-dimensional set of stationary solutions decaying as $x \to \pm\infty$, given in the transversal case by* $\mathrm{Span}\{\partial_{\alpha_j}\bar{U}^\alpha\}$. *Undercompressive connections ($d \le 0$) are never transversal.*

The following is another fundamental fact:

Lemma 5.1. *Under (A1)–(A3), (H0)–(H2):*

$$\Re\sigma(-i\xi A - \xi^2 B)_\pm \le -\frac{\theta|\xi|^2}{1+|\xi|^2} \qquad \theta > 0, \quad \forall \xi \in \mathbb{R}. \tag{5.9}$$

Consequently: (i) the essential spectrum $\sigma_{ess}(L)$ lies to the left of a curve $\lambda(\xi) = -\frac{\theta|\xi|^2}{1+|\xi|^2}$, $\theta > 0$, (ii) there exist essential spectra $\lambda_*(\xi)$ lying to the right of $\lambda(\xi) = -\frac{\theta-2|\xi|^2}{1+|\xi|^2}$, $\theta_2 > \theta > 0$, (iii) the eigenvalue equations satisfy (h0)–(h1).

Corollary 5.1. *The Evans function $D(\lambda)$ is well defined and is analytic on $\{\Re\lambda \ge 0\}$, with at least ℓ zeros at $\lambda = 0$.*

Definition 5.1. *We define the Evans stability condition as:*

(D) *$D(\cdot)$ has precisely ℓ zeros on $\{\Re\lambda \geq 0\}$, necessarily at $\lambda = 0$.*

We see that it is more difficult to verify the stability condition than disprove it, since stability is verified on an uncountable set of λ whereas, to show instability it suffices to find one eigenvalue with $\Re\lambda > 0$.

5.1. Relation to inviscid stability. As $\lambda = 0$ is an embedded eigenvalue in the essential spectrum of L, the meaning of the Evans function at $\lambda = 0$ is not immediately clear. The answer is given by the following fundamental lemma describing the low-frequency behavior of the Evans function. Recall in the Lax case that inviscid shock stability is determined [Er3, M1, M2, M3, Me] by the *Lopatinski determinant*:

$$\Delta(\lambda) := \lambda \det(r_1^-, \ldots, r_{p-1}^-, r_{p+1}^+, \ldots, r_n^+, (U_+ - U_-)), \tag{5.10}$$

where $r_1^-, \ldots r_{p-1}^-$ denote eigenvectors of $dF(U_-)$ associated with negative eigenvalues and $r_{p+1}^+, \ldots r_n^+$ denote eigenvectors of $dF(U_+)$ associated with positive eigenvalues. The shock is stable if $\Delta/\lambda \neq 0$.

Lemma 5.2 ([GZ, ZS]). *Under (A1)–(A3), (H0)–(H3),*

$$D(\lambda) = \beta\Delta(\lambda) + o(|\lambda|^\ell), \tag{5.11}$$

where β is a constant transversality coefficient which in the Lax or overcompressive case is a Wronskian of the linearized traveling-wave ODE measuring transverality of the connecting orbit $\bar U(\cdot)$, with $\beta \neq 0$ equivalent to transversality, and $\Delta(\cdot)$ is a positive homogeneous degree ℓ function which in the Lax or undercompressive case is exactly the Lopatinski determinant determing inviscid stability. In the undercompressive case, β measures "maximal transversality", or transversality of connections within the subspace spanned by the tangent spaces of stable and unstable manifolds.

It follows that a shock wave satisfies the viscous (Evans) stabilty condition *only if* it satisfies the inviscid (Lopatinski) condition along with transversality of the viscous traveling-wave connection [GZ, ZS].

5.2. Basic nonlinear stability result. We now introduce a basic viscous stability result applying to the Lax or overcompressive case. For the undercompressive case, more complicated pointwise analyses [HZ, RZ] again confirm that Evans implies nonlinear stability. For a proof, see [MaZ4], or the discussion in Section 8.1 of the more general Theorem 8.2.

Theorem 5.1 ([MaZ4, R]). *For Lax or overcompressive shocks, under (A1)–(A3), (H0)–(H3), and (D), $\bar U$ is nonlinearly orbitally stable in $L^1 \cap H^3$ with respect to perturbations with $L^1 \cap H^3$ norm and L^1-first moment $E_0 := \|\tilde U(\cdot, 0) - \hat U\|_{L^1\cap H^3}$ and $E_1 := \||x|\,|\tilde U(\cdot, 0) - \hat U|\|_{L^1}$ sufficiently small, in the sense that, for some $\alpha(\cdot) \in \mathbb{R}^\ell$, any $\varepsilon > 0$, and all $p \geq 1$,*

$$
\begin{aligned}
|U(x,t) - \bar U^{\alpha(t)}(x)|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}(E_0+E_1),\\
|U(x,t) - \bar U^{\alpha(t)}(x)|_{H^3} &\le C(1+t)^{-\frac{1}{4}}(E_0+E_1),\\
\dot\alpha(t) &\le C(1+t)^{-1+\varepsilon}(E_0+E_1),\\
\alpha(t) &\le C(1+t)^{-1/2+\varepsilon}(E_0+E_1).
\end{aligned}
\tag{5.12}
$$

Note that the rates of convergence given in (5.12) are time-algebraic rather than -exponential, in accordance with the absence of a spectral gap between neutral and decaying modes. This is a substantial technical difference from the finite-dimensional ODE theory described in Section 3.

For shock waves with $|U_+ - U_-|$ sufficiently small, and $U_\pm$ approaching a base state U_0 for which $dF(U_0)$ has a single "genuinely nonlinear" zero eigenvalue, the existence of traveling waves may be studied as a bifurcation from the constant solution, both for inviscid [L, Sm] and viscous [MP, Pe] shock waves, reducing by the Implicit Function Theorem to the scalar, Burgers case – in particular, a classical Lax-type wave. In this case, the conditions in Theorem 5.1 may be verified via the same reduction [PZ, FS, HuZ1]. Small-amplitude waves bifurcating from multiple zero eigenvalues are typically of nonclassical over- or undercompressive type, and may be unstable [AMPZ, GZ, Z6]. However, their stability should be determinable in principle by similar reduction/singular perturbation techniques to those used in the Lax case, an interesting open problem.

Except in isolated special cases [Z6], stability of large-amplitude shock waves for the moment must be verified either numerically or else in certain asymptotic limits. Whether or not there exist structural conditions for large-amplitude stability, connected with existence of a convex entropy perhaps, remains a fundamental open question, as does even the profile existence problem for large-amplitude shocks [Z1, Z2]. In what follows, we describe a workable general approach based on numerical approximation and asymptotic ODE theory.

6. Numerical approximation of the Evans function. We now discuss the general question of numerical stability analysis by approximation of the Evans function. An efficient method introduced in [EF, Er1, Er2] is the "Nyquist diagram" approach familiar from control theory, based on the Principle of the Argument of Complex Analysis. Specifically, taking the winding number around a contour $\Gamma = \partial\Lambda \subset \{\Re\lambda \ge 0\}$, where Λ is a set outside which eigenvalues may be excluded by other methods (e.g. energy estimates or asymptotic ODE theory), counts the number of unstable eigenvalues in Λ of the linearized operator about the wave, with zero winding number corresponding to stability [Br1, Br2, BrZ, BDG, HSZ, HuZ2, BHRZ, HLZ, CHNZ, HLyZ1, HLyZ2, BHZ]. Alternatively, one may use Mueller's method or any number of root-finding methods for analytic functions to locate individual roots [OZ1, LS].

A first important detail is the choice of approximate spatial infinities $L_\pm$ determining the computational domain $x \in [-L_-, L_+]$. These must be sufficiently small that computational costs remain reasonable, but sufficiently large that the solution of the variable-coefficient Evans system remains within a desired relative error of its value at $\pm\infty$ (which is a constant solution of the limiting constant-coefficient system). Indeed, the same fixed-point argument by which the exact, variable coefficient is defined (the conjugation lemma) shows that a good rule of thumb for $|A - A_\pm| \sim C_\pm e^{-\theta_\pm |x|}$, is:

$$L_\pm \sim (\log C_\pm + |\log TOL|)/\theta_\pm \tag{6.1}$$

where TOL is desired relative error tolerance [HLyZ1].

Further, the numerical approximation of Evans function breaks into 2 steps below. In both, it is important to preserve analyticity in λ, which concerns *numerical propagation of subspaces*, thus tying into large bodies of theory in numerical linear algebra [ACR, DDF, DE1, DE2, DF] and hydrodynamic stability theory [Dr, Da, NR1, NR2, NR3, NR4, Ba].

Step (i). Computation of analytic bases for stable (resp. unstable) subspaces of A_+ (resp. A_-) [BrZ, HSZ, HuZ2, Z10, Z11]. Choose a set of mesh points λ_j, $j = 0, \dots, J$ along a path $\Gamma \subset \Lambda$ and denote by $\Pi_j := \Pi(\lambda_j)$ and R_j the approximation of $R(\lambda_j)$. Typically, $\lambda_0 = \lambda_J$, i.e., Γ is a closed contour.

Given a matrix A, one may efficiently ($\sim 32n^3$ operations [GvL, SB]) compute by "ordered" Schur decomposition ($A = QUQ^{-1}$, Q orthogonal and U upper triangular, for which also the diagonal entries of U are ordered in increasing real part) an orthonormal basis $\check R_u := (Q_{k+1}, \dots, Q_n)$, of its unstable subspace, where $Q_{k+1}, \dots, Q_n$ are the last $n-k$ columns of Q, and $n-k$ is the dimension of the unstable subspace. Performing the same procedure for $-A$, A^*, and $-A^*$ we obtain orthonormal bases $\check R_s$, $\check L_u$, $\check L_s$ also for the stable subspace of A and the unstable and stable subspaces of A^*, from which we compute the stable and unstable eigenprojections in numerically well-conditioned manner via:

$$\Pi_s := \check R_s (\check L_s^* \check R_s)^{-1} \check L_s^*, \quad \Pi_u := \check R_u (\check L_u^* \check R_u)^{-1} \check L_u^*. \tag{6.2}$$

Applying this to matrices $A_j^\pm := A_\pm(\lambda_j)$, we obtain the projectors $\Pi_j^\pm := \Pi_\pm(\lambda_j)$. Hereafter, we consider $\Pi_j^\pm$ as known quantities.

Approximating $\Pi'(\lambda_j)$ to first order by the finite difference $(\Pi_{j+1} - \Pi_j)/(\lambda_{j+1} - \lambda_j)$ and substituting this into a first-order Euler scheme gives:

$$R_{j+1} = R_j + (\lambda_{j+1} - \lambda_j) \frac{\Pi_{j+1} - \Pi_j}{\lambda_{j+1} - \lambda_j} R_j,$$

or $R_{j+1} = R_j + \Pi_{j+1} R_j - \Pi_j R_j$, yielding by the property $\Pi_j R_j = R_j$ (preserved exactly by the scheme) the simple greedy algorithm:

$$R_{j+1} = \Pi_{j+1} R_j. \tag{6.3}$$

It is a remarkable fact [Z10, Z11] (a consequence of Lemma 4.1) that, up to numerical error, evolution of (6.3) about a closed loop $\lambda_0 = \lambda_J$ yields the original value $R_J = R_0$.

To obtain a second-order discretization of (4.2), we approximate

$$R_{j+1} - R_j \approx \Delta\lambda_j \Pi'_{j+1/2} R_{j+1/2},$$

where $\Delta\lambda_j := \lambda_{j+1} - \lambda_j$. Noting that $R_{j+1/2} \approx \Pi_{j+1/2} R_j$ to second order by (6.3), and approximating $\Pi_{j+1/2} \approx \frac{1}{2}(\Pi_{j+1} + \Pi_j)$, and $\Pi'_{j+1/2} \approx (\Pi_{j+1} - \Pi_j)/\Delta\lambda_j$, we obtain, combining and rearranging: $R_{j+1} = R_j + \frac{1}{2}(\Pi_{j+1} - \Pi_j)(\Pi_{j+1} + \Pi_j)R_j$. Stabilizing by following with a projection Π_{j+1}, we obtain the reduced second-order explicit scheme:

$$R_{j+1} = \Pi_{j+1}[I + \frac{1}{2}\Pi_j(I - \Pi_{j+1})]R_j. \tag{6.4}$$

This is the version we recommend for serious computations. For individual numerical experiments the simpler greedy algorithm (6.3) suffices.

Step (ii). Propagation of the bases by ODE (4.1) **on a sufficiently large interval** $x \in [M, 0]$ **(resp.** $x \in [-M, 0]$**).** Our basic principles for efficient numerical integration of (4.1) are readily motivated by consideration of the simpler constant-coefficient case $W_x = AW$ with $A \equiv \text{const}$. We must avoid two main potential pitfalls, which are: the wrong direction of integration, and existence of "parasitic modes" [Z10].

For general systems of equations, the dimension of the stable subspace of A typically involves two or more eigenmodes, with distinct decay rates. In practice, parasitic faster-decaying modes will tend to take over slower-decaying modes, preventing their resolution. Degradation of results from parasitic modes is of the same rough order as that resulting from integrating in the wrong spatial direction.

Resolution one: the centered exterior product scheme. Denote $R^+ = (R_1^+, \dots, R_k^+)$ and $R^- = (R_1^-, \dots, R_{n-k}^-)$. Define the initializing wedge products:

$$\mathcal{R}_{S_+} := R_1^+ \wedge \cdots \wedge R_k^+ \quad \text{and} \quad \mathcal{R}_{S_-} := R_1^- \wedge \cdots \wedge R_{n-k}^-. \tag{6.5}$$

The Evans determinant is then recovered through the isomorphism:

$$\begin{aligned} &\det(W_1^+, \cdots, W_k^+, W_{k+1}^-, \cdots, W_n^-) \\ &\qquad\qquad \sim (W_1^+ \wedge \cdots \wedge W_k^+) \wedge (W_{k+1}^- \wedge \cdots \wedge W_n^-); \end{aligned} \tag{6.6}$$

see [AS, Br1, Br2, BrZ, BDG, AlB] and ancestors [GB, NR1, NR2, NR3, NR4]. This reduces the problem to the case $k = 1$. We may further optimize by factoring out the expected asymptotic decay rate $e^{\mu x}$ of the single decaying mode and solving the "centered" equation:

$$Z' = (A - \mu \text{Id})Z, \quad Z(+\infty) = r : A_+ r = \mu r \tag{6.7}$$

for $Z = e^{-\mu x}W$, which is now asymptotically an equilibrium as $x \to +\infty$. With these preparations, one obtains excellent results [BDG, HuZ2]; however, omitting any one of them leads to a loss of efficiency of as much as an order of magnitude [HuZ3, BZ].

Resolution two: the polar coordinate method. Unfortunately, the dimension $\binom{n}{k}$ of the phase space for the exterior product in center exterior product scheme grows exponentially with n, since k is $\sim n/2$ in typical applications. This limits its usefulness to $n \leq 10$ or so, whereas the Evans system arising in compressible MHD is size $n = 15$, $k = 7$ [BHZ], giving a phase space of size $\binom{n}{k} = 6,435$: clearly impractical. A more compact, but nonlinear, alternative is the polar coordinate method of [HuZ2], in which the exterior products of the columns of $W_\pm$ are represented in "polar coordinates" $(\Omega, \gamma)_\pm$, where the columns of $\Omega_+ \in \mathbb{C}^{n\times k}$ and $\Omega_- \in \mathbb{C}^{(n-k)\times k}$ are orthonormal bases of the subspaces spanned by the columns of $W_+ := \begin{pmatrix} W_1^+ & \cdots & W_k^+ \end{pmatrix}$ and $W_- := \begin{pmatrix} W_{k+1}^- & \cdots & W_n^- \end{pmatrix}$.

For each $\lambda \in \Lambda$, compute matrices $\Omega_\pm(\lambda)$ whose columns form orthonormal bases for $S_\pm$, by the same ordered Schur decomposition used in the computation of $\Pi_\pm$. Equating: $\Omega_+\tilde\alpha_+(\lambda) = R_+(\lambda)$, $\Omega_-\tilde\alpha_-(\lambda) = R_-(\lambda)$, for some $\tilde\alpha_\pm$, we obtain:

$$\tilde\alpha_+(\lambda) = \Omega_+^* R_+(\lambda), \quad \tilde\alpha_-(\lambda) = \Omega_-^* R_-(\lambda),$$

and therefore the exterior product of the columns of $R_\pm$ is equal to the exterior product of the columns of $\Omega_\pm$ times $\tilde\gamma_\pm(\lambda) := \det(\Omega^* R)_\pm(\lambda)$.

Imposing now the choice $\Omega^*\Omega'$ to fix a specific realization (similar to $PP' = 0$ determining Kato's ODE), we obtain after a brief calculation [HuZ2] the block-triangular system:

$$\begin{aligned} \Omega' &= (I - \Omega\Omega^*)A\Omega, \\ (\log\tilde\gamma)' &= \text{trace }(\Omega^* A\Omega) - \text{trace }(\Omega^* A\Omega)(\pm\infty), \end{aligned} \tag{6.8}$$

$\tilde\gamma := \tilde\gamma e^{-\text{trace }(\Omega^* A\Omega)(\pm\infty)x}$, for which the "angular" Ω-equation is exactly the continuous orthogonalization method of Drury [Dr, Da], and the "radial" $\tilde\gamma$-equation, given Ω, may be solved by simple quadrature. Note that for constant A, the invariant subspaces Ω of A are equilibria of the flow, and solutions of the $\tilde\gamma$-equation are constant, so that any first-order or higher numerical scheme resolves $\log\tilde\gamma$ exactly; thus, the $\tilde\gamma$-equation may in practice be solved together with the Ω-equation with good results.

The Evans function is recovered, finally, through the relation:

$$\begin{aligned} D(\lambda) &= \det(W_1^+, \cdots, W_k^+, W_{k+1}^-, \cdots, W_n^-)|_{x=0} \\ &= \tilde\gamma_+\tilde\gamma_- \det(\Omega^+, \Omega^-)|_{x=0}. \end{aligned} \tag{6.9}$$

This method performs comparably to the exterior product method for small systems [HuZ2], but does not suffer from exponential growth in complexity with respect to the system's size.

As the integration of Kato's ODE (6.8) is carried out on a bounded closed curve, standard error estimates apply and convergence is essentially automatic. In contrast to integration in x of the Evans system, integration in λ of the Kato system is a one-time cost, so not a rate-determining factor in the performance of the overall code. However, the computation time *is* sensitive to the number of mesh points in λ at which the Evans function is computed, so this should be held down as much as possible.

7. Global stability analysis: a case study. To show the full power of the Evans function approach, we now carry out a global stability analysis for the equations of isentropic gas dynamics (1.6) by a combination of energy estimates, numerical Evans function evaluation, and asymptotic ODE. The result is that for an isentropic γ-law gas, $\gamma \in [1,3]$, viscous shock profiles are (numerically) unconditionally stable.

Step (i). Preliminary estimates. Taking $(x,t,v,u,a_0) \mapsto (-\varepsilon s(x-st), \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s), a_0\varepsilon^{-\gamma-1}s^{-2})$, with ε so that $0 < v_+ < v_- = 1$, we consider stationary solutions $(\bar v, \bar u)(x)$ of:

$$v_t + v_x - u_x = 0, \qquad u_t + u_x + (av^{-\gamma})_x = \Big(\frac{u_x}{v}\Big)_x \tag{7.1}$$

where $a = a_0\varepsilon^{-\gamma-1}s^{-2}$. Steady shock profiles of (7.1) satisfy:

$$v' = H(v, v_+) := v(v - 1 + a(v^{-\gamma} - 1)), \tag{7.2}$$

where a is found by $H(v_+, v_+) = 0$, yielding Rankine-Hugoniot condition:

$$a = -\frac{v_+ - 1}{v_+^{-\gamma} - 1} = v_+^\gamma \frac{1 - v_+}{1 - v_+^\gamma}. \tag{7.3}$$

Evidently, $a \to \gamma^{-1}$ in the weak shock limit $v_+ \to 1$, while $a \sim v_+^\gamma$ in the strong shock limit $v_+ \to 0$. In this scaling, large-amplitude limit corresponds to $v_+ \to 0$, or $\rho_+ = 1/v_+ \to \infty$. Linearizing (7.1) about the profile $(\bar v, \bar u)$ and integrating with respect to x, we obtain:

$$\lambda v + v' - u' = 0, \qquad \lambda u + u' - \frac{h(\bar v)}{\bar v^{\gamma+1}} v' = \frac{u''}{\bar v}, \tag{7.4}$$

where $h(\bar v) = -\bar v^{\gamma+1} + a(\gamma-1) + (a+1)\bar v^\gamma$. Spectral stability of $\hat v$ corresponds to nonexistence of solutions of (7.4) decaying at $x = \pm\infty$ for $\Re\lambda \ge 0$ [HuZ2, BHRZ, HLZ].

PROPOSITION 7.1 ([BHRZ]). *For each $\gamma \ge 1$, $0 < v_+ \le 1$, (7.2) has a unique (up to translation) monotone decreasing solution $\hat v$ decaying to its endstates with a uniform exponential rate. For $0 < v_+ \le \frac{1}{12}$ and $\hat v(0) = v_+ + \frac{1}{12}$, there holds:*

$$|\bar v(x) - v_+| \le \Big(\frac{1}{12}\Big) e^{-\frac{3x}{4}} \quad \forall x \ge 0, \qquad |\bar v(x) - v_-| \le \Big(\frac{1}{4}\Big) e^{\frac{x+12}{2}} \quad \forall x \le 0. \tag{7.5}$$

Above, existence and monotonicity follow trivially by the fact that (7.2) is a scalar first-order ODE with convex righthand side. Exponential convergence as $x \to +\infty$ follows by:

$$H(v, v_+) = (v - v_+)\left(v - \left(\frac{1 - v_+}{1 - v_+^\gamma}\right)\left(\frac{1 - (\frac{v_+}{v})^\gamma}{1 - (\frac{v_+}{v})}\right)\right),$$

whence $v - \gamma \le \frac{H(v,v_+)}{v - v_+} \le v - (1 - v_+)$ by $1 \le \frac{1-x^\gamma}{1-x} \le \gamma$ for $0 \le x \le 1$.

Writing now (7.4) as $U_t + A(x)U_x = B(x)U_{xx}$, with $B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\bar v} \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -1 \\ -\frac{h(\bar v)}{\bar v^{\gamma+1}} & 1 \end{pmatrix}$, we see that $S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\bar v^{\gamma+1}}{h(\bar v)} \end{pmatrix}$ symmetrizes A, B. Taking the L^2 complex inner product of SU against the equations yields $\Re\lambda\langle U, SU\rangle + \langle U', SBU'\rangle = -\langle u, g(\hat v)u\rangle$, where $g > 0$ under condition (7.6) below; see [MN, BHRZ]. Consequently:

PROPOSITION 7.2 ([MN]). *Viscous shocks of* (1.6) *are spectrally stable whenever:*

$$\Big(\frac{v_+^{\gamma+1}}{a\gamma}\Big)^2 + 2(\gamma - 1)\Big(\frac{v_+^{\gamma+1}}{a\gamma}\Big) - (\gamma - 1) \ge 0. \tag{7.6}$$

In particular, stability condition (7.6) holds for $|v_+ - 1| << 1$. Further energy estimates related to those of Proposition 7.2 yield:

PROPOSITION 7.3 ([BHRZ]). *Nonstable eigenvalues λ of* (7.4)*, i.e., eigenvalues with nonnegative real part, are confined for any $\gamma \ge 1$, $0 < v_+ \le 1$ to the region Λ defined by*

$$\Re(\lambda) + |\Im(\lambda)| \le \Big(\sqrt{\gamma} + \frac{1}{2}\Big)^2. \tag{7.7}$$

Step (ii). The Evans function formulation. We express (7.4) as a first-order system: $W' = A(x, \lambda)W$, where:

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda \bar v & \lambda \bar v & f(\bar v) - \lambda \end{pmatrix}, \quad W = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}, \quad ' = \frac{d}{dx}, \tag{7.8}$$

$$f(\bar v) = 2\bar v - (\gamma - 1)\left(\frac{1 - v_+}{1 - v_+^\gamma}\right)\left(\frac{v_+}{\bar v}\right)^\gamma - \left(\frac{1 - v_+}{1 - v_+^\gamma}\right)v_+^\gamma - 1. \tag{7.9}$$

Eigenvalues of (7.4) correspond to nontrivial solutions W for which the boundary conditions $W(\pm\infty) = 0$ are satisfied. Because $A(x, \lambda)$ as a function of $\bar{v}$ is asymptotically constant in x, the behavior of (7.8) near $x = \pm\infty$ is governed by the limiting systems: $W' = A_\pm(\lambda)W$, where $A_\pm(\lambda) = A(\pm\infty, \lambda)$. From this we readily find on the (nonstable) domain $\Re\lambda \geq 0$, $\lambda \neq 0$ that there is a one-dimensional unstable manifold $W_1^-(x)$ of solutions decaying at $x = -\infty$ and a two-dimensional stable manifold $W_2^+(x) \wedge W_3^+(x)$ of solutions decaying at $x = +\infty$, analytic in λ, with asymptotic behavior $W_j^\pm(x,\lambda) \sim e^{\mu_\pm(\lambda)x} V_j^\pm(\lambda)$ as $x \to \pm\infty$, where $\mu_\pm(\lambda)$ and $V_j^\pm(\lambda)$ are eigenvalues and analytically chosen eigenvectors of $A_\pm(\lambda)$.

A standard choice of eigenvectors $V_j^\pm$ [BrZ], uniquely specifying D (up to constant factor) is obtained by Kato's ODE, as in section 4.1, whose solution can be alternatively characterized by the property that there exist corresponding left eigenvectors $\tilde{V}_j^\pm$ such that:

$$(\tilde{V} \cdot V)^\pm \equiv const, \quad (\tilde{V} \cdot V')^\pm \equiv 0.$$

Defining the *Evans function* D associated with operator L as:

$$D(\lambda) = \det(W_1^- W_2^+ W_3^+)_{|x=0}, \tag{7.10}$$

we find [Z2] that D is analytic on $\Re\lambda \geq 0$, with eigenvalues of L corresponding in location and multiplicity to zeroes of D.

7.1. Main results. Taking a formal limit as $v_+ \to 0$ of (7.1) and recalling that $a \sim v_+^\gamma$, we get:

$$v_t + v_x - u_x = 0, \qquad u_t + u_x = \left(\frac{u_x}{v}\right)_x \tag{7.11}$$

corresponding to a *pressure-less gas*, or $\gamma = 1$. The limiting profile equation $v' = v(v-1)$ has solution $\hat{v}_0(x) = (1 - \tanh(x/2))/2$, and the limiting eigenvalue system is $W' = A^0(x,\lambda)W$:

$$\begin{aligned} A^0(x,\lambda) &= \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\bar{v}_0 & \lambda\bar{v}_0 & f_0(\bar{v}_0) - \lambda \end{pmatrix}, \\ f_0(\bar{v}_0) &= 2\bar{v}_0 - 1 = -\tanh(x/2). \end{aligned} \tag{7.12}$$

Observe that $A_+^0(\lambda) := A^0(+\infty, \lambda)$ is nonhyperbolic for all λ, having eigenvalues $0, 0, -1-\lambda$. In particular, the stable manifold drops to dimension one in the limit $v_+ \to 0$, and so the prescription of an associated Evans function is *underdetermined*.

This difficulty is resolved by a careful boundary-layer analysis in [HLZ], determining a special "slow stable" mode $V_2^+ \pm (1,0,0)^T$ – the common limiting direction of slow stable and unstable modes as $v_+ \to 0$, collapsing to a Jordan block– augmenting the "fast stable" mode $V_3 := (a^{-1}(\lambda/a + 1), a^{-1}, 1)^T$ associated with the single stable eigenvalue $a = -1-\lambda$ of A_+^0. This determines a *limiting Evans function* $D^0(\lambda)$ by the prescription (7.10).

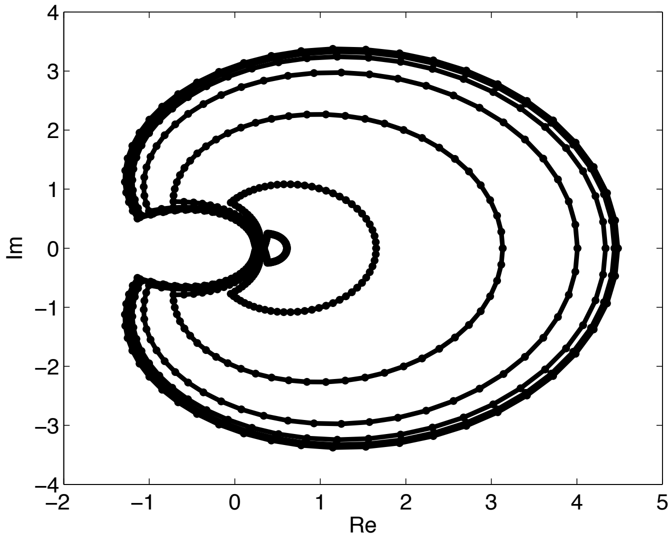

FIG. 1. *Convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas (reproduced from [HLZ] with permission of the authors).*

THEOREM 7.1 ([HLZ]). *For λ in any compact subset of $\Re\lambda \geq 0$, $D(\lambda)$ converges uniformly to $D^0(\lambda)$ as $v_+ \to 0$. Moreover, the limiting function D^0 is nonzero on $\Re\lambda \geq 0$.*

Consequently, for any $\gamma \geq 1$, isentropic Navier–Stokes shocks are stable in the strong shock limit, i.e., for v_+ sufficiently small.

Stability in the weak shock limit is known [MN].[2] For the intermediate strong shocks, one needs to consider a bounded parameter range on which the Evans function may be effiently computed numerically [BrZ]. Specifically, we may map a semicircle $\partial\{\Re\lambda \geq 0\} \cap \{|\lambda| \leq 10\}$ enclosing Λ for $\gamma \in [1, 3]$ by D^0 and compute the winding number of its image about the origin to determine the number of zeroes of D^0 within the semicircle, and thus within Λ. Such a study was carried out systematically in [BHRZ] on the parameter range $\gamma \in [1, 3]$, for shocks with Mach number $M \in [1, 3,000]$, which corresponds on $\gamma \in [1, 2.5]$ to $v_+ \geq 10^{-3}$, with the result of stability for all values tested.

In Figure 1, we superimpose on the numerically computed image of a semicircle by D^0, its image by D, for a monatomic gas $\gamma \approx 1.66$ at successively higher Mach numbers $v+$ =`1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`, showing convergence of D to D^0 in the strong shock limit as $v_+ \to 0$ and its convergence to a constant in the weak shock limit $v_+ \to 1$.

[2]For an extension to general hyperbolic-parabolic systems, see [HuZ1].

The displayed contours are, to the scale visible by eye, "monotone" in v_+, or nested, one within the other, with lower-Mach number contours essentially "trapped" within higher-Mach number contours, and all contours interpolating smoothly between this and the inner, constant limit. Behavior for other $\gamma \in [0, 3]$ is entirely similar [HLZ].

8. Conditional stability of viscous shock waves [Z4, Z7, Z8]. We next consider the situation of a viscous shock *with one or more strictly unstable but no neutrally unstable eigenvalues*, and seek to describe the nearby phase portrait in terms of invariant manifolds and behavior therein. Specifically, for shock waves of systems of conservation laws with artificial viscosity, we construct a center stable manifold and show that the shock is conditionally (nonlinearly) stable with respect to this codimension p set of initial data, where p is the number of unstable eigenvalues. Note that this includes a rigorous *nonlinear instability* result in the case of unstable eigenvalues. Such conditionally stable shock waves play an important role in asymptotic behavior as metastable states [AMPZ, GZ].

For simplicity of exposition, we treat the easier case of a Lax shock of a semilinear, identity viscosity system [Z7]. For the general case, including in particular the class of equations described in Section 5, see [Z8, Z9].

Consider a viscous shock $u(x,t) = \bar{u}(x)$, $\lim_{z\to\pm\infty} \bar{u}(z) = u_\pm$ of:

$$u_t + f(u)_x = u_{xx}, \qquad u,\, f \in \mathbb{R}^n, \quad x\,, t \in \mathbb{R}, \tag{8.1}$$

under the basic assumptions:

(H0) $f \in C^{k+2}$, $k \geq 2$.

(H1) $A_\pm := df(u_\pm)$ have simple, real, nonzero eigenvalues.

Linearizing (8.1) about $\bar{u}$ yields equations with generator L [He, Sa, ZH]:

$$u_t = Lu := -(df(\bar{u})u)_x + u_{xx}, \tag{8.2}$$

possessing both a translational zero-eigenvalue and essential spectrum tangent at zero to the imaginary axis. The absence of a spectral gap between neutral (zero real part) and stable (negative real part) spectra of L prevents the usual ODE-type decomposition of the flow near $\bar{u}$ into invariant stable, center, and unstable manifolds. Our first result asserts that we can still determine a center stable manifold, and that this may be chosen to respect the underlying translation-invariance of (8.1). We omit the proof of this standard observation; for a particularly short treatment, see [Z7].

Theorem 8.1 ([Z7]). *Under assumptions (H0)–(H1), there exists, in an H^2 neighborhood of the set of translates of $\bar{u}$, a codimension-p translation invariant C^k (with respect to H^2) center stable manifold $\mathcal{M}_{cs}$, tangent at $\bar{u}$ to the center stable subspace Σ_{cs} of L. It is (locally) invariant under the forward time-evolution of* (8.1) *and contains all solutions that remain bounded and sufficiently close to a translate of $\bar{u}$ in forward time, where p is the (necessarily finite) number of unstable eigenvalues of L.*

We add to (H0)–(H1) the hypothesis that $\bar u$ be a *Lax-type shock*:

(H2) The dimensions of the unstable subspace of $df(u_-)$ and the stable subspace of $df(u_+)$ sum to $n+1$.

We assume further the following *spectral genericity* conditions.

(D1) L has no nonzero imaginary eigenvalues.

(D2) The orbit $\bar u(\cdot)$ is a transversal connection of standing wave equation $\bar u_x = f(\bar u) - f(u_-)$.

(D3) The associated inviscid shock (u_-, u_+) is hyperbolically stable, i.e.,

$$\det(r_1^-, \dots, r_{p-1}^-, r_{p+1}^+, \dots, r_n^+, (u_+ - u_-)) \neq 0, \tag{8.3}$$

where $r_1^-, \dots r_{p-1}^-$ denote eigenvectors of $df(u_-)$ associated with negative eigenvalues and $r_{p+1}^+, \dots r_n^+$ denote eigenvectors of $df(u_+)$ associated with positive eigenvalues.

As discussed in [ZH, MaZ1], (D2)–(D3) correspond in the absence of a spectral gap to a generalized notion of simplicity of the embedded eigenvalue $\lambda = 0$ of L. Thus, (D1)–(D3) together yield that there are no additional (usual or generalized) eigenvalues on the imaginary axis other than the translational eigenvalue at $\lambda = 0$. Hence the shock is not in transition between different degrees of stability, but has stability properties that are insensitive to small variations in parameters.

With these assumptions, we obtain our second main result characterizing stability properties of $\bar u$. In the case $p = 0$, this reduces to the nonlinear orbital stability result established in [ZH, MaZ1, MaZ2, MaZ3, Z2, Z4].

Theorem 8.2 ([Z7]). *Under (H0)–(H2) and (D1)–(D3), $\bar u$ is nonlinearly orbitally stable for all its sufficiently small perturbations in $L^1 \cap H^2$ lying on $\mathcal{M}_{cs}$ and its translates. That is:*

$$\begin{aligned} |u(x,t) - \bar u(x - \alpha(t))|_{L^p} &\le C(1+t)^{-\frac12(1-\frac1p)} |u(x,0) - \bar u(x)|_{L^1 \cap H^2}, \\ |u(x,t) - \bar u(x - \alpha(t))|_{H^2} &\le C(1+t)^{-\frac14} |u(x,0) - \bar u(x)|_{L^1 \cap H^2}, \\ \dot\alpha(t) &\le C(1+t)^{-\frac12} |u(x,0) - \bar u(x)|_{L^1 \cap H^2}, \\ \alpha(t) &\le C |u(x,0) - \bar u(x)|_{L^1 \cap H^2}, \end{aligned} \tag{8.4}$$

for some $\alpha(\cdot)$, and all L^p. Moreover, $\bar u$ is orbitally unstable with respect to small H^2 perturbations not lying in $\mathcal{M}_{cs}$, in the sense that the corresponding solution leaves a fixed-radius neighborhood of the set of translates of $\bar u$ in finite time.

8.1. Conditional stability analysis. Define the perturbation variable $v(x,t) := u(x + \alpha(t), t) - \bar u(x)$ for u a solution of (8.1), where α is to be specified later in a way appropriate for the task at hand. Subtracting the equations for $u(x + \alpha(t), t)$ and $\bar u(x)$, we obtain:

$$v_t - Lv = N(v)_x + \partial_t \alpha (\phi + \partial_x v), \tag{8.5}$$

where $L := -\partial_x df(\bar u) + \partial_x^2$ as in (8.2) denotes the linearized operator about $\bar u$, $\phi = \bar u_x$, and:

$$N(v) := -(f(\bar u + v) - f(\bar u) - df(\bar u)v). \tag{8.6}$$

So long as $|v|_{H^1}$ (hence $|v|_{L^\infty}$ and $|u|_{L^\infty}$) remains bounded, there holds:

$$N(v)=O(|v|^2),\ \partial_x N(v)=O(|v||\partial_x v|),\ \partial_x^2 N(v)=O(|\partial_x|^2+|v||\partial_x^2 v|). \tag{8.7}$$

LEMMA 8.1 ([Z7]). *Let Π_u denote the eigenprojection of L onto its unstable subspace Σ_u, and $\Pi_{cs} = \mathrm{Id} - \Pi_u$ the eigenprojection onto its center stable subspace Σ_{cs}. Assuming (H0)–(H1), there exists $\tilde\Pi_j$ such that $\Pi_j \partial_x = \partial_x \tilde\Pi_j$ for $j = u$, cs and, for all $1 \le p \le \infty$, $0 \le r \le 4$:*

$$|\Pi_u|_{L^p \to W^{r,p}}, |\tilde\Pi_u|_{L^p \to W^{r,p}} \le C, \quad |\tilde\Pi_{cs}|_{Wr,p \to W^{r,p}},\ |\tilde\Pi_{cs}|_{Wr,p \to W^{r,p}} \le C.$$

Let now $G_{cs}(x,t;y) := \Pi_{cs} e^{Lt} \delta_y(x)$ denote the Green kernel of the linearized solution operator on the center stable subspace Σ_{cs}. The following is a consequence of the detailed pointwise bounds established in [TZ2, MaZ1].

THEOREM 8.3 ([TZ2, MaZ1]). *Assuming (H0)–(H2), (D1)–D(3), the center stable Green function may be decomposed as $G_{cs} = E + \tilde G$, where*

$$E(x,t;y) = \partial_x \bar u(x) e(y,t), \tag{8.8}$$

$$e(y,t) = \sum_{a_k^- > 0} \left(errfn\left(\frac{y + a_k^- t}{\sqrt{4(t+1)}} \right) - errfn\left(\frac{y - a_k^- t}{\sqrt{4(t+1)}} \right) \right) l_k^-(y) \tag{8.9}$$

for $y \le 0$ and symmetrically for $y \ge 0$, $l_k^- \in \mathbb{R}^n$ constant. Above, $a_j^\pm$ are the eigenvalues of $df(u_\pm)$, $x^\pm$ denotes the positive/negative part of x. Moreover:

$$\left| \int_{-\infty}^{+\infty} \partial_x^s \tilde G(\cdot,t;y) f(y) dy \right|_{L^p} \le C(1 + t^{-\frac{s}{2}}) t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} |f|_{L^q}, \tag{8.10}$$

$$\left| \int_{-\infty}^{+\infty} \partial_x^s \tilde G_y(\cdot,t;y) f(y) dy \right|_{L^p} \le C(1 + t^{-\frac{s}{2}}) t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} |f|_{L^q}, \tag{8.11}$$

for all $t \ge 0$, $0 \le s \le 2$, some $C > 0$, for any $1 \le q \le p$ (equivalently, $1 \le r \le p$) and $f \in L^q$, where $1/r + 1/q = 1 + 1/p$. The kernel e satisfies:

$$|e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p} \le C t^{-\frac{1}{2}(1-1/p)}, \qquad |e_{ty}(\cdot,t)|_{L^p} \le C t^{-\frac{1}{2}(1-1/p)-1/2},$$

for all $t > 0$. Moreover, for $y \le 0$ we have the pointwise bounds:

$$|e_y(y,t)|, |e_t(y,t)| \le C(t+1)^{-\frac{1}{2}} \sum_{a_k^- > 0} \left(e^{-\frac{(y + a_k^- t)^2}{M(t+1)}} + e^{-\frac{(y - a_k^- t)^2}{M(t+1)}} \right),$$

$$|e_{ty}(y,t)| \le C(t+1)^{-1} \sum_{a_k^->0} \Big(e^{-\frac{(y+a_k^- t)^2}{M(t+1)}} + e^{-\frac{(y-a_k^- t)^2}{(t+1)t}}\Big),$$

for $M>0$ sufficiently large, and symmetrically for $y \ge 0$.

8.2. Reduced equations and shock location. Recalling that $\partial_x \bar u$ is a stationary solution of the equations $u_t = Lu$, so that $L\partial_x \bar u = 0$, or:

$$\int_{-\infty}^{\infty} G(x,t;y)\bar u_x(y)dy = e^{Lt}\bar u_x(x) = \partial_x \bar u(x),$$

we have, applying Duhamel's principle to (8.5):

$$\begin{aligned} v(x,t) = &\int_{-\infty}^{\infty} G(x,t;y)v_0(y)\,dy \\ &- \int_0^t \int_{-\infty}^{\infty} G_y(x,t-s;y)(N(v)+\dot\alpha v)(y,s)\,dy\,ds + \alpha(t)\partial_x \bar u(x). \end{aligned}$$

Following [ZH, Z4, MaZ2, MaZ3], define (with e as in (8.9)):

$$\begin{aligned} \alpha(t) = & -\int_{-\infty}^{\infty} e(y,t)v_0(y)\,dy \\ & + \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(N(v)+\dot\alpha\, v)(y,s)dyds. \end{aligned} \tag{8.12}$$

Recalling that $G = E + G_u + \tilde G$, we obtain the *reduced equations*:

$$\begin{aligned} v(x,t) = &\int_{-\infty}^{\infty} (G_u+\tilde G)(x,t;y)v_0(y)\,dy \\ &- \int_0^t \int_{-\infty}^{\infty} (G_u+\tilde G)_y(x,t-s;y)(N(v)+\dot\alpha v)(y,s)dy\,ds. \end{aligned} \tag{8.13}$$

Differentiating (8.12) in t, and noting that $e_y(y,s)\to 0$ as $s\to 0$, we get:

$$\begin{aligned} \dot\alpha(t) = & -\int_{-\infty}^{\infty} e_t(y,t)v_0(y)\,dy \\ & + \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(v)+\dot\alpha v)(y,s)\,dy\,ds. \end{aligned} \tag{8.14}$$

As discussed in [Go, Z4, MaZ2, MaZ3, GMWZ1, BeSZ], α may be considered as defining a notion of approximate shock location.

We further obtain the following nonlinear damping estimate:

Proposition 8.1 ([MaZ3]). *Assume (H0)-(H3), let $v_0 \in H^2$, and suppose that for $0 \le t \le T$, the H^2 norm of a perturbation v remains bounded by a sufficiently small constant, for u a solution of* (8.1)*. Then, for some constants $\theta_{1,2}>0$, for all $0\le t\le T$:*

$$\|v(t)\|_{H^2}^2 \le Ce^{-\theta_1 t}\|v(0)\|_{H^2}^2 + C\int_0^t e^{-\theta_2(t-s)}(|v|_{L^2}^2 + |\dot\alpha|^2)(s)\,ds. \tag{8.15}$$

Proof. ([Z7]) Subtracting the equations for $u(x+\alpha(t),t)$ and $\bar u(x)$, we may write the perturbation equation for v alternatively as:

$$v_t + \left(\int_0^1 df\big(\bar u(x) + \tau v(x,t)\big)\, d\tau\, v\right)_x - v_{xx} = \dot\alpha(t)\partial_x \bar u(x) + \dot\alpha(t)\partial_x v. \quad (8.16)$$

Observing that $\partial_x^j(\partial_x \bar u)(x) = O(e^{-\eta|x|})$ is bounded in L^1 norm for $j \le 2$, we take the L^2 product in x of $\sum_{j=0}^2 \partial_x^{2j} v$ against (8.16), integrate by parts and arrive at:

$$\partial_t \|v\|_{H^2}^2(t) \le -\theta \|\partial_x^3 v\|_{L^2}^2 + C\left(\|v\|_{H^2}^2 + |\dot\alpha(t)|^2\right),$$

for $\theta > 0$, for $C > 0$ sufficiently large, and so long as $\|v\|_{H^2}$ remains bounded. Using the Sobolev interpolation inequality

$$\|v\|_{H^2}^2 \le \tilde C^{-1} \|\partial_x^3 v\|_{L^2}^2 + \tilde C \|v\|_{L^2}^2$$

for $\tilde C > 0$ sufficiently large, we obtain:

$$\partial_t \|v\|_{H^2}^2(t) \le -\tilde\theta \|v\|_{H^2}^2 + C\left(\|v\|_{L^2}^2 + |\dot\alpha(t)|^2\right),$$

from which (8.15) follows by Gronwall's inequality. ∎

8.3. Proof of Theorem 8.2 ([Z7]). Decompose now the nonlinear perturbation v as: $v(x,t) = w(x,t) + z(x,t)$, where $w = \Pi_{cs} v$, $z = \Pi_u v$. Applying Π_{cs} to (8.13) and by Lemma 8.1:

$$\begin{aligned} w(x,t) = &\int_{-\infty}^{\infty} \tilde G(x,t;y) w_0(y)\, dy \\ &- \int_0^t \int_{-\infty}^{\infty} \tilde G_y(x,t-s;y) \tilde\Pi_{cs}(N(v) + \dot\alpha v)(y,s)\, dy\, ds \end{aligned} \quad (8.17)$$

for the flow along the center stable manifold, parametrized by $w \in \Sigma_{cs}$. On the other hand:

LEMMA 8.2 ([Z7]). *Assuming (H0)–(H1), for v lying initially on the center stable manifold $\mathcal{M}_{cs}$:*

$$|z|_{W^{r,p}} \le C|w|_{H^2}^2 \quad \textit{for some } C > 0 \quad \forall 1 \le p \le \infty,\ 0 \le r \le 4, \quad (8.18)$$

so long as $|w|_{H^2}$ remains sufficiently small.

Indeed, by tangency of the center stable manifold to Σ_{cs}, we have immediately $|z|_{H^2} \le C|w|_{H^2}^2$, whreas (8.18) follows by equivalence of norms for finite-dimensional vector spaces, applied to the p-dimensional space Σ_u.

Recalling by Theorem 8.1 that solutions remaining for all time in a sufficiently small radius neighborhood $\mathcal{N}$ of the set of translates of $\bar u$ lie in the center stable manifold $\mathcal{M}_{cs}$, we obtain that solutions not originating in $\mathcal{M}_{cs}$ must exit $\mathcal{N}$ in finite time, verifying the final assertion of orbital instability with respect to perturbations not in $\mathcal{M}_{cs}$.

Consider now a solution $v \in \mathcal{M}_{cs}$, or, equivalently, a solution $w \in \Sigma_{cs}$ of (8.17) with $z = \Phi_{cs}(w) \in \Sigma_u$. Define:

$$\zeta(t) := \sup_{0 \le s \le t} \Big(|w|_{H^2}(1+s)^{\frac{1}{4}} + (|w|_{L^\infty} + |\dot\alpha(s)|)(1+s)^{\frac{1}{2}} \Big). \tag{8.19}$$

We shall establish that for all $t \ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds:

$$\zeta(t) \le C_2(E_0 + \zeta(t)^2) \quad \text{for} \quad E_0 := |v_0|_{L^1 \cap H^2}. \tag{8.20}$$

From this result, provided $E_0 < 1/4C_2^2$, we infer that $\zeta(t) \le 2C_2E_0$ implies $\zeta(t) < 2C_2E_0$, and so we may conclude by continuous induction that $\zeta(t) < 2C_2E_0$for all $t \ge 0$, from which we readily obtain the stated bounds by continuous induction.

It remains to prove the claim (8.20). By Lemma 8.1, $|w_0|_{L^1\cap H^2} = |\Pi_{cs} v_0|_{L^1 \cap H^2} \le CE_0$. Likewise, by Lemma 8.2, (8.19), (8.7), and Lemma 8.1, for $0 \le s \le t$:

$$|\tilde\Pi_{cs}(N(v) + \dot\alpha v)(y,s)|_{L^2} \le C\zeta(t)^2(1+s)^{-\frac{3}{4}}. \tag{8.21}$$

Combining the latter bounds with (8.17) and (8.14) and applying Theorem 8.3, we obtain:

$$\begin{aligned} |w(x,t)|_{L^p} \le & \Big| \int_{-\infty}^{\infty} \tilde G(x,t;y) w_0(y)\, dy \Big|_{L^p} \\ & + \Big| \int_0^t \int_{-\infty}^{\infty} \tilde G_y(x,t-s;y) \tilde\Pi_{cs}(N(v) + \dot\alpha v)(y,s) dy\, ds \Big|_{L^p} \\ & \le E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} + C\zeta(t)^2 \int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{2p}}(1+s)^{-\frac{3}{4}} dy\, ds \\ & \le C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \end{aligned} \tag{8.22}$$

and, similarly, using Hölder's inequality and applying further Theorem 8.3:

$$\begin{aligned} |\dot\alpha(t)| \le & \int_{-\infty}^{\infty} |e_t(y,t)||v_0(y)|\, dy \\ & + \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y,t-s)||(N(v) + \dot\alpha v)(y,s)|\, dy\, ds \\ & \le |e_t|_{L^\infty}|v_0|_{L^1} + C\zeta(t)^2 \int_0^t |e_{yt}|_{L^2}(t-s)|(N(v) + \dot\alpha v)|_{L^2}(s) ds \\ & \le E_0(1+t)^{-\frac{1}{2}} + C\zeta(t)^2 \int_0^t (t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{4}} ds \\ & \le C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{8.23}$$

By Lemma 8.2: $|z|_{H^2}(t) \le C|w|_{H^2}^2(t) \le C\zeta(t)^2$, so in particular, $|z|_{L^2}(t) \le C\zeta(t)^2(1+t)^{-\frac{1}{2}}$. Applying Proposition 8.1, (8.22) and (8.23), we obtain:

$$|w|_{H^2}(t) \le C(E_0 + \zeta(t)^2)(1+t)^{-\frac{1}{4}}. \tag{8.24}$$

Combining (8.22), (8.23), and (8.24), we obtain (8.20) as claimed. Finally, a computation parallel to (8.23) (see [MaZ3, Z2]) yields $|\alpha(t)| \le C(E_0 + \zeta(t)^2)$, from which the last remaining bound on $|\alpha(t)|$ follows.

9. Cellular instability for flow in a duct [Z9]. We conclude by briefly discussing multidimensional stability and bifurcation of flow in a finite cross-sectional duct, or "shock tube", extending infinitely in the axial direction [Z9]. It is well known both experimentally and numerically [BE, MT, BMR, FW, MT, ALT, AT, F1, F2, KS] that shock and detonation waves propagating in a finite cross-section duct can exhibit time-oscillatory or "cellular" instabilities, in which the initially nearly planar shock takes on nontrivial transverse geometry.

In this section, following [Z9], we combine the analyses of [BSZ, TZ4, Z4] to make an explicit connection between stability of planar shocks on the whole space, and Hopf bifurcation in a finite cross-section duct. We point out that violation of the *refined stability condition* of [ZS, Z1, BSZ], a viscous correction of the inviscid planar stability condition of Majda [M1]–[M4], is generically associated with Hopf bifurcation corresponding to the observed cellular instability, for cross-section M sufficiently large. Indeed, we show more, that this is associated with a cascade of bifurcations to higher and higher wave numbers and more and more complicated solutions, with features on finer and finer length/time scales.

Consider a planar viscous shock solution:

$$u(x,t) = \bar{u}(x_1 - st) \tag{9.1}$$

of a two-dimensional system of viscous conservation laws:

$$u_t + \sum f^j(u)_{x_j} = \Delta_x u, \qquad u \in \mathbb{R}^n,\ x \in \mathbb{R}^2,\ t \in \mathbb{R}^+. \tag{9.2}$$

on the whole space. This may be viewed alternatively as a planar traveling-wave solution on an infinite channel $\mathcal{C} := \{x : (x_1, x_2) \in \mathbb{R}^1 \times [-M, M]\}$, under periodic boundary conditions:

$$u(x_1, M) = u(x_1, -M). \tag{9.3}$$

We take this as a simplified mathematical model for compressible flow in a duct, in which we have neglected boundary-layer phenomena along the wall $\partial\Omega$ in order to isolate the oscillatory phenomena of our main interest. Following [TZ2], consider a one-parameter family of standing planar viscous shock solutions $\bar{u}^\varepsilon(x_1)$ of a smooth family od PDEs:

$$u_t = \mathcal{F}(\varepsilon, u) := \Delta_x u - \sum_{j=1}^{2} F^j(\varepsilon, u)_{x_j}, \qquad u \in \mathbb{R}^n \tag{9.4}$$

in the fixed channel $\mathcal{C}$, with periodic boundary conditions (typically shifts $\sum F^j(\varepsilon, u)_{x_j} := \sum f^j(u)_{x_j} - s(\varepsilon) u_{x_1}$ of a single equation (9.2) written in coordinates $x_1 \mapsto x_1 - s(\varepsilon)t$ moving with traveling-wave solutions of varying speeds $s(\varepsilon)$), with linearized operators $L(\varepsilon) := \partial\mathcal{F}/\partial u|_{u=\bar u^\varepsilon}$.

Profiles $\bar u^\varepsilon$ satisfy the standing-wave ODE:

$$u' = F^1(\varepsilon, u) - F^1(\varepsilon, u_-). \tag{9.5}$$

Let $A^1_\pm(\varepsilon) := \lim_{z\to\pm\infty} F^1_u(\varepsilon, \bar u^\varepsilon)$. Following [Z1, TZ2, Z4], we assume:

(H0) $F^j \in C^k$, $k \geq 2$.

(H1) $\sigma(A^1_\pm(\varepsilon))$ are real, distinct, nonzero, and $\sigma(\sum \xi_j A^j_\pm(\varepsilon))$ real and semisimple for $\xi \in \mathbb{R}^d$.

For most of our results, we require also:

(H2) Considered as connecting orbits of (9.5), $\bar u^\varepsilon$ are transverse and unique up to translation, with dimensions of stable subpace $S(A^1_+)$ and unstable subspace $U(A^1_-)$ summing, for each ε, to $n+1$.

(H3) $\det(r_1^-, \dots, r_c^-, r_{c+1}^+, \dots, r_n^+, u_+ - u_-) \neq 0$, where $r_1^-, \dots, r_c^-$ are eigenvectors of A^1_- with negative eigenvalues and $r_{c+1}^-, \dots, r_n^-$ are eigenvectors of A^1_+ with positive eigenvalues.

Hypothesis (H2) asserts in particular that $\bar u^\varepsilon$ is of standard *Lax type*, meaning that the axial hyperbolic convection matrices $A^1_+(\varepsilon)$ and $A^1_-(\varepsilon)$ at plus and minus spatial infinity have, respectively, $n-c$ positive and $c-1$ negative real eigenvalues for $1 \leq c \leq n$, where c is the characteristic family associated with the shock. Hypothesis (H3) is the *Liu–Majda condition* corresponding to one-dimensional stability of the associated inviscid shock. In the present, viscous, context, this, together with transverality, (H2), plays the role of a spectral nondegeneracy condition corresponding in a generalized sense [ZH, Z1] to simplicity of the embedded zero eigenvalue associated with eigenfunction $\partial_{x_1}\bar u$ and translational invariance.

9.1. Stability and bifurcation conditions. Our first set of results characterizes stability/instability of waves $\bar u^\varepsilon$ in terms of the spectrum of the linearized operator $L(\varepsilon)$. Fixing ε, we suppress the parameter ε. We start with the routine observation that the semilinear parabolic equation (9.2) has a center-stable manifold about the equilibrium solution $\bar u$.

Proposition 9.1 ([Z9]). *Under assumptions (H0)–(H1), there exists in an H^2 neighborhood of the set of translates of $\bar u$ a codimension-p translation invariant C^k (with respect to H^2) center stable manifold $\mathcal{M}_{cs}$, tangent at $\bar u$ to the center stable subspace Σ_{cs} of L, that is (locally) invariant under the forward time-evolution of* (9.2)–(9.3) *and contains all solutions that remain bounded and sufficiently close to a translate of $\bar u$ in forward time, where p is the (necessarily finite) number of unstable, i.e., positive real part, eigenvalues of L.*

Introduce now the *nonbifurcation condition*:

(D1) L has no nonzero imaginary eigenvalues.

As discussed above, (H2)–(H3) correspond to a generalized notion of simplicity of the embedded eigenvalue $\lambda = 0$ of L. Thus, (D1) together with (H2)–(H3) correspond to the assumption that there are no additional (usual or generalized) eigenvalues on the imaginary axis other than the translational eigenvalue at $\lambda = 0$.

THEOREM 9.1 ([Z9]). *Under (H0)–(H3) and (D1), $\bar u$ is nonlinearly orbitally stable as a solution of* (9.2)–(9.3) *under sufficiently small perturbations of $\bar u$ in $L^1 \cap H^2$ lying on $\mathcal{M}_{cs}$ and its translates, in the sense that, for some $\alpha(\cdot)$, and all L^p:*

$$\begin{aligned} |u(x,t) - \bar u(x - \alpha(t))|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\ |u(x,t) - \bar u(x - \alpha(t))|_{H^2} &\le C(1+t)^{-\frac{1}{4}}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\ \dot\alpha(t) &\le C(1+t)^{-\frac{1}{2}}|u(x,0) - \bar u(x)|_{L^1\cap H^2}, \\ \alpha(t) &\le C|u(x,0) - \bar u(x)|_{L^1\cap H^2}\ . \end{aligned} \tag{9.6}$$

Moreover, $\bar u$ is orbitally unstable with respect to small H^2 perturbations not lying in $\mathcal{M}_{cs}$, in the sense that the corresponding solution leaves a fixed-radius neighborhood of the set of translates of $\bar u$ in finite time.

Proof. Observing that transverse modes $\xi \ne 0$ due to finite tranvserse cross-section are exponentially damped, reduces the analysis essentially to the one-dimensional case $\xi = 0$ treated in Theorem 8.2; see [Z9]. ∎

Define the *Hopf bifurcation condition*:

(D2) Outside the essential spectrum of $L(\varepsilon)$, for ε and $\delta > 0$ sufficiently small, the only eigenvalues of $L(\varepsilon)$ with real part of absolute value less than δ are a crossing conjugate pair $\lambda_\pm(\varepsilon) := \gamma(\varepsilon) \pm i\tau(\varepsilon)$ of $L(\varepsilon)$, with $\gamma(0) = 0$, $\partial_\varepsilon\gamma(0) > 0$, and $\tau(0) \ne 0$.

PROPOSITION 9.2 ([TZ2, TZ3]). *Let $\bar u^\varepsilon$,* (9.4) *be a family of traveling-waves and systems satisfying assumptions (H0)–(H3) and (D2), and $\eta > 0$ sufficiently small. Then, for $a \ge 0$ sufficiently small and $C > 0$ sufficiently large, there are C^1 functions $\varepsilon(a)$, $\varepsilon(0) = 0$, and $T^*(a)$, $T^*(0) = 2\pi/\tau(0)$, and a C^1 family of solutions $u^a(x_1,t)$ of* (9.4) *with $\varepsilon = \varepsilon(a)$, time-periodic of period $T^*(a)$, such that:*

$$C^{-1}a \le \sup_{x_1\in\mathbb{R}} e^{\eta|x_1|}\left|u^a(x,t) - \bar u^{\varepsilon(a)}(x_1)\right| \le Ca \qquad \text{for all } t \ge 0. \tag{9.7}$$

Up to fixed translations in x, t, for ε sufficiently small, these are the only nearby solutions as measured in norm $\|f\|_{X_1} := \|(1+|x_1|)f(x)\|_{L^\infty(x)}$ that are time-periodic with period $T \in [T_0, T_1]$, for any fixed $0 < T_0 < T_1 < +\infty$. Indeed, they are the only nearby solutions of form $u^a(x,t) = \mathbf{u}^a(x - \sigma^a t, t)$ with $\mathbf{u}^a$ periodic in its second argument.

Proof. Nonstandard Lyapunov–Schmidt reduction making use of nonlinear cancellation estimates similar to those appearing in the stability theory [TZ2]. Exponential damping of transverse modes $\xi \ne 0$ again reduces the analysis essentially to that of the one-dimensional case $\xi = 0$. ∎

Together with Theorem 9.1, Proposition 9.2 implies that, under the Hopf bifurcation assumption (D2) together with the further assumption that $L(\varepsilon)$ have no strictly positive real part eigenvalues other than possibly $\lambda_\pm$, waves $\bar u^\varepsilon$ are linearly and nonlinearly *stable* for $\varepsilon < 0$ and *unstable* for $\varepsilon > 0$, with bifurcation/exchange of stability at $\varepsilon = 0$.

9.2. Longitudinal vs. transverse bifurcation. The analysis of [TZ2] in fact gives slightly more information. Denote by:

$$\Pi^\varepsilon f := \sum_{j=\pm} \phi_j^\varepsilon(x)\langle \tilde\phi_j^\varepsilon, f\rangle \tag{9.8}$$

the $L(\varepsilon)$-invariant projection onto oscillatory eigenspace $\Sigma^\varepsilon := \mathrm{Span}\{\phi_\pm^\varepsilon\}$, where $\phi_\pm^\varepsilon$ are the eigenfunctions associated with $\lambda_\pm(\varepsilon)$. Then, we have the following result:

Proposition 9.3 ([TZ2]). *Under the assumptions of Proposition 9.2:*

$$\sup_{x_1} e^{\eta|x_1|}|u^a - \bar u - \Pi^\varepsilon(u^a - \bar u)| \le Ca^2 \qquad \text{for all } t \ge 0. \tag{9.9}$$

Bounds (9.7) and (9.9) together yield the standard finite-dimensional property that bifurcating solutions lie to quadratic order in the direction of the oscillatory eigenspace of $L(\varepsilon)$. From this, we may draw the following additional conclusions about the structure of bifurcating waves. By separation of variables, and x_2-independence of the coefficients of $L(\varepsilon)$, we have that the eigenfunctions ψ of $L(\varepsilon)$ decompose into families: $e^{i\xi x_2}\psi(x_1)$, $\xi = \frac{\pi k}{M}$, associated with different integers k, where M is cross-sectional width. Thus, there are two very different cases: (i) (*longitudinal instability*) the bifurcating eigenvalues $\lambda_\pm(\varepsilon)$ are associated with wave-number $k = 0$, or (ii) (*transverse instability*) the bifurcating eigenvalues $\lambda_\pm(\varepsilon)$ are associated with wave-numbers $\pm k \ne 0$.

Corollary 9.1 ([Z9]). *Under the assumptions of Proposition 9.2, u^a depend nontrivially on x_2 iff the bifurcating eigenvalues $\lambda_\pm$ are associated with transverse wave-numbers $\pm k \ne 0$.*

Bifurcation through longitudinal instability corresponds to “galloping” or “pulsating” instabilities described in detonation literature, while symmetry-breaking bifurcation through transverse instability corresponds to “cellular” instabilities introducing nontrivial transverse geometry to the structure of the propagating wave.

9.3. Review of multidimensional stability conditions. Inviscid stability analysis for shocks centers about the *Lopatinski determinant*:

$$\Delta(\tilde\xi, \lambda) := \begin{pmatrix} \mathcal{R}_1^- & \cdots & \mathcal{R}_{p-1}^- & \mathcal{R}_{p+1}^+ & \cdots & \mathcal{R}_n^+ & \lambda[u] + i\tilde\xi[f^2] \end{pmatrix}, \tag{9.10}$$

$\tilde\xi \in \mathbb{R}^1$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant whose zeroes correspond to normal modes $e^{\lambda t}e^{i\tilde\xi x_2}w(x_1)$ of the constant-coefficient linearized equations about the discontinuous shock solution. Here, $\{\mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+\}$ and $\{\mathcal{R}_1^-, \dots, \mathcal{R}_{p-1}^-\}$ denote bases for the unstable/resp. stable subspaces of: $\mathcal{A}_+(\tilde\xi, \lambda) := (\lambda I + i\tilde\xi df^2(u_\pm))(df^1(u_\pm))^{-1}$.

Weak stability $|\Delta| > 0$ for $\tau > 0$ is clearly necessary for linearized stability, while strong, or uniform stability, $|\Delta|/|(\tilde\xi, \lambda)| \geq c_0 > 0$, is sufficient for nonlinear stability. Between strong instability, or failure of weak stability, and strong stability, there lies a region of neutral stability corresponding to the appearance of surface waves propagating along the shock front, for which Δ is nonvanishing for $\Re\lambda > 0$ but has one or more roots $(\tilde\xi_0, \lambda_0)$ with $\lambda_0 = \tau_0$ pure imaginary. This region of neutral inviscid stability typically occupies an open set in physical parameter space [M1, M2, M3, BRSZ, Z1, Z2]. For details, see, e.g., [Er1, M1, M2, M3, Me, Se1, Se2, Se3, ZS, Z1, Z2, Z3, BRSZ], and references therein.

Viscous stability analysis for shocks in the whole space centers about the *Evans function* $D(\tilde\xi, \lambda)$, $\tilde\xi \in \mathbb{R}^1$, $\lambda = \gamma + i\tau \in \mathbb{C}$, $\tau > 0$, a spectral determinant analogous to the Lopatinski determinant of the inviscid theory, whose zeroes correspond to normal modes $e^{\lambda t}e^{i\tilde\xi x_2}w(x_1)$, of the linearized equations about $\bar u$ (now variable-coefficient), or spectra of the linearized operator about the wave. The main result of [ZS], establishing a rigorous relation between viscous and inviscid stability, was the expansion:

$$D(\tilde\xi, \lambda) = \gamma\Delta(\tilde\xi, \lambda) + o(|(\tilde\xi, \lambda)|) \tag{9.11}$$

of D about the origin $(\tilde\xi, \lambda) = (0, 0)$, where γ is a constant measuring tranversality of $\bar u$ as a connecting orbit of the traveling-wave ODE. Equivalently, considering $D(\tilde\xi, \lambda) = D(\rho\tilde\xi_0, \rho\lambda_0)$ as a function of polar coordinates $(\rho, \tilde\xi_0, \lambda_0)$, we have:

$$D|_{\rho=0} = 0 \text{ and } (\partial/\partial\rho)|_{\rho=0} D = \gamma\Delta(\tilde\xi_0, \lambda_0). \tag{9.12}$$

An important consequence of (9.11) is that *weak inviscid stability,* $|\Delta| > 0$, *is necessary for weak viscous stability*, $|D| > 0$ (an evident necessary condition for linearized viscous stability). For, (9.11) implies that the zero set of D is tangent at the origin to the cone $\{\Delta = 0\}$ (recall, (9.10), that Δ is homogeneous, degree one), hence enters $\{\tau > 0\}$ if $\{\Delta = 0\}$ does. Moreover, in case of *neutral inviscid stability* $\Delta(\xi_0, i\tau_0) = 0$, $(\xi_0, i\tau_0) \neq (0, 0)$, one may extract a further, *refined stability condition*:

$$\beta := -D_{\rho\rho}/D_{\rho\lambda}|_{\rho=0} \geq 0 \tag{9.13}$$

necessary for weak viscous stability. For, (9.12) then implies $D_\rho|_{\rho=0} = \gamma\Delta(\xi_0, i\tau_0) = 0$, whence Taylor expansion of D yields that the zero level set of D is concave or convex toward $\tau > 0$ according to the sign of β [ZS]. As discussed in [ZS, Z1], the constant β has a heuristic interpretation as an effective diffusion coefficient for surface waves moving along the front.

As shown in [ZS, BSZ], the formula (9.13) is well-defined whenever Δ is analytic at $(\tilde\xi_0, i\tau)$, in which case D considered as a function of polar coordinates is analytic at $(0, \tilde\xi_0, i\tau_0)$, and $i\tau_0$ is a simple root of $\Delta(\tilde\xi_0, \cdot)$. The determinant Δ in turn is analytic at $(\tilde\xi_0, i\tau_0)$, for all except a finite set of branch singularities $\tau_0 = \tilde\xi_0 \eta_j$. As discussed in [BSZ, Z2, Z3], the apparently nongeneric behavior that the family of holomorphic functions Δ^ε associated with shocks $(u_+^\varepsilon, u_-^\varepsilon)$ have roots $(\xi_0^\varepsilon, i\tau_0(\varepsilon))$ with $i\tau_0$ pure imaginary on an open set of ε is explained by the fact that, on certain components of the complement on the imaginary axis of this finite set of branch singularities, $\Delta^\varepsilon(\xi_0^\varepsilon, \cdot)$ takes the imaginary axis to itself. Thus, zeros of odd multiplicity persist on the imaginary axis, by consideration of the topological degree of Δ^ε as a map from the imaginary axis to itself.

Moreover, the same topological considerations show that a simple imaginary root of this type can only enter or leave the imaginary axis at a branch singularity of $\Delta^\varepsilon(\tilde\xi^\varepsilon, \cdot)$ or at infinity, which greatly aids in the computation of transition points for inviscid stability [BSZ, Z1, Z2, Z3]. As described in [Z2, Z3, Se1], escape to infinity is always associated with transition to strong instability. Indeed, using real homogeneity of Δ, we may rescale by $|\lambda|$ to find in the limit as $|\lambda| \to \infty$ that $0 = |\lambda_0|^{-1}\Delta(\tilde\xi_0, \lambda_0) = \Delta(\tilde\xi_0/|\lambda_0|, \lambda_0/|\lambda_0|) \to \Delta(0, i)$, which, by the complex homogeneity $\Delta(0, \lambda) \equiv \lambda\Delta(0, 1)$ of the one-dimensional Lopatinski determinant $\Delta(0, \cdot)$, yields *one-dimensional instability* $\Delta(0, 1) = 0$. As described in [Z1], Section 6.2, this is associated not with surface waves, but the more dramatic phenomenon of *wave-splitting*, in which the axial structure of the front bifurcates from a single shock to a more complicated multi-wave Riemann pattern.

We note in passing that one-dimensional inviscid stability $\Delta^\varepsilon(0, 1) \neq 0$ is equivalent to (H3) through the relation

$$\Delta^\varepsilon(0, \lambda) = \lambda \det(r_1^-, \dots, r_c^-, r_{c+1}^+, \dots, r_n^+, u_+ - u_-). \tag{9.14}$$

9.4. Transverse bifurcation of flow in a duct. We now make an elementary observation connecting cellular bifurcation of flow in a duct to stability of shocks in the whole space. Assume for the family $\bar u^\varepsilon$:

(B1) For ε sufficiently small, the inviscid shock $(u_+^\varepsilon, u_-^\varepsilon)$ is weakly stable; more precisely, $\Delta^\varepsilon(1, \lambda)$ has no roots $\Re\lambda \geq 0$ but a single simple pure imaginary root $\lambda(\varepsilon) = i\tau_*(\varepsilon) \neq 0$ lying away from the singularities of Δ^ε.

(B2) Coefficient $\beta(\varepsilon)$ defined in (9.13) satisfies $\Re\beta(0) = 0$, $\partial_\varepsilon \Re\beta(0) < 0$.

LEMMA 9.1 ([ZS, Z1]). *Let (H0)–(H2) and (B1). For ε, $\tilde\xi$ sufficiently small, there exist a smooth family of roots $(\tilde\xi, \lambda_*^\varepsilon(\tilde\xi))$ of $D(\tilde\xi, \lambda)$ with:*

$$\lambda_*^\varepsilon(\tilde\xi) = i\tilde\xi\tau_*(\varepsilon) - \tilde\xi^2\beta(\varepsilon) + \delta(\varepsilon)\tilde\xi^3 + r(\varepsilon, \tilde\xi)\tilde\xi^4, \qquad r \in C^1(\varepsilon, \tilde\xi). \tag{9.15}$$

Moreover, these are the unique roots of D satisfying $\Re\lambda \geq -|\tilde\xi|/C$ for some $C > 0$ and $\rho = |(\tilde\xi, \lambda)|$ sufficiently small.

Consequently, there is a unique C^1 function $\mathcal{E}(\tilde\xi) \neq 0$, $\mathcal{E}(0) = O$, such that $\Re\lambda_*^\varepsilon(\tilde\xi) = 0$ for $\varepsilon = \mathcal{E}(\tilde\xi)$. In the generic case $\Re\delta(0) \neq 0$, moreover:

$$\mathcal{E}(\tilde\xi) \sim (\Re\delta(0)/\partial_\varepsilon\beta(0))\tilde\xi. \tag{9.16}$$

To (B1) and (B2), adjoin now the additional assumptions:

(B3) $\delta(0) \neq 0$.

(B4) At $\varepsilon = 0$, the Evans function $D(\tilde\xi, \lambda)$ has no roots $\tilde\xi \in \mathbb{R}$, $\Re\lambda \geq 0$ outside a sufficiently small ball about the origin.

Then, we have the following main result:

THEOREM 9.2 ([Z9]). *Assuming (H0)–(H2), (B1)-(B4), for $\varepsilon_{max} > 0$ sufficiently small and each cross-sectional width M sufficiently large, there is a finite sequence $0 < \varepsilon_1(M) < \cdots < \varepsilon_k(M) < \cdots \leq \varepsilon_{max}$, with $\varepsilon_k(M) \sim (\Re\delta(0)/\partial_\varepsilon\beta(0))\frac{\pi k}{M}$, such that, as ε crosses successive ε_k from the left, there occur a series of transverse (i.e., "cellular") Hopf bifurcations of $\bar u^\varepsilon$ associated with wave-numbers $\pm k$, with successively smaller periods $T_k(\varepsilon) \sim \tau_*(0)\frac{2M}{k}$.*

Proof. ([Z9]) By (B4), for $|\varepsilon| \leq \varepsilon_{max}$ sufficiently small, we have by continuity that there exist no roots of $D(\tilde\xi, \lambda)$ for $\Re\lambda \geq -1/C$, $C > 0$, outside a small ball about the origin. By Lemma 9.1, within this small ball, there are no roots other than possibly $(\tilde\xi, \lambda^\varepsilon(\tilde\xi))$ with $\Re\lambda \geq 0$: in particular, *no nonzero purely imaginary spectra are possible other than at values $\lambda^\varepsilon(\tilde\xi)$* for operator $L(\varepsilon)$ acting on functions on the whole space.

Considering L instead as an operator acting on functions on the channel $\mathcal{C}$, we find by discrete Fourier transform/separation of variables that its spectra are exactly the zeros of $D(\xi_k, \lambda)$, as $\xi_k = \frac{\pi k}{L}$ runs through all integer wave-numbers k. Applying Lemma 9.1 and using (B3), we find, therefore, that pure imaginary eigenvalues of $L(\varepsilon)$ with $|\varepsilon| \leq \varepsilon_{max}$ sufficiently small occur precisely at values $\varepsilon = \varepsilon_k$, and consist of crossing conjugate pairs $\lambda_\pm^k(\varepsilon)$ associated with wave-numbers $\pm k$, satisfying Hopf bifurcation condition (D2) with:

$$\Im\lambda_\pm^k(\varepsilon) \sim \tau_*(0)\Big)\pi k/L.$$

Applying Proposition 9.2, the result follows. ∎

10. Discussion and open problems. We have described a comprehensive program, centered around the Evans function, for the study of stability and dynamics of viscous shock waves and related nonlinear phenomena: in particular, determination of stability, invariant manifolds, and bifurcation/exchange of stability. A novel feature of our approach is the systematic incorporation in our numerical studies of rigorous asymptotic analysis in high-frequency and other limits, allowing us to carry out global studies in parameter space, as in the arbitrary-amplitude study of Sec. 7.

The particular analyses presented here are special to their contexts. However, the basic approach is quite general. For large-amplitude/model parameter stability studies in the more complicated settings of MHD and nonisentropic gas dynamics, see [BHZ, BLZ, HLyZ1, HLyZ2]. For applications to viscoelastic shocks, see [BLeZ]. Similar programs have been carried out for noncharacteristic boundary layers [CHNZ, NZ1, NZ2, GMWZ2, Z3], detonation waves [LyZ1, LyZ2, LRTZ, TZ4, G1, W, Z12, Z13, HuZ1, BZ, HLyZ3], and solitary waves in shallow-water flow [BJRZ].

Extensions to spatially periodic traveling waves have been carried out in [G2, OZ1, OZ2, OZ3, OZ4, BJNRZ1, BJNRZ2, JZ1, JZ2, JZ3, JZ4, JZ5, JZB, JZN], and to temporally periodic waves in [BeSZ]. Other technically substantial extensions include the case of coupled hyperbolic-elliptic (e.g., radiative) shocks [LMNPZ, NPZ] and shocks of semidiscrete [B, BHu, BHuR, BeHSZ] and discrete [God] numerical schemes. Pointwise bounds sharpening Theorem 5.1 are given in [HZ, RZ, HR, HRZ].

Recall from the discussion at the end of Section 5.1, that inviscid shock stability is *necessary* for viscous stability [ZS]; this has the somewhat counterintuitive consequence that the inclusion of viscosity *may destabilize but never stabilize* an inviscid shock wave. A fundamental open question is whether in practice such destabilization can in fact occur; that is, to find, presumably numerically, a physical example for which viscous effects hasten the onset of instability as model parameters (say, shock amplitude) are varied, or to show analytically that this cannot occur.

Likewise, it would be very interesting to further investigate numerically the mechanism for cellular instabilities proposed in Section 9, in particular, to compare to alternative predictions of Majda et al [MR1, MR2, AM] based on hyperbolic effects (nonlinear rather than viscous corrrection), obtained by weakly nonlinear geometric optics.

More generally, the larger themes connecting the various topics presented here are stability analysis of solutions of systems of pde with large numbers of variables, and or about which the linearized equations have no spectral gap between decaying and neutral modes. As the main directions for the future we see the analysis of complex systems possessing orders of magnitude more variables, and of more complicated, genuinely multi-dimensional hdyrodynamic flows in unbounded domains. The recent studies [GLZ, Z12] represent first, preliminary, steps in these directions.

Extension to time-varying flows such as Riemann patterns is another important direction, as is the extension to infinite-dimensional kinetic systems such as Boltzmann's equation; see [MeZ3] and references therein.

Finally, we mention the problems of numerical proof and of purely analytical verification of stability in important cases. Though the basic ingredients (rigorous convergence theory, numerical well-conditioning, and mature algorithmic development) are present for numerical proof, this is of course a separate problem with its own challenges. However, the problems considered seem sufficiently physically fundamental to warrant such an investigation.

Regarding analytical proof, we point to the surprising recent result of Matsumura and Wang [MW], in which the authors establish spectral stability of arbitrary amplitude shock waves of the isentropic gas dynamics equations with density-dependent viscosity of form $\mu(v) = Cv^{-(\gamma-1)\beta}$, $\beta \geq 1/2$ including that ($\beta = 1/2$) predicted by statistical mechanics/Boltzmann's equation in the rarefied gas limit,[3] by an elegant energy estimate quite similar to that of Proposition 7.2. This demonstrates by example that, with sufficient skill and understanding, analytical verification of spectral stability may yet be possible.

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