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Stability of Isentropic Navier–Stokes Shocks in the High-Mach Number Limit*

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Abstract: By a combination of asymptotic ODE estimates and numerical Evans function calculations, we establish stability of viscous shock solutions of the isentropic compressible Navier–Stokes equations with γ-law pressure (i) in the limit as Mach number M goes to infinity, for any $\gamma \geq 1$ (proved analytically), and (ii) for $M \geq 2,500$, $\gamma \in [1, 2.5]$ or $M \geq 13,000$, $\gamma \in [2.5, 3]$ (demonstrated numerically). This builds on and completes earlier studies by Matsumura–Nishihara and Barker–Humpherys–Rudd–Zumbrun establishing stability for low and intermediate Mach numbers, respectively, indicating unconditional stability, independent of shock amplitude, of viscous shock waves for γ-law gas dynamics in the range $\gamma \in [1, 3]$. Other γ-values may be treated similarly, but have not been checked numerically. The main idea is to establish convergence of the Evans function in the high-Mach number limit to that of a pressureless, or "infinitely compressible", gas with additional upstream boundary condition determined by a boundary-layer analysis. Recall that low-Mach number behavior is formally incompressible.

Contents

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1. Introduction

The isentropic compressible Navier-Stokes equations in one spatial dimension expressed in Lagrangian coordinates take the form

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= \left(\frac{u_x}{v}\right)_x, \end{aligned} \tag{1.1}$$

where v is specific volume, u is velocity, and p pressure. We assume an adiabatic pressure law

$$p(v) = a_0 v^{-\gamma} \tag{1.2}$$

corresponding to a γ-law gas, for some constants $a_0 > 0$ and $\gamma \geq 1$. In the thermodynamical rarified gas approximation, $\gamma > 1$ is the average over constituent particles of $\gamma = (n+2)/n$, where n is the number of internal degrees of freedom of an individual particle [4]: $n = 3$ ($\gamma = 5/3$) for monatomic, $n = 5$ ($\gamma = 7/5$) for diatomic gas. For dense fluids, γ is typically determined phenomenologically [19]. In general, γ is usually taken within $1 \leq \gamma \leq 3$ in models of gas-dynamical flow, whether phenomenological or derived by statistical mechanics [42,43,45].

It is well known that these equations support *viscous shock waves*, or asymptotically-constant traveling-wave solutions

$$(v, u)(x, t) = (\hat{v}, \hat{u})(x - st), \qquad \lim_{z \to \pm\infty} (\hat{v}, \hat{u})(z) = (v, u)_\pm, \tag{1.3}$$

in agreement with physically-observed phenomena. In nature, such waves are seen to be quite stable, even for large variations in pressure between $v_\pm$. However, it is a long-standing mathematical question to what extent this is reflected in the continuum-mechanical model (1.1), that is, for which choice of parameters $(v_\pm, u_\pm, \gamma)$ are solutions of (1.3) time-evolutionarily stable in the sense of PDE; see, for example, the discussions in [3,25].

The first result on this problem was obtained by Matsumura and Nishihara in 1985 [36] using clever energy estimates on the integrals of perturbations in v and u, by which they established stability with respect to the restricted class of perturbations with "zero mass", i.e., perturbations whose integral is zero, for shocks with sufficiently small amplitude

$$|p(v_+) - p(v_-)| \le C(v_-, \gamma),$$

with $C \to \infty$ as $\gamma \to 1$, but $C << \infty$ for $\gamma \ne 1$. There followed a number of works by Liu, Goodman, Szepessy-Xin, and others [14,15,30,31,46] toward the treatment of general, nonzero mass perturbations; see [48–50] and references therein. A complete result of nonlinear stability with respect to general $L^1 \cap H^3$ perturbations of small-amplitude shocks of system (1.1) was finally obtained in 2004 by Mascia and Zumbrun [35] using pointwise semigroup techniques introduced by Zumbrun and Howard [47,50] in the strictly parabolic case. (For a precise statement, including rates of convergence, see the subsuming Proposition 1.1 below.)

The result of [35], together with the small-amplitude spectral stability result of Humpherys and Zumbrun [23][1] generalizing that of [36], in fact yields stability of small-amplitude shocks of general symmetric hyperbolic-parabolic systems, largely settling the problem of small-amplitude shock stability for continuum mechanical systems.

However, there remains the interesting question of *large-amplitude stability*. The main result in this direction, following a general strategy proposed in [50], is a "refined Lyapunov theorem"[2] established by Mascia and Zumbrun [34,49] for general symmetric hyperbolic–parabolic systems, stating that linearized $L^1 \cap H^3 \to L^1 \cap H^3$ stability is equivalent to and nonlinear $L^1 \cap H^3 \to L^1 \cap H^3$ stability is implied by *spectral stability*, defined as nonexistence of nonstable (nonnegative real part) eigenvalues of the linearized operator L about the wave, other than at $\lambda = 0$ (where there is always an eigenvalue, due to translational invariance of the underlying equations), together with *transversality* of the traveling-wave connection and *hyperbolic stability* of the corresponding discontinuous shock. Moreover, these same conditions imply also asymptotic orbital stability, defined as convergence to the family of translates of the unperturbed wave, in L^p, $p > 1$, with optimal rates.

For concreteness, we state below the specialization of this result to system (1.1), along with an extension of Raoofi [40] asserting phase-asymptotic stability, or convergence to a specific translate of the unperturbed wave, under the additional assumption that the initial perturbation has small L^1-first moment. These results hold for general hyperbolic-parabolic systems and waves under the additional assumptions of transversality/hyperbolic stability. Under our hypotheses, transversality/hyperbolic stability hold always for traveling waves of (1.1), as discussed in detail in the introduction of [34].

Proposition 1.1 ([34]). *For any p such that $p' < 0$ and $p''(0) > 0$, in particular for p as in* (1.2) *with $\gamma \ge 1$, let $(\hat v, \hat u)(x - st)$ be a spectrally stable traveling-wave* (1.3) *of* (1.1). *Then, for any solution $(\tilde v, \tilde u)(x, t)$ of* (1.1) *with $L^1 \cap H^3$ initial difference*

[1] The result of [23] is obtained by energy estimates combining the techniques of [36] with those of [14,15]; a similar approach has been used in [32] to obtain small-amplitude zero-mass stability of Boltzmann shocks. See [12,39] for an alternative approach based on asymptotic ODE methods.

[2] "Refined" because the linearized operator L does not possess a spectral gap, hence e^{Lt} decays time-algebraically and not exponentially; see [49,50] for further discussion.

$E_0 := \|(\tilde{v}, \tilde{u})(\cdot, 0) - (\hat{v}, \hat{u})\|_{L^1 \cap H^3}$ sufficiently small and some uniform $C > 0$, $(\tilde{v}, \tilde{u})$ exists for all $t \geq 0$, with

$$\|(\tilde{v}, \tilde{u})(\cdot, t) - (\hat{v}, \hat{u})(\cdot - st)\|_{L^1 \cap H^3} \leq CE_0 \quad \text{(stability)}. \tag{1.4}$$

Moreover, there exists $|\alpha(t)| \leq CE_0$, $|\dot{\alpha}(t)| \leq CE_0(1+t)^{-1/2}$ such that

$$\|(\tilde{v}, \tilde{u})(\cdot, t) - (\hat{v}, \hat{u})(\cdot - st - \alpha(t))\|_{L^p} \leq CE_0(1+t)^{-(1/2)(1-1/p)} \tag{1.5}$$

for all $1 \leq p \leq \infty$ (asymptotic orbital stability).

Proposition 1.2 ([40]). *For $(\tilde{v}, \tilde{u})$, $(\hat{v}, \hat{u})$ as in Proposition 1.1, if the initial difference has in addition a sufficiently small L^1-first moment*

$$E_1 := \| |x| |(\tilde{v}, \tilde{u})(\cdot, 0) - (\hat{v}, \hat{u})| \|_{L^1}, \tag{1.6}$$

then also α converges to a limit α_∞ as $t \to +\infty$, with

$$|\alpha - \alpha_\infty|(t),\ (1+t)^{1/2}|\dot{\alpha}(t)| \leq C(\epsilon)\max\{E_0, E_1\}(1+t)^{-1/2+\epsilon}, \tag{1.7}$$

$\epsilon > 0$ arbitrary (phase-asymptotic orbital stability).

Remark 1.3. Under the additional assumption that the initial perturbation and its first two derivatives decay as $E_0(1+|x|)^{-3/2}$, E_0 sufficiently small, one may obtain also sharp pointwise bounds on the solution, along with the sharp rate (1.7), $\epsilon = 0$ for the phase; see [20,21]. Such a localization condition, or the weaker (1.6) is necessary in order to obtain a rate of convergence for α; smallness in $L^1 \cup H^3$ is not enough, as may be seen by the example of a small initial perturbation localized arbitrarily far from $x = 0$, which takes arbitrarily long to reach and substantially affect the location of the shock profile.

Propositions 1.1–1.2 reduce the problem of large-amplitude stability to the study of the associated eigenvalue equation $(L - \lambda)u = 0$, a standard analytically and numerically well-posed (boundary value) problem in ODE, which can be attacked by the large body of techniques developed for asymptotic, exact, and numerical study of ODE. In particular, there exist well-developed and efficient numerical algorithms to determine the number of unstable roots for any specific linearized operator L, independent of its origins in the PDE setting; see, e.g., [8–11,24] and references therein. In this sense, the problem of determining stability of any single wave is satisfactorily resolved, or, for that matter, of any compact family of waves. To determine stability of a family of waves across an unbounded parameter regime, however, is another matter. It is this issue that we confront in attempting to decide the stability of general isentropic Navier–Stokes shocks.

As pointed out in [23,50], zero-mass stability implies (and in a practical sense is roughly equivalent to) spectral stability. Thus, the original results of Matsumura and Nishida [36] imply small-amplitude shock stability for general γ and large-amplitude stability as $\gamma \to 1$. Recently, Barker, Humpherys, Rudd, and Zumbrun [2] have carried out a numerical Evans function study indicating stability on the large, but still bounded, parameter range $\gamma \in [1, 3]$, $1 \leq M \leq 3{,}000$, where M is the Mach number associated with the shock. For discussion of the Evans function, see Sect. 2.4. Recall that Mach number is an alternative measure of shock strength, with 1 corresponding to $|p(v_+) - p(v_-)| = 0$ and $M \to \infty$ corresponding to $|p(v_+) - p(v_-)| \to \infty$; see Appendix A, [2], or Sect. 2.1 below. Mach 3, 000 is far beyond the hypersonic regime

$M \sim 10^1$ encountered in current aerodynamics. However, the mathematical question of stability across arbitrary γ, M has remained open up to now.

In this paper, we resolve this question, using a combination of asymptotic ODE estimates and numerical Evans function calculations to conclude, first, stability of isentropic Navier–Stokes shocks in the large Mach number limit $M \to \infty$ for any $\gamma \geq 1$, and, second, stability for all $M \geq 2,500$ for $\gamma \in [1, 2.5]$ (for $\gamma \in [1, 2]$, we obtain in fact stability for $M \geq 500$) and for all $M \geq 13,000$ for $\gamma \in [2.5, 3]$. The first result is obtained analytically, the second by a systematic numerical study. Together with the numerical results of [2] (supplemented with additional computations for $\gamma \in [2.5, 3]$), this gives convincing numerical evidence of *unconditional stability for* $\gamma \in [1, 3]$*, independent of shock amplitude*. As in [2], our numerical study is not a numerical proof, but contains the necessary ingredients for one; see discussion, Sect. 6. The restriction to $\gamma \in [1, 3]$ is an arbitrary one coming from the choice of parameters on which the numerical study [2] was carried out; stability for other γ can be easily checked as well. (Note that all analytical results are for any $\gamma \geq 1$.)

In particular, we establish without aid of numerics the following theorem, an immediate consequence of Corollary 3.3 and Proposition 3.5 below. The rest of our results, both analytical and numerical, are stated after some preliminary preparations and discussion in Sect. 3.

Theorem 1.4. *For any* $\gamma \geq 1$*, isentropic Navier–Stokes shocks are spectrally stable for Mach number* M *sufficiently large* (*equivalently,* v_+ *sufficiently small, taking without loss of generality* $v_- > v_+ > 0$ *in* (1.3))*, hence nonlinearly stable in* $L^1 \cap H^3$*, with bounds* (1.4)–(1.5)*,* (1.7).

Our method of analysis is straightforward, though somewhat delicate to carry out. Working with the rescaled and conveniently rearranged versions of the equations introduced in [2], we observe that the associated eigenvalue equations converge uniformly as Mach number goes to infinity on a "regular region" $x \leq L$, for any fixed $L > 0$, to a limiting system that is well-behaved (hence treatable by the standard methods of [2,33,39]) in the sense that its coefficient matrix converges uniformly exponentially in x to limits at $x = \pm\infty$, but is underdetermined at $x = +\infty$.

On the complementary "singular region" $x \geq L$, the convergence is only pointwise due to a fast "inner structure" featuring rapid variation of the converging coefficient matrices near $x = +\infty$, but the behavior at $x = +\infty$ is of course determinate. Performing a boundary-layer analysis on the singular region and matching across $x = L$, we are able to show convergence of the Evans function of the original system as the Mach number goes to infinity to an Evans function of the limiting system with an appropriately imposed additional condition at $x = +\infty$, upstream of the shock. This reduces the question of stability in the high-Mach number limit to existence or nonexistence of zeroes of the limiting Evans function on $\Re e\lambda \geq 0$, a question that can be resolved by routine numerical computation as in [2], or by energy estimates as in Appendix B.

The limiting system can be recognized as the eigenvalue equation associated with a pressureless ($\gamma = 0$) gas, that is, the "infinitely-compressible" limit one might expect as the Mach number goes to infinity. Recall that behavior in the low Mach number limit is incompressible [18,27,29]. However, the upstream boundary condition has to our knowledge no such simple interpretation. Indeed, to carry out the boundary-layer analysis by which we derive this condition is the main technical difficulty of the paper.

Our results on the large-amplitude limit allow us to complete a *global stability analysis* for traveling-waves of a nontrivial, physically relevant, and much-studied system of equations, allowing us to conclude for this canonical model that isentropic

Navier–Stokes shocks are stable for a γ-law gas with $\gamma \in [1, 3]$. More, the large amplitude limit appears to serve as an organizing center governing behavior also of shocks of large but relatively modest size. See, for example, Figs. 2–4, from which universal stability may be deduced essentially by inspection. Indeed, together with the small-amplitude limit, this appears to essentially govern by interpolation behavior of shocks of all amplitudes. The study of this limit thus has importance apart from the specific physical interest of the high-Mach number regime.

Besides their independent interest, the results of this paper seem significant as prototypes for future analyses. Our calculations use some properties specific to the structure of (1.1). In particular, we make use of surprisingly strong energy estimates carried out in [2] in confining unstable eigenvalues to a bounded set independent of shock strength (or Mach number). Also, we use the extremely simple structure of the eigenvalue equation to carry out the key analysis of the eigenvalue flow in the singular region near $x = +\infty$ essentially by hand. However, these appear to be conveniences rather than essential aspects of the analysis. It is our hope that the basic argument structure of this paper together with [2] can serve as a blueprint for the global study of large-amplitude stability in more general situations. In particular, we expect that the analysis will carry over to the full (nonisentropic) equations of gas dynamics with ideal gas equation of state.

2. Preliminaries

We begin by recalling a number of preliminary steps carried out in [2]. Making the standard change of coordinates $x \to x - st$, we consider instead stationary solutions $(v, u)(x, t) \equiv (\hat{v}, \hat{u})(x)$ of

$$\begin{aligned} v_t - s v_x - u_x &= 0, \\ u_t - s u_x + (a_0 v^{-\gamma})_x &= \left(\frac{u_x}{v}\right)_x . \end{aligned} \tag{2.1}$$

Under the rescaling

$$(x, t, v, u) \to (-\varepsilon s x, \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s)), \tag{2.2}$$

where ε is chosen so that $0 < v_+ < v_- = 1$, our system takes the convenient form

$$\begin{aligned} v_t + v_x - u_x &= 0, \\ u_t + u_x + (a v^{-\gamma})_x &= \left(\frac{u_x}{v}\right)_x , \end{aligned} \tag{2.3}$$

where $a = a_0 \varepsilon^{-\gamma-1} s^{-2}$.

2.1. Profile equation. Steady shock profiles of (2.3) satisfy

$$\begin{aligned} v' - u' &= 0, \\ u' + (a v^{-\gamma})' &= \left(\frac{u'}{v}\right)', \end{aligned}$$

subject to boundary conditions $(v, u)(\pm\infty) = (v_\pm, u_\pm)$, or, simplifying,

$$v' + (av^{-\gamma})' = \left(\frac{v'}{v}\right)'.$$

Integrating from $-\infty$ to x, we get the profile equation

$$v' = H(v, v_+) := v(v - 1 + a(v^{-\gamma} - 1)), \tag{2.4}$$

where a is found by setting $x = +\infty$, thus yielding the Rankine-Hugoniot condition

$$a = -\frac{v_+ - 1}{v_+^{-\gamma} - 1} = v_+^\gamma \frac{1 - v_+}{1 - v_+^\gamma}. \tag{2.5}$$

Evidently, $a \to \gamma^{-1}$ in the weak shock limit $v_+ \to 1$, while $a \sim v_+^\gamma$ in the strong shock limit $v_+ \to 0$. The associated Mach number M may be computed as in [2], Appendix A, as

$$M = (\gamma a)^{-1/2} \tag{2.6}$$

so that $M \sim \gamma^{-1/2} v_+^{-\gamma/2} \to +\infty$ as $v_+ \to 0$ and $M \to 1$ as $v_+ \to 1$; *that is, the high-Mach number limit corresponds to the limit* $v_+ \to 0$.

2.2. Eigenvalue equations. Linearizing (2.3) about the profile $(\hat{v}, \hat{u})$, we obtain the eigenvalue problem

$$\begin{aligned} \lambda v + v' - u' &= 0, \\ \lambda u + u' - \left(\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v\right)' &= \left(\frac{u'}{\hat{v}}\right)', \end{aligned} \tag{2.7}$$

where

$$h(\hat{v}) = -\hat{v}^{\gamma+1} + a(\gamma - 1) + (a + 1)\hat{v}^\gamma. \tag{2.8}$$

We seek nonstable eigenvalues $\lambda \in \{\Re e(\lambda) \geq 0\} \setminus \{0\}$, i.e., λ for which (2.7) possess a nontrivial solution (v, u) decaying at plus and minus spatial infinity. As pointed out in [23,50], by divergence form of the equations, such solutions necessarily satisfy $\int_{-\infty}^{+\infty} v(x)dx = \int_{-\infty}^{+\infty} u(x)dx = 0$, from which we may deduce that

$$\tilde{u}(x) = \int_{-\infty}^{x} u(z)dz, \quad \tilde{v}(x) = \int_{-\infty}^{x} v(z)dz$$

and their derivatives decay exponentially as $x \to \infty$. Substituting and then integrating, we find that $(\tilde{u}, \tilde{v})$ satisfies the *integrated eigenvalue equations* (suppressing the tilde)

$$\lambda v + v' - u' = 0, \tag{2.9a}$$

$$\lambda u + u' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v' = \frac{u''}{\hat{v}}. \tag{2.9b}$$

This new eigenvalue problem differs spectrally from (2.7) only at $\lambda = 0$, hence spectral stability of (2.7) is equivalent to spectral stability of (2.9). Moreover, since (2.9) has no eigenvalue at $\lambda = 0$, one can expect more uniform stability estimates for the integrated equations in the vicinity of $\lambda = 0$ [14,36,50].

2.3. Preliminary estimates. The following estimates established in [2] indicate the suitability of the rescaling chosen in Sect. 2.1. For completeness, we recall their proofs in Appendix A.

Proposition 2.1 ([2]). *For each $\gamma \geq 1$, $0 < v_+ \leq 1$, (2.4) has a unique (up to translation) monotone decreasing solution $\hat{v}$ decaying to its endstates with a uniform exponential rate, independent of v_+, γ. In particular, for $0 < v_+ \leq \frac{1}{12}$ and $\hat{v}(0) := v_+ + \frac{1}{12}$,*

$$|\hat{v}(x) - v_+| \leq \left(\frac{1}{12}\right) e^{-\frac{3x}{4}} \quad x \geq 0, \tag{2.10a}$$

$$|\hat{v}(x) - v_-| \leq \left(\frac{1}{4}\right) e^{\frac{x+12}{2}} \quad x \leq 0. \tag{2.10b}$$

Proposition 2.2 ([2]). *Nonstable eigenvalues λ of (2.9), i.e., eigenvalues with nonnegative real part, are confined for any $0 < v_+ \leq 1$ to the region*

$$\Lambda := \{\lambda : \Re e(\lambda) + |\Im m(\lambda)| \leq \left(\sqrt{\gamma} + \frac{1}{2}\right)^2\}. \tag{2.11}$$

2.4. Evans function formulation. Following [2], we may express (2.9) concisely as a first-order system

$$W' = A(x, \lambda)W, \tag{2.12}$$

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\hat{v} & \lambda\hat{v} & f(\hat{v}) - \lambda \end{pmatrix}, \quad W = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}, \quad ' = \frac{d}{dx}, \tag{2.13}$$

where

$$f(\hat{v}) = \hat{v} - \hat{v}^{-\gamma} h(\hat{v}) = 2\hat{v} - a(\gamma - 1)\hat{v}^{-\gamma} - (a + 1), \tag{2.14}$$

with h as in (2.8) and a as in (2.5), or, equivalently,

$$f(\hat{v}) = 2\hat{v} - (\gamma - 1)\left(\frac{1 - v_+}{1 - v_+^\gamma}\right)\left(\frac{v_+}{\hat{v}}\right)^\gamma - \left(\frac{1 - v_+}{1 - v_+^\gamma}\right) v_+^\gamma - 1. \tag{2.15}$$

Eigenvalues of (2.9) correspond to nontrivial solutions W for which the boundary conditions $W(\pm\infty) = 0$ are satisfied. Because $A(x, \lambda)$ as a function of $\hat{v}$ is asymptotically constant in x, the behavior near $x = \pm\infty$ of solutions of (2.13) is governed by the limiting constant-coefficient systems

$$W' = A_\pm(\lambda)W, \qquad A_\pm(\lambda) := A(\pm\infty, \lambda), \tag{2.16}$$

from which we readily find on the (nonstable) domain $\Re\lambda \geq 0, \lambda \neq 0$ of interest that there is a one-dimensional unstable manifold $W_1^-(x)$ of solutions decaying at $x = -\infty$ and a two-dimensional stable manifold $W_2^+(x) \wedge W_3^+(x)$ of solutions decaying at $x = +\infty$, analytic in λ, with asymptotic behavior

$$W_j^\pm(x, \lambda) \sim e^{\mu_\pm(\lambda)x} V_j^\pm(\lambda) \tag{2.17}$$

as $x \to \pm\infty$, where $\mu_\pm(\lambda)$ and $V_j^\pm(\lambda)$ are eigenvalues and associated analytically chosen eigenvectors of the limiting coefficient matrices $A_\pm(\lambda)$. A standard choice of eigenvectors $V_j^\pm$ [8,11,13,22], uniquely specifying $W_j^\pm$ (up to constant factor) is obtained by Kato's ODE [26], a linear, analytic ODE whose solution can be alternatively characterized by the property that there exist corresponding left eigenvectors $\tilde V_j^\pm$ such that

$$(\tilde V_j \cdot V_j)^\pm \equiv \text{constant}, \quad (\tilde V_j \cdot \dot V_j)^\pm \equiv 0, \tag{2.18}$$

where "$\dot{}$" denotes $d/d\lambda$; for further discussion, see [13,22,26].

Defining the *Evans function* D associated with operator L as the analytic function

$$D(\lambda) := \det(W_1^- W_2^+ W_3^+)_{|x=0}, \tag{2.19}$$

we find that eigenvalues of L correspond to zeroes of D both in location and multiplicity; moreover, the Evans function extends analytically to $\lambda = 0$, i.e., to all of $\Re\lambda \geq 0$. See [1,13,33,49] for further details.

Equivalently, following [2,38], we may express the Evans function as

$$D(\lambda) = \left(\widetilde W_1^+ \cdot W_1^-\right)_{|x=0}, \tag{2.20}$$

where $\widetilde W_1^+(x)$ spans the one-dimensional unstable manifold of solutions decaying at $x = +\infty$ (necessarily orthogonal to the span of $W_2^+(x)$ and $W_3^+(x)$) of the adjoint eigenvalue ODE,

$$\widetilde W' = -A(x,\lambda)^* \widetilde W. \tag{2.21}$$

The simpler representation (2.20) is the one that we shall use here.

3. Description of the Main Results

We can now state precisely our main results.

3.1. Limiting equations. Under the strategic rescaling (2.2), both profile and eigenvalues equations converge *pointwise* as $v_+ \to 0$ to limiting equations at $v_+ = 0$. The limiting profile equation (the limit of (2.4)) is evidently

$$v' = v(v-1), \tag{3.1}$$

with explicit solution

$$\hat v_0(x) = \frac{1 - \tanh(x/2)}{2}, \tag{3.2}$$

while the limiting eigenvalue system (the limit of (2.13)) is

$$W' = A^0(x,\lambda) W, \tag{3.3}$$

$$A^0(x,\lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda \hat v_0 & \lambda \hat v_0 & f_0(\hat v_0) - \lambda \end{pmatrix}, \tag{3.4}$$

where

$$f_0(\hat{v}_0) = 2\hat{v}_0 - 1 = -\tanh(x/2). \tag{3.5}$$

Indeed, this convergence is *uniform* on any interval $\hat{v}_0 \geq \epsilon > 0$, or, equivalently, $x \leq L$, for L any positive constant, where the sequence is therefore a *regular perturbation* of its limit. We will call $x \in (-\infty, L]$ the "regular region" or "regular side". For $\hat{v}_0 \to 0$ on the other hand, or $x \to \infty$, the limit is less well-behaved, as may be seen by the fact that $\partial f/\partial \hat{v} \sim \hat{v}^{-1}$ as $\hat{v} \to v_+$, a consequence of the appearance of $\left(\frac{v_+}{\hat{v}}\right)$ in the expression (2.15) for f. Similarly, in contrast to $\hat{v}$, $A(x, \lambda)$ does not converge to $A_+(\lambda)$ as $x \to +\infty$ with uniform exponential rate independent of v_+, γ, but rather as $C\hat{v}^{-1}e^{-x/2}$. We call $x \in [L, +\infty)$ therefore the "singular region" or "singular side". (This is not a singular perturbation in the usual sense but a weaker type of singularity, at least as we have framed the problem here.)

3.2. Limiting Evans function. We should now like to define a limiting Evans function following the asymptotic Evans function framework introduced in [39] and establish convergence to this function in the $v_+ \to 0$ limit, thus reducing the stability problem as $v_+ \to 0$ to the study of the (fixed) limiting Evans function. However, we face certain difficulties due to the (mild) singularity of the limit, as can be seen even at the first step of defining an Evans function for the limiting system.

For, the limiting coefficient matrix

$$A_+^0(\lambda) := A^0(+\infty, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix} \tag{3.6}$$

is nonhyperbolic (in ODE sense) for all λ, having eigenvalues $0, 0, -1-\lambda$; in particular, the stable manifold drops to dimension one in the limit $v_+ \to 0$. Thus, the subspace in which W_2^+ and W_3^+ should be initialized at $x = +\infty$ is not self-determined by (3.6), but must be deduced by a careful study of the double limit $v_+ \to 0$, $x \to +\infty$. But, the computation

$$\lim_{v_+ \to 0} A(+\infty, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -\gamma-\lambda \end{pmatrix} \neq A_+^0(\lambda) = \lim_{x\to\infty} \lim_{v_+ \to 0} A(x, \lambda)$$

shows that these limits do not commute, except in the special case $\gamma = 1$ already treated in [36] by other methods.

The rigorous treatment of this issue is the main work of the paper. However, the end result can be easily motivated on heuristic grounds. A study of $\lim_{v_+\to 0} A(+\infty, \lambda)$ on the set $\Re e\lambda \geq 0$ of interest reveals that eigenmodes decouple into a single "fast" (stable subspace) decaying mode $(*, *, 1)^T$ associated with eigenvalue $-\gamma - \lambda$ of strictly negative real part and a two-dimensional (center) subspace of neutral modes $(r, 0)^T$ associated with Jordan block $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$, of which there is only a single genuine eigenvector $(1, 0, 0)^T$. For v_+ small, therefore, $A_+(\lambda)$ has also a single fast, decaying, eigenmode with eigenvalue near $-\gamma - 1$ and two slow eigenmodes with eigenvalues near zero, one decaying and the other growing (recall, Sect. 2.4, that the stable subspace of A_+ has dimension two for $\Re e\lambda \geq 0$, $\lambda \neq 0$ and the unstable subspace dimension one).

Focusing on the single slow decaying eigenvector of A_+, and considering its limiting behavior as $v_+ \to 0$, we see immediately that it must converge in direction to $\pm(1, 0, 0)^T$. For, the sequence of direction vectors, since continuously varying and restricted to a compact set, has a nonempty, connected set of accumulation points, and these must be eigenvectors of $\lim_{v_+ \to 0} A_+$ with eigenvalues near zero. Since $\pm(1, 0, 0)^T$ are the unique candidates, we obtain the result. Indeed, both growing and decaying slow eigenmodes must converge to this common direction, making the limiting analysis trivial.

The same argument shows that $\pm(1, 0, 0)^T$ is the limiting direction of the slow stable eigenmode of $A^0(x, \lambda)$ as $x \to +\infty$, that is, in the alternate limit

$$\lim_{x \to \infty} \lim_{v_+ \to 0} A(x, \lambda).$$

Since $V_2^+ := (1, 0, 0)^T$ is the common limit of the slow decaying eigenmode in either of the two alternative limits $\lim_{v_+ \to 0} A_+$ and $\lim_{v_+ \to 0} A_+$, it thus seems a reasonable choice to use this limiting slow direction to define an Evans function for the limiting system (3.4). On the other hand, the stable eigenmode

$$V_3 := (b^{-1}(\lambda/b + 1), b^{-1}, 1)^T,$$

$b = -1 - \lambda$, of A_+^0 is forced on us by the system itself, independent of the limiting process.

Combining these two observations, we require that solutions W_2^{0+} and W_3^{0+} of the limiting eigenvalue system (3.4) lie asympotically in directions V_2 and V_3, respectively, thus determining a limiting, or "reduced" Evans function

$$D^0(\lambda) := \det(W_1^{0-} W_2^{0+} W_3^{0+})_{|x=0}, \tag{3.7}$$

or alternatively

$$D^0(\lambda) = \left(\widetilde{W}_1^{0+} \cdot W_1^{0-}\right)_{|x=0}, \tag{3.8}$$

with $\widetilde{W}_1^{0+}$ defined analogously as a solution of the adjoint limiting system lying asymptotically at $x = +\infty$ in direction

$$\widetilde{V}_1 := (0, 1, \bar{b}^{-1})^T = (0, 1, (1 + \bar{\lambda})^{-1})^T \tag{3.9}$$

orthogonal to the span of V_2 and V_3, where "¯" denotes complex conjugate. (The prescription of W_1^{0-} in the regular region is straightforward: it must lie on the one-dimensional unstable manifold of A_-^0 as in the $v_+ > 0$ case.)

3.3. Physical interpretation. Alternatively, the limiting equations may be derived by taking a formal limit as $v_+ \to 0$ of the rescaled equations (2.3), recalling that $a \sim v_+^\gamma$, to obtain a *limiting evolution equation*

$$\begin{aligned} v_t + v_x - u_x &= 0, \\ u_t + u_x &= \left(\frac{u_x}{v}\right)_x \end{aligned} \tag{3.10}$$

corresponding to a *pressureless gas*, or $\gamma = 0$, then deriving profile and eigenvalue equations from (3.10) in the usual way. This gives some additional insight on behavior, of which we make important mathematical use in Appendix B. Physically, it has the interpretation that, in the high-Mach number limit $v_+ \to 0$, effects of pressure are concentrated near $x = +\infty$ on the infinite-density side, as encoded in the special upstream boundary condition $(u, u', v, v') \to c(1, 0, 0, 0)$ as $x \to +\infty$ for the integrated eigenvalue equation, which may be seen to be equivalent to the conditions imposed on W_j^+ in the previous subsection.

3.4. Analytical results. Defining D^0 as in (3.7)–(3.8), we have the following main theorems, all valid for arbitrary $\gamma \geq 1$.

Theorem 3.1. *For each $\gamma \geq 1$ and λ in any compact subset of $\Re e\lambda \geq 0$, $D(\lambda)$ converges uniformly to $D^0(\lambda)$ as $v_+ \to 0$.*

Corollary 3.2. *For each $\gamma \geq 1$ and any compact subset Λ of $\Re e\lambda \geq 0$, D is nonvanishing on Λ for v_+ sufficiently small if D^0 is nonvanishing on Λ, and is nonvanishing on the interior of Λ only if D^0 is nonvanishing there.*

Proof. Standard properties of uniform limit of analytic functions. □

Corollary 3.3. *For each $\gamma \geq 1$, isentropic Navier–Stokes shocks are spectrally stable in the high-Mach number limit $v_+ \to 0$ if D^0 is nonvanishing on the wedge*

$$\Lambda : \Re e(\lambda) + |\Im m(\lambda)| \leq \left(\sqrt{\gamma} + \frac{1}{2}\right)^2, \quad \Re e\lambda \geq 0 \tag{3.11}$$

and only if D^0 is nonvanishing on the interior of Λ.

Proof. Corollary 3.2 together with Proposition 2.2. □

Remark 3.4. Likewise, on any compact subset of $\Re e\lambda \geq 0$, $|D|$ is uniformly bounded from zero for v_+ sufficiently small (M sufficiently large) if and only if $|D^0|$ is uniformly bounded from zero. Thus, isentropic Navier–Stokes shocks are "uniformly stable" for sufficiently small v_+, in the sense that $|D|$ is bounded from below independent of v_+, if and only if D^0 is nonvanishing on Λ as defined in (3.11).

The following result completes our abstract stability analysis. The proof, given in Appendix B, is by an energy estimate analogous to that of [36].[3]

Proposition 3.5. *The limiting Evans function D^0 (note: independent of γ) is nonzero on $\Re e\lambda \geq 0$.*

From Proposition 3.5 and Corollary 3.3 we obtain Main Theorem 1.4 stated in the Introduction.

[3] Stability for $\gamma = 1$, proved in [36], already implies nonvanishing of D^0 outside the imaginary interval $[-i\sqrt{3/2}, +i\sqrt{3/2}]$, by Corollary 3.3.

3.5. Numerical computations. Unfortunately, the energy estimate used to establish Proposition 3.5, though mathematically elegant, yields only the stated, abstract result and not quantitative estimates. A simpler and more general approach, that does yield quantitative information, is to compute the reduced Evans function numerically. We carry out this by-now routine numerical computation using the methods of [2]. Specifically, we map a semicircle $\partial(\{\Re e\lambda \geq 0\} \cap \{|\lambda| \leq 10\})$ enclosing Λ for $\gamma \in [1, 3]$ by D^0 and compute the winding number of its image about the origin to determine the number of zeroes of D^0 within the semicircle, and thus within Λ. For details of the numerical algorithm, see [2,11,23].

The result is displayed in Fig. 1, clearly indicating stability. More precisely, the minimum of $|D|$ on the semicircle is found to be ≈ 0.2433. Together with Rouchés Theorem, this gives explicit bounds on the size of the Mach number for which shocks are stable, as displayed in Table 1, Sect. 6. Specifically, computing numerically the first value of M at which $|D - D^0|/|D^0| < 1/2$ on the entire semicircle, we obtain the lower bounds $M \geq 500$ for $\gamma \in [1, 2]$, $M \geq 2,400$ for $\gamma \in [2, 2.5]$, and $M \geq 13,000$ for $\gamma \in [2.5, 3]$, all corresponding approximately to $v_+ = 10^{-3}$. Recall by Rouchés Theorem that winding number is unaffected by relative error $|D - D^0|/|D^0| < 1$.

This is essentially a convergence study, since we rely on the assumed convergence of the relative error to zero in concluding that relative error remains < 1 also for Mach numbers larger than this minimum value. In Fig. 2, we superimpose on the image of the semicircle by D^0 its (again numerically computed) image by the full Evans function D, for a monatomic gas $\gamma = 5/3$ at successively higher Mach numbers v_+ =`1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`, graphically demonstrating the convergence of D to D^0 as v_+ approaches zero and (numerically) verifying the conclusions of Table 1.

In fact, we can see a great deal more from Fig. 2. For, note that the displayed contours are, to the scale visible by eye, "monotone" in v_+, or nested, one within the other (they do not appear so at smaller scales). Thus, lower-Mach number contours are essentially "trapped" within higher-Mach number contours, with the worst-case, outmost contour corresponding to the limiting Evans function D^0. From this observation, we may conclude with confidence stability down to the smallest value $M \approx 5.5$ displayed in the figure, corresponding (see Table 2) to $v_+ = 10^{-1}$. Note, further, that the low-Mach number contours appear to be shrinking to a point as $v_+ \to 1$. This is indeed the case, as can be confirmed by the small-amplitude analysis of [39]; see Remark 3.6.

That is, a great deal of topological information is encoded in the analytic family of Evans functions indexed by v_+, from which stability may be deduced almost by inspection. Behavior for other $\gamma \in [1, 3]$ is similar. See, for example, the case $\gamma = 3$ displayed in Fig. 4, which is virtually identical to Fig. 2.[4]

Such topological information does not seem to be available from other methods of investigating stability such as direct discretation of the linearized operator about the wave [28] or studies based on linearized time-evolution or power methods [5,6]. This represents in our view a significant advantage of the Evans function formulation.

Remark 3.6. Recall that the Evans function is not determined uniquely, but only up to a nonvanishing analytic factor [1,13]. The simple contour structure in Fig. 2 is thus partly due to a favorable choice of D induced by the initialization at $\pm\infty$ by Kato's ODE [26], as described in [2,11,24]. A canonical algorithm for tracking bases of evolving subspaces, this in some sense minimizes "action"; see [22] for further discussion.

[4] In particular, Fig. 4 indicates stability down to $v_+ = 10^{-1}$, or Mach number ~ 20, from which we may conclude unconditional stability on the whole range $\gamma \in [1, 3]$ of [2].

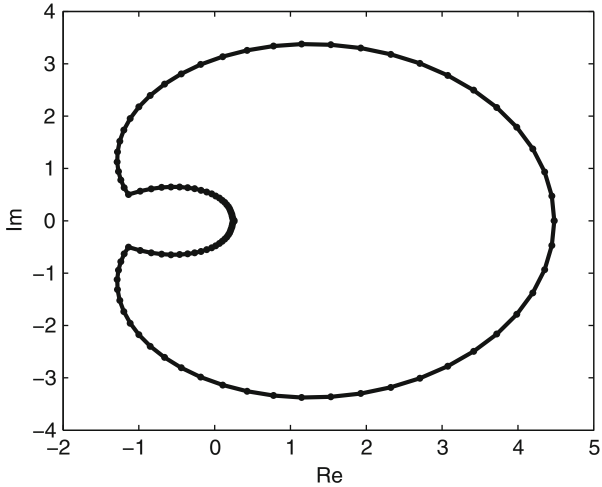

Fig. 1. The image of the semi-circular contour via the Evans function for the reduced system. Note that the winding number of this graph is zero. Hence, there are no unstable eigenvalues in the semi-circle

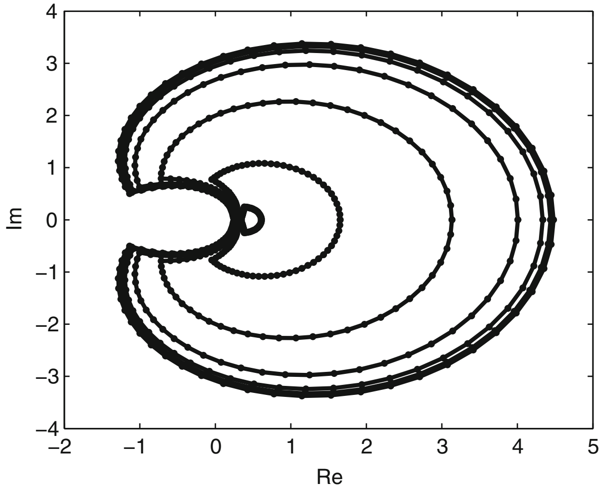

Fig. 2. Convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas, $\gamma = 5/3$. The contours depicted, going from inner to outer, are images of the semicircle under D for $v_+ =$`1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`. The outermost contour is the image under D^0, which is nearly indistinguishable from the image for $v_+ =$`1e-6`

In particular, this leads to the surprisingly simple constant small-amplitude limit, as we show by explicit computation in Appendix D for the case of a Burgers shock. The small-amplitude analysis of [39] shows that the Evans function in the general case converges to a multiple of the Burgers Evans function, yielding the result in the general case.

Remark 3.7. Note that the limiting equations, and the limiting Evans function D^0 are both independent of γ. To study high-Mach number stability for $\gamma > 3$, therefore, requires only to examine D^0 on successively larger semicircles. Thus, our methods in combination with the those of [2], allow us to determine stability in principle over any bounded interval in γ, for $\gamma \geq 1$ and for all Mach numbers $M \geq 1$.

Remark 3.8. As Fig. 2 suggests, an alternative method for determining stability, without reference to D^0, is to compute the full Evans function for sufficiently high Mach number. That is, nonvanishing of D^0, and thus stability of sufficiently high Mach number shocks for $\gamma \in [1, 3]$, can already be concluded by large-but-finite Mach number study of [2] together with the fact that a limit $D \to D^0$ exists (see also Remark 3.4).

3.6. Conclusions. The analytical result of Theorem 1.4 guarantees stability for $\gamma \geq 1$, M sufficiently large. For $\gamma \in [1, 2.5]$, our numerical results indicate stability for $M \geq$ 2,500 by a crude Rouche bound, and indeed much lower if further structure is taken into account. Together with the small and intermediate Mach number studies of [2,36] for $M \leq 3,000$, this yields unconditional stability of isentropic Navier–Stokes shocks for $\gamma \in [1, 2.5]$ and $M \geq 1$. Additional intermediate-strength computations supplementing [2] extend this result to $\gamma \in [1, 3]$. There is no inherent restriction to $\gamma \in [1, 3]$; as discussed in Remark 3.7, numerical computations can be carried out for any value of γ to determine stability (or instability) for all $M \geq 1$. Indeed, our method of analysis indicates that the large-γ limit is quite analogous to the high-Mach number limit $v_+ \to 0$, suggesting the possibility to establish still more general results encompassing all $\gamma \geq 1$, $M \geq 1$.

Our numerical results reveal also an unexpected "universality" of behavior in the high-Mach number regime, beyond just convergence to the limiting system. Namely, we see (cf. Figs. 2 and 4) that behavior for a given v_+ is virtually independent of the value of γ. This also indicates that v_+ and not M is the more useful measure of shock strength in this regime.

4. Boundary-Layer Analysis

We now carry out the main work of the paper, analyzing the flow of (2.13) in the singular region. Our starting point is the observation that

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\hat{v} & \lambda\hat{v} & f(\hat{v}) - \lambda \end{pmatrix} \tag{4.1}$$

is approximately block upper-triangular for $\hat{v}$ sufficiently small, with diagonal blocks $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ and $\big(f(\hat{v}) - \lambda\big)$ that are uniformly spectrally separated on $\Re e\lambda \geq 0$, as follows by

$$f(\hat{v}) \leq 2\hat{v} - 1 \leq -1/2. \tag{4.2}$$

We exploit this structure by a judicious coordinate change converting (2.13) to a system in exact upper triangular form, for which the decoupled "slow" upper lefthand 2×2 block undergoes a *regular perturbation* that can be analyzed by standard tools introduced in [39]. Meanwhile, the fast, lower righthand 1×1 block, since scalar, may be solved exactly.

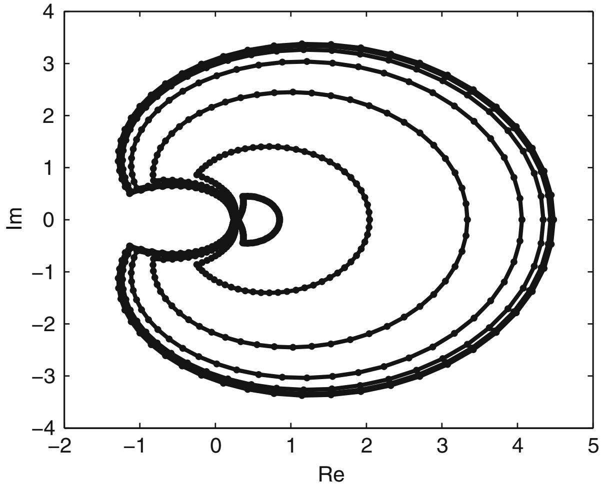

Fig. 3. Convergence to the limiting Evans function as $v_+ \to 0$ for $\gamma = 1$. The contours depicted, going from inner to outer, are images of the semicircle under D for $v_+ =$ `1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`. The outermost contour is the image under D^0

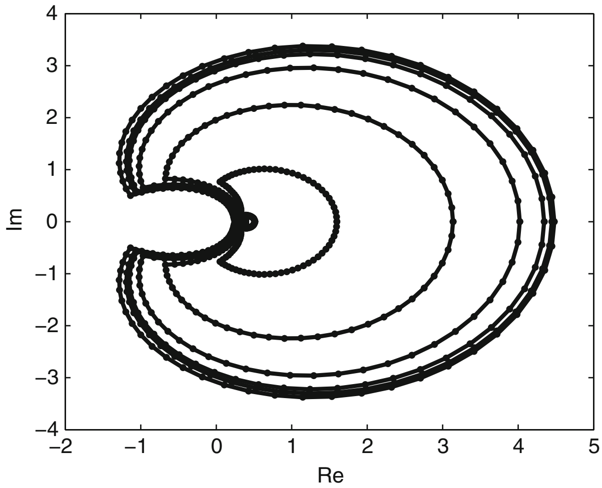

Fig. 4. Convergence to the limiting Evans function as $v_+ \to 0$ for $\gamma = 3$. The contours depicted, going from inner to outer, are images of the semicircle under D for $v_+ =$ `1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`. The outermost contour is the image under D^0

The global structure of this argument loosely follows the general strategy introduced in [39] of first decoupling fast and slow modes, then treating slow modes by regular perturbation methods. However, there are interesting departures that may be of use in other degenerate situations. First, we only partially decouple the system, to block-triangular rather than block-diagonal form as in more standard cases, and second, we introduce a more stable method of block-reduction taking account of usually negligible derivative terms in the definition of block-triangularizing transformations, which, if ignored, would in this case lead to unacceptably large errors.

4.1. Preliminary transformation. We first block upper-triangularize by a static (constant) coordinate transformation the limiting matrix

$$A_+ = A(+\infty, \lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda v_+ & \lambda v_+ & f(v_+) - \lambda \end{pmatrix} \tag{4.3}$$

at $x = +\infty$ using special block lower-triangular transformations

$$R_+ := \begin{pmatrix} I & 0 \\ \lambda v_+ \theta_+ & 1 \end{pmatrix}, \qquad L_+ := R_+^{-1} = \begin{pmatrix} I & 0 \\ -\lambda v_+ \theta_+ & 1 \end{pmatrix}, \tag{4.4}$$

where I denotes the 2×2 identity matrix and $\theta_+ \in \mathbb{C}^{1\times 2}$ is a 1×2 row vector.

Lemma 4.1. *For each $\gamma \geq 1$, on any compact subset of $\Re e\lambda \geq 0$, for each $v_+ > 0$ sufficiently small, there exists a unique $\theta_+ = \theta_+(v_+, \lambda)$ such that $\hat{A}_+ := L_+ A_+ R_+$ is upper block-triangular,*

$$\hat{A}_+ = \begin{pmatrix} \lambda(J + v_+ \mathbb{1}\theta_+) & \mathbb{1} \\ 0 & f(v_+) - \lambda - \lambda v_+ \theta_+ \mathbb{1} \end{pmatrix}, \tag{4.5}$$

where $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, satisfying a uniform bound

$$|\theta_+| \leq C. \tag{4.6}$$

Proof. Setting the (2, 1) block of $\hat{A}_+$ to zero, we obtain the matrix equation

$$\theta_+(bI - \lambda J) = -\mathbb{1}^T + \lambda v_+ \theta_+ \mathbb{1} \theta_+,$$

where $b = f(v_+) - \lambda$, or, equivalently, the fixed-point equation

$$\theta_+ = \left(-\mathbb{1}^T + \lambda v_+ \theta_+ \mathbb{1} \theta_+\right)(bI - \lambda J)^{-1}. \tag{4.7}$$

By $\det(bI - \lambda J) = b^2 \neq 0$, $(bI - \lambda J)^{-1}$ is uniformly bounded on compact subsets of $\Re e\lambda \geq 0$ (indeed, it is uniformly bounded on all of $\Re e\lambda \geq 0$), whence, for $|\lambda|$ bounded and v_+ sufficiently small, there exists a unique solution by the Contraction Mapping Theorem, which, moreover, satisfies (4.6). □

4.2. Dynamic triangularization. Defining now $Y := L_+ W$ and

$$\begin{aligned} \hat{A}(x, \lambda) &= L_+ A(x, \lambda) R_+ \\ &= \begin{pmatrix} \lambda(J + v_+ \mathbb{1}\theta_+) & \mathbb{1} \\ \lambda(\hat{v} - v_+)\mathbb{1}^T + \lambda v_+ (f(\hat{v}) - f(v_+))\theta_+ & f(\hat{v}) - \lambda - \lambda v_+ \theta_+ \mathbb{1} \end{pmatrix}, \end{aligned}$$

we have converted (2.13) to an asymptotically block upper-triangular system

$$Y' = \hat{A}(x, \lambda) Y, \tag{4.8}$$

with $\hat{A}_+ = \hat{A}(+\infty, \lambda)$ as in (4.5). Our next step is to choose a *dynamic* transformation of the same form

$$\tilde{R} := \begin{pmatrix} I & 0 \\ \tilde{\Theta} & 1 \end{pmatrix}, \qquad \tilde{L} := \tilde{R}^{-1} = \begin{pmatrix} I & 0 \\ -\tilde{\Theta} & 1 \end{pmatrix}, \tag{4.9}$$

converting (4.8) to an exactly block upper-triangular system, with $\tilde{\Theta}$ uniformly exponentially decaying at $x = +\infty$: that is, a *regular perturbation* of the identity.

Lemma 4.2. *For each $\gamma \geq 1$, on any compact subset of $\Re e\lambda \geq 0$, for L sufficiently large and each $v_+ > 0$ sufficiently small, there exists a unique $\tilde{\Theta} = \tilde{\Theta}(x, \lambda, v_+)$ such that $\tilde{A} := \tilde{L}\hat{A}(x, \lambda)\tilde{R} + \tilde{L}'\tilde{R}$ is upper block-triangular,*

$$\tilde{A} = \begin{pmatrix} \lambda(J + v_+ \mathbb{1}\theta_+) + \mathbb{1}\tilde{\Theta} & \mathbb{1} \\ 0 & f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1} - \tilde{\Theta}\mathbb{1} \end{pmatrix}, \tag{4.10}$$

and $\tilde{\Theta}(L) = 0$, satisfying a uniform bound

$$|\tilde{\Theta}(x, \lambda, v_+)| \leq Ce^{-\eta x}, \qquad \eta > 0,\ x \geq L, \tag{4.11}$$

independent of the choice of L, v_+.

Proof. Setting the (2, 1) block of $\tilde{A}$ to zero and computing

$$\tilde{L}'\tilde{R} = \begin{pmatrix} 0 & 0 \\ -\tilde{\Theta}' & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{\Theta} & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\tilde{\Theta}' & 0 \end{pmatrix},$$

we obtain the matrix equation

$$\tilde{\Theta}' - \tilde{\Theta}\,(bI - \lambda(J + v_+\mathbb{1}\theta_+)) = \zeta + \tilde{\Theta}\mathbb{1}\tilde{\Theta}, \tag{4.12}$$

where $b(x, \lambda, v_+) := f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1}$ and the forcing term

$$\zeta := \lambda(\hat{v} - v_+)\mathbb{1}^T + \lambda v_+(f(\hat{v}) - f(v_+))\theta_+$$

by derivative estimate $df/d\hat{v} \leq C\hat{v}^{-1}$ together with the Mean Value Theorem is uniformly exponentially decaying:

$$|\zeta| \leq C|\hat{v} - v_+| \leq C_2e^{-\eta x}, \qquad \eta > 0. \tag{4.13}$$

Initializing $\tilde{\Theta}(L) = 0$, we obtain by Duhamel's Principle/Variation of Constants the representation (supressing the argument λ)

$$\tilde{\Theta}(x) = \int_L^x S^{y\to x}(\zeta + \tilde{\Theta}\mathbb{1}\tilde{\Theta})(y)\,dy, \tag{4.14}$$

where $S^{y\to x}$ is the solution operator for the homogeneous equation

$$\tilde{\Theta}' - \tilde{\Theta}\,(bI - \lambda(J + v_+\mathbb{1}\theta_+)) = 0,$$

or, explicitly,

$$S^{y\to x} = e^{\int_y^x b(y)dy}e^{-\lambda(J+v_+\mathbb{1}\theta_+)(x-y)}.$$

For $|\lambda|$ bounded and v_+ sufficiently small, we have by matrix perturbation theory that the eigenvalues of $-\lambda(J + v_+ 1\!\!1\theta_+)$ are small and the entries are bounded, hence

$$|e^{-\lambda(J+v_+ 1\!\!1\theta_+)z}| \le Ce^{\epsilon z}$$

for $z \ge 0$. Recalling the uniform spectral gap $\Re eb = f(\hat{v}) - \Re e\lambda \le -1/2$ for $\Re e\lambda \ge 0$, we thus have

$$|S^{y\to x}| \le Ce^{-\eta(x-y)} \tag{4.15}$$

for some $C, \eta > 0$. Combining (4.13) and (4.15), we obtain

$$\begin{aligned}\left|\int_L^x S^{y\to x}\zeta(y)\,dy\right| &\le \int_L^x C_2 e^{-\eta(x-y)}e^{-(\eta/2)y}dy\\ &= C_3 e^{-(\eta/2)x}.\end{aligned} \tag{4.16}$$

Defining $\tilde{\Theta}(x) =: \tilde{\theta}(x)e^{-(\eta/2)x}$ and recalling (4.14) we thus have

$$\tilde{\theta}(x) = f + e^{(\eta/2)x}\int_L^x S^{y\to x}e^{-\eta y}\tilde{\theta}\, 1\!\!1\tilde{\theta}(y)\,dy, \tag{4.17}$$

where $f := e^{(\eta/2)x}\int_L^x S^{y\to x}\zeta(y)\,dy$ is uniformly bounded, $|f| \le C_3$, and

$$e^{(\eta/2)x}\int_L^x S^{y\to x}e^{-\eta y}\tilde{\theta}\, 1\!\!1\tilde{\theta}(y)\,dy$$

is contractive with arbitrarily small contraction constant $\epsilon > 0$ in $L^\infty[L, +\infty)$ for $|\tilde{\theta}| \le 2C_3$ for L sufficiently large, by the calculation

$$\begin{aligned}&\left|e^{(\eta/2)x}\int_L^x S^{y\to x}e^{-\eta y}\tilde{\theta}_1 1\!\!1\tilde{\theta}_1(y) - e^{(\eta/2)x}\int_L^x S^{y\to x}e^{-\eta y}\tilde{\theta}_2 1\!\!1\tilde{\theta}_2(y)\right|\\ &\quad\le \left|e^{(\eta/2)x}\int_L^x Ce^{-\eta(x-y)}e^{-\eta y}\,dy\right| \|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty\\ &\quad\le e^{-(\eta/2)L}\left|\int_L^x Ce^{-(\eta/2)(x-y)}\,dy\right| \|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty\\ &\quad= C_3 e^{-(\eta/2)L}\|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty.\end{aligned}$$

It follows by the Contraction Mapping Principle that there exists a unique solution $\tilde{\theta}$ of fixed point equation (4.17) with $|\tilde{\theta}(x)| \le 2C_3$ for $x \ge L$, or, equivalently (redefining the unspecified constant η), (4.11). □

Remark 4.3. The above calculation is the most delicate part of the analysis, and the main technical point of the paper. The interested reader may verify that a "quasi-static" transformation treating term $\tilde{\Theta}'$ in (4.12) as an error, as is typically used in situations of slowly-varying coefficients (see for example [33,39]), would lead to unacceptable errors of magnitude

$$O(|(f(\hat{v}))'||\hat{v} - v_+|) = O(|df/d\hat{v}||\hat{v} - v_+|) = O(|\hat{v}|^{-1}|\hat{v} - v_+|).$$

One may think of the exact ODE solution (4.9) as "averaging" the effects of rapidly-varying coefficients by integration of (4.12). Note that success of this approach depends in principle only on spectral separation at infinity and not everywhere along the profile; thus, it may be considered as interpolating between the tracking and conjugation lemmas of [50,33,48,49,39], used respectively to treat slowly-varying- and asymptotically constant- coefficient systems. This seems likely to be of use also in other delicate situations.

Remark 4.4. One may combine the static and the dynamic triangularization used in this section. We may define

$$\Theta = \tilde{\Theta} + \lambda v_+ \theta_+, \tag{4.18}$$

and obtain the two different terms $\tilde{\Theta}$ and $\lambda v_+ \theta_+$ as follows. Consider $\tilde{A}$ being defined as $L'R + L\hat{A}R$, where $L = \begin{pmatrix} I & 0 \\ -\Theta & 1 \end{pmatrix}$ and $R := L^{-1}$. One obtains

$$\tilde{A} = \begin{pmatrix} \lambda J + 1\!\!1 \Theta & 1\!\!1 \\ -\Theta' + (f(\hat{v}) - \lambda - \Theta 1\!\!1)\Theta - \lambda \Theta J + \lambda \hat{v} 1\!\!1^T & f(\hat{v}) - \lambda - \Theta 1\!\!1 \end{pmatrix}.$$

A necessary and sufficient condition for $\tilde{A}$ to be upper block-triangular is that Θ solve

$$\Theta' = (f(\hat{v}) - \lambda - \Theta . 1\!\!1)\Theta - \lambda \Theta J + \lambda \hat{v} 1\!\!1^T. \tag{4.19}$$

If a solution Θ is bounded at $+\infty$ and $\Theta' \to 0$, then the limit of Θ, called Θ_+, solves $(f(v_+) - \lambda - \Theta_+ 1\!\!1)\Theta_+ - \lambda \Theta_+ J + \lambda v_+ 1\!\!1^T = 0$. The unique solution of this equation that is uniformly bounded by $C|\lambda| v_+$ for $0 < v_+ < 1/C$ and λ in a compact subset of $\Re e \lambda \geq 0$ is $\Theta_+ = \lambda v_+ \theta_+$. Considering now $\tilde{\Theta} = \Theta - \lambda v_+ \theta_+$ and replacing in Eq. (4.19), one obtains the equation (4.12) for $\tilde{\Theta}$.

The preliminary, static, transformation is thus a way of subtracting the behavior at infinity (and choosing this behavior) in the dynamic triangularization and can be used in general. For each possible behavior at infinity, however, we have a different solution $\tilde{\Theta}$ and our construction is the only one that ensures the uniform bounds in v_+ at infinity.

4.3. Fast/Slow dynamics. Making now the further change of coordinates

$$Z = \tilde{L} Y$$

and computing

$$(\tilde{L}Y)' = \tilde{L}Y' + \tilde{L}'Y = (\tilde{L}A_+ + \tilde{L}')Y = (\tilde{L}A_+\tilde{R} + \tilde{L}'\tilde{R})Z,$$

we find that we have converted (4.8) to a block-triangular system

$$Z' = \tilde{A}Z = \begin{pmatrix} \lambda(J + v_+ 1\!\!1 \theta_+) + 1\!\!1 \tilde{\Theta} & 1\!\!1 \\ 0 & f(\hat{v}) - \lambda - \lambda v_+ \theta_+ 1\!\!1 - \tilde{\Theta} 1\!\!1 \end{pmatrix} Z, \tag{4.20}$$

related to the original eigenvalue system (2.13) by

$$W = LZ, \quad R := R_+ \tilde{R} = \begin{pmatrix} I & 0 \\ \Theta & 1 \end{pmatrix}, \quad L := R^{-1} = \begin{pmatrix} I & 0 \\ -\Theta & 1 \end{pmatrix}. \tag{4.21}$$

Since it is triangular, (4.20) may be solved completely if we can solve the component systems associated with its diagonal blocks. The *fast system*

$$z' = \Big(f(\hat{v}) - \lambda - \lambda v_+ \theta_+ \mathbb{1} - \tilde{\Theta}\mathbb{1}\Big) z$$

associated to the lower righthand block features rapidly-varying coefficients. However, because it is scalar, it can be solved explicitly by exponentiation.

The *slow system*

$$z' = \Big(\lambda(J + v_+ \mathbb{1}\theta_+) + \mathbb{1}\tilde{\Theta}\Big) z \tag{4.22}$$

associated to the upper lefthand block, on the other hand, by (4.11), is an exponentially decaying perturbation of a constant-coefficient system

$$z' = \lambda(J + v_+ \mathbb{1}\theta_+) z \tag{4.23}$$

that can be explicitly solved by exponentiation, and thus can be well-estimated by comparison with (4.23). A rigorous version of this statement is given by the *conjugation lemma* of [37]:

Proposition 4.5 ([37]). *Let $M(x, \lambda) = M_+(\lambda) + \Theta(x, \lambda)$, with M_+ continuous in λ and $|\Theta(x, \lambda)| \le Ce^{-\eta x}$, for λ in some compact set Λ. Then, there exists a globally invertible matrix $P(x, \lambda) = I + Q(x, \lambda)$ such that the coordinate change $z = Pv$ converts the variable-coefficient ODE $z' = M(x, \lambda)z$ to a constant-coefficient equation*

$$v' = M_+(\lambda)v,$$

satisfying for any L, $0 < \hat{\eta} < \eta$ a uniform bound

$$|Q(x, \lambda)| \le C(L, \hat{\eta}, \eta, \max|(M_+)_{ij}|, \dim M_+)e^{-\hat{\eta}x} \quad \text{for } x \ge L. \tag{4.24}$$

Proof. See [37,49], or Appendix C. □

By Proposition 4.5, the solution operator for (4.22) is given by

$$P(x, \lambda)e^{\lambda(J + v_+ \mathbb{1}\theta_+(\lambda, v_+))(x-y)} P(y, \lambda)^{-1}, \tag{4.25}$$

where P is a uniformly small perturbation of the identity for $x \ge L$ and $L > 0$ sufficiently large.

5. Proof of the Main Theorem

With these preparations, we turn now to the proof of the main theorem.

5.1. Boundary estimate. We begin by establishing the following key estimates on $\widetilde{W}_1^+(L)$, that is, the value of the dual mode $\widetilde{W}_1^+$ appearing in (2.20) at the boundary $x = L$ between regular and singular regions.

Lemma 5.1. *For each $\gamma \geq 1$, for λ on any compact subset of $\Re e\lambda \geq 0$, and $L > 0$ sufficiently large, with $\widetilde{W}_1^+$ normalized as in* [13,39,2],

$$|\widetilde{W}_1^+(L) - \widetilde{V}_1| \leq Ce^{-\eta L} \tag{5.1}$$

as $v_+ \to 0$, uniformly in λ, where $C, \eta > 0$ are independent of L and

$$\widetilde{V}_1 := (0, 1, (1 + \bar{\lambda})^{-1})^T$$

is the limiting direction vector (3.9) appearing in the definition of D^0.

Corollary 5.2. *For each $\gamma \geq 1$, under the hypotheses of Lemma 5.1,*

$$|\tilde{W}_1^{0+}(L) - \widetilde{V}_1| \leq Ce^{-\eta L} \tag{5.2}$$

and

$$|\widetilde{W}_1^+(L) - \widetilde{W}_1^{0+}(L)| \leq Ce^{-\eta L} \tag{5.3}$$

as $v_+ \to 0$, uniformly in λ, where $C, \eta > 0$ are independent of L and $\widetilde{W}_1^{0+}$ is the solution of the limiting adjoint eigenvalue system appearing in definition (3.8) of D^0.

Proof of Lemma 5.1. Making the coordinate-change

$$\tilde{Z} := R^* \tilde{W}, \tag{5.4}$$

R as in (4.21), reduces the adjoint equation $\tilde{W}' = -A^* \tilde{W}$ to block lower-triangular form,

$$\begin{aligned} \tilde{Z}' &= -\tilde{A}^* \tilde{Z} \\ &= \begin{pmatrix} -\left(\lambda J + \lambda v_+ 1\!\!1 \theta_+) + 1\!\!1 \tilde{\Theta}\right)^* & 0 \\ -1\!\!1^T & -f(\hat{v}) + \bar{\lambda} + \bar{\lambda} v_+ (\theta_+ 1\!\!1)^* + (\tilde{\Theta} 1\!\!1)^* \end{pmatrix} Z, \end{aligned} \tag{5.5}$$

with "$^-$" denoting complex conjugate.

Denoting by $\tilde{V}_1^+$ a suitably normalized element of the one-dimensional (slow) stable subspace of $-\tilde{A}^*$, we find, similarly as in the discussion of Sect. 3.2 that, without loss of generality,

$$\tilde{V}_1^+ \to (0, 1, (\gamma + \bar{\lambda})^{-1})^T \tag{5.6}$$

as $v_+ \to 0$, while the associated eigenvalue $\tilde{\mu}_1^+ \to 0$, uniformly for λ on an compact subset of $\Re e\lambda \geq 0$. The dual mode $\tilde{Z}_1^+ = R^* \tilde{W}_1^+$ is uniquely determined by the property that it is asymptotic as $x \to +\infty$ to the corresponding constant-coefficient solution $e^{\tilde{\mu}_1^+} \tilde{V}_1^+$ (the standard normalization of [2,13,39]).

By lower block-triangular form (5.5), the equations for the slow variable $\tilde{z}^T := (\tilde{Z}_1, \tilde{Z}_2)$ decouples as a slow system

$$\tilde{z}' = -\left(\lambda(J + v_+ 1\!\!1 \theta_+) + 1\!\!1 \tilde{\Theta}\right)^* \tilde{z} \tag{5.7}$$

dual to (4.22), with solution operator

$$P^*(x,\lambda)^{-1} e^{-\bar\lambda (J+v_+\mathbb{1}\theta_+)^*)(x-y)} P(y,\lambda)^* \tag{5.8}$$

dual to—that is, the adjoint inverse of—(4.25), i.e. (fixing $y = L$, say), having solutions of general form

$$\tilde z(\lambda, x) = P^*(x,\lambda)^{-1} e^{-\bar\lambda (J+v_+\mathbb{1}\theta_+)^*)x} \tilde v, \tag{5.9}$$

$\tilde v \in \mathbb{C}^2$ arbitrary.

Denoting by

$$\tilde Z_1^+(L) := R^* \tilde W_1^+(L),$$

therefore, the unique (up to constant factor) decaying solution at $+\infty$, and $\tilde v_1^+ := ((\tilde V_1^+)_1, (\tilde V_1^+)_2)^T$, we thus have evidently

$$\tilde z_1^+(x,\lambda) = P^*(x,\lambda)^{-1} e^{-\bar\lambda (J+v_+\mathbb{1}\theta_+)^*)x} \tilde v_1^+,$$

which, as $v_+ \to 0$, is uniformly bounded by

$$|\tilde z_1^+(x,\lambda)| \le C e^{\epsilon x} \tag{5.10}$$

for arbitrarily small $\epsilon > 0$ and, by (5.6), converges for x less than or equal to any fixed X simply to

$$\lim_{v_+\to 0} \tilde z_1^+(x,\lambda) = P^*(x,\lambda)^{-1}(0,1)^T. \tag{5.11}$$

Defining by $\tilde q := (\tilde Z_1^+)_3$ the fast coordinate of $\tilde Z_1^+$, we have, by (5.5),

$$\tilde q' + \Big(f(\hat v) - \bar\lambda - (\lambda v_+ \theta_+ \mathbb{1} + \tilde\Theta \mathbb{1})^* \Big) \tilde q = \mathbb{1}^T \tilde z_1^+,$$

whence, by Duhamel's principle, any decaying solution is given by

$$\tilde q(x,\lambda) = \int_x^{+\infty} e^{\int_y^x \tilde b(z,\lambda,v_+)dz} \mathbb{1}^T z_1^+(y)\, dy,$$

where

$$\tilde b(y,\lambda,v_+) := -\bar b(y,\lambda,v_+) = -\Big(f(\hat v) - \bar\lambda - (\lambda v_+\theta_+\mathbb{1} + \tilde\Theta\mathbb{1})^* \Big),$$

b as in (4.12). Recalling, for $\Re e\lambda \ge 0$, that $\Re e\tilde b = -\Re b \ge 1/2$, combining (5.10) and (5.11), and noting that $\tilde b$ converges uniformly on $y \le Y$ as $v_+ \to 0$ for any $Y > 0$ to

$$\begin{aligned} \tilde b_0(y,\lambda) &:= -f_0(\hat v) + \bar\lambda + (\tilde\Theta_0 \mathbb{1})^* \\ &= (1+\bar\lambda) + O(e^{-\eta y}), \end{aligned}$$

we obtain by the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned} \tilde q(L,\lambda) &\to \int_L^{+\infty} e^{\int_y^L \tilde b_0(z,\lambda)dz} \mathbb{1}^T (0,1)^T\, dy \\ &= \int_L^{+\infty} e^{(1+\bar\lambda)(L-y) + \int_y^L O(e^{-\eta z})dz}\, dy \\ &= (1+\bar\lambda)^{-1}(1 + O(e^{-\eta L})). \end{aligned}$$

Recalling, finally, (5.11), and the fact that

$$|P - Id|(L, \lambda),\ |R - Id|(L, \lambda) \leq Ce^{-\eta L}$$

for v_+ sufficiently small, we obtain (5.1) as claimed. □

Proof of Corollary 5.2. Applying Proposition 4.5 to the limiting adjoint system,

$$\tilde{W}' = -(A^0)^* \tilde{W} = \begin{pmatrix} 0 & 0 & 0 \\ -\bar{\lambda} & 0 & 0 \\ -1 & -1 & 1 + \bar{\lambda} \end{pmatrix} \tilde{W} + O(e^{-\eta x}) \tilde{W},$$

we find that, up to an $Id + O(e^{-\eta x})$ coordinate change, $\tilde{W}_1^{0+}(x)$ is given by the exact solution $\tilde{W} \equiv \tilde{V}_1$ of the limiting, constant-coefficient system

$$\tilde{W}' = -(A^0)^* \tilde{W} = \begin{pmatrix} 0 & 0 & 0 \\ -\bar{\lambda} & 0 & 0 \\ -1 & -1 & 1 + \bar{\lambda} \end{pmatrix} \tilde{W}.$$

This yields immediately (5.2), which, together with (5.1), yields (5.3). □

Remark 5.3. Noting that (5.2) is sharp, we see from (5.3) that the error between $\tilde{W}_1^{+}(L)$ and $\tilde{W}_1^{0+}(L)$ is already within the error tolerance of the numerical scheme used to approximate D^0, in which $\widetilde{W}_1^{0+}$ is initialized at $x = L$ with approximate value $\tilde{V}_1$ [2,11,39]. Thus, so long as the flow on the regular region $x \leq L$ well-approximates the exact limiting flow as $v_+ \to 0$, we can expect convergence of D to D^0 based on the known convergence of the numerical approximation scheme.

5.2. Convergence to D^0. As hinted by Remark 5.3, the rest of our analysis is standard if not entirely routine.

Lemma 5.4. *For each $\gamma \geq 1$, on $x \leq L$ for any fixed $L > 0$, there exists a coordinate-change $W = TZ$ conjugating (2.13) to the limiting equations (3.4), $T = T(x, \lambda, v_+)$, satisfying a uniform bound*

$$|T - Id| \leq C(L) v_+ \tag{5.12}$$

for all $v_+ > 0$ sufficiently small.

Proof. For $x \in (-\infty, 0]$, this is a consequence of the *Convergence Lemma* of [39], a variation on Proposition 4.5, together with uniform convergence of the profile and eigenvalue equations. For $x \in [0, L]$, it is essentially continuous dependence; more precisely, observing that $|A - A^0| \leq C_1(L) v_+$ for $x \in [0, L]$, setting $S := T - Id$, and writing the homological equation expressing conjugacy of (2.13) and (3.4), we obtain

$$S' - (AS - SA^0) = (A - A^0),$$

which, considered as an inhomogeneous linear matrix-valued equation, yields an exponential growth bound

$$S(x) \leq e^{Cx} (S(0) + C^{-1} C_1(L) v_+)$$

for some $C > 0$, giving the result. □

Proof of Theorem 3.1. Lemma 5.4, together with convergence as $v_+ \to 0$ of the unstable subspace of A_- to the unstable subspace of A_-^0 at the same rate $O(v_+)$ (as follows by spectral separation of the unstable eigenvalue of A^0 and standard matrix perturbation theory) yields

$$|W_1^-(0,\lambda) - W_1^{0-}(0,\lambda)| \le C(L)v_+. \tag{5.13}$$

Likewise, Lemma 5.4 gives

$$\begin{aligned}|\tilde{W}_1^+(0,\lambda) - \tilde{W}_1^{0+}(0,\lambda)| &\le C(L)v_+|\tilde{W}_1^+(0,\lambda)| \\ &\quad + |S_0^{L\to 0}||\tilde{W}_1^+(L,\lambda) - \tilde{W}_1^{0+}(L,\lambda)|,\end{aligned} \tag{5.14}$$

where $S_0^{y\to x}$ denotes the solution operator of the limiting adjoint eigenvalue equation $\tilde{W}' = -(A^0)^*\tilde{W}$. Applying Proposition 4.5 to the limiting system, we obtain

$$|S_0^{L\to 0}| \le C_2 e^{-A_+^0 L} \le C_2 L|\lambda|$$

by direct computation of $e^{-A_+^0 L}$, where C_2 is independent of $L > 0$. Together with (5.3) and (5.14), this gives

$$|\tilde{W}_1^+(0,\lambda) - \tilde{W}_1^{0+}(0,\lambda)| \le C(L)v_+|\tilde{W}_1^+(0,\lambda)| + L|\lambda|C_2Ce^{-\eta L},$$

hence, for $|\lambda|$ bounded,

$$\begin{aligned}|\tilde{W}_1^+(0,\lambda) - \tilde{W}_1^{0+}(0,\lambda)| &\le C_3(L)v_+|\tilde{W}_1^{0+}(0,\lambda)| + LC_4e^{-\eta L} \\ &\le C_5(L)v_+ + LC_4e^{-\eta L}.\end{aligned} \tag{5.15}$$

Taking first $L \to \infty$ and then $v_+ \to 0$, we obtain therefore convergence of $W_1^+(0,\lambda)$ and $\tilde{W}_1^+(0,\lambda)$ to $W_1^{0+}(0,\lambda)$ and $\tilde{W}_1^{0+}(0,\lambda)$, yielding the result by definitions (2.20) and (3.8).

6. Numerical Convergence

Having established analytically the convergence of D to D^0 as $v_+ \to 0$ ($M \to \infty$), we turn finally to numerics to obtain quantitative information yielding a concrete stability threshold. Specifically, for fixed γ, we numerically compute the "Rouché bound" consisting of the value of v_+ at which the maximum relative error $|D - D^0|/|D^0|$ over the semicircular contour $\partial\{\Re e\lambda \ge 0,\ |\lambda| \le 10\}$ around which we perform our winding number calculations becomes $1/2$. Recall that Rouché's Theorem guarantees for relative error < 1 that the winding number of D is equal to the winding number of D^0, which we have shown to be zero, hence we may conservatively conclude stability for v_+ less than or equal to this bound, or M greater than or equal to the corresponding Mach number. Computations are performed using the algorithm of [2]; results are displayed in Table 1.

More detailed results are displayed for the monatomic gas case $\gamma = 5/3$ in Table 2. Results are similar for other $\gamma \in [1, 3]$, as may be seen by comparing Figs. 2–4.

From the quantitative gap and conjugation estimates given in Appendix C, which in turn yield a quantitative version of the Convergence Lemma of [39], one could in principle establish quantitative convergence rates for $|D - D^0|$, by tracking constants carefully through the estimates of the previous sections. Indeed, one could do much better than the rather crude bounds stated for the general case by taking into account the

Table 1. Rouché bounds for various γ

γ	v_+	Relative Error	Mach Number
3.0	1.27e-3	.5009	12765
2.5	1.36e-3	.5006	2423
2.0	1.49e-3	.5001	474
1.5	1.75e-3	.4999	95.5
1.0	2.8e-3	.4995	18.9

Table 2. Maximum relative and absolute differences between D and D^0, for $\gamma = 5/3$ and λ on the semicircle of radius 10

v_+	Mach Number	Relative Difference	Absolute Difference
1.0 (−6)	7.71 (4)	0.1221	0.0601
1.0 (−5)	1.13 (4)	0.1236	0.1445
1.0 (−4)	1.64 (3)	0.1487	0.4714
1.0 (−3)	2.44 (2)	0.4098	1.3464
1.0 (−2)	36.1	0.9046	2.8253
1.0 (−1)	5.50	1.2386	3.8688

eigenstructure of the actual matrices $A_\pm$, $A^0_\pm$ appearing in our analysis, and (crucial for good estimates, since bounds grow exponentially with dimension n) by observing that it suffices to use the gap and not the full conjugation lemma for the estimation of the single dual mode at plus infinity. That is, there are contained in our analysis, as in the study of [2], all of the ingredients needed for a numerical proof. Given the fundamental nature of the problem studied, this would be a very interesting program to carry out.

7. Discussion and Open Problems

Besides long-time stability, our results have application also to existence of shock layers in the small-viscosity limit, which likewise reduces to the question of stability of the Evans function [17,41]. Indeed, spectral stability has been a key missing piece in several directions [48,49]. Our methods should have application also to spectral stability of large-amplitude noncharacteristic boundary layers, completing the investigations of [16,37,44]. It may be hoped that they will extend also to full gas dynamics and multi-dimensions, two important directions for further investigation. As discussed in the text, the problems of numerical proof and of stability in the large-γ limit are two other interesting directions for further study.

More speculatively, our results suggest the possibility of a large-variation version of the results obtained by quite different methods in [7] on general viscous solutions (including not only noninteracting shocks, but shocks, rarefactions, and their interactions), and, through the physical insight provided into the high-Mach number limit, perhaps even a hint toward possible methods of analysis. Note that the results of [7] include not only convergence in the small-viscosity limit but also bounded L^1 stability for constant viscosity of general small-variation solutions. This would be an extremely interesting direction for further investigation.

Appendix A. Proofs of Preliminary Estimates

Proof of Proposition 2.1. Existence and monotonicity follow trivially by the fact that (2.4) is a scalar first-order ODE with convex righthand side. Exponential convergence as $x \to +\infty$ follows, for example, by the computation

$$\begin{aligned} H(v, v_+) &= v\left((v-1) - \frac{(v_+-1)(v^{-\gamma}-1)}{v_+^{-\gamma}-1}\right) \\ &= v\left((v-v_+) + \left(\frac{1-v_+}{1-v_+^{\gamma}}\right)\left(\left(\frac{v_+}{v}\right)^{\gamma}-1\right)\right) \\ &= (v-v_+)\left(v - \left(\frac{1-v_+}{1-v_+^{\gamma}}\right)\left(\frac{1-\left(\frac{v_+}{v}\right)^{\gamma}}{1-\left(\frac{v_+}{v}\right)}\right)\right), \end{aligned}$$

yielding

$$v - \gamma \le \frac{H(v, v_+)}{v - v_+} \le v - (1 - v_+)$$

by the elementary estimate $1 \le \frac{1-x^{\gamma}}{1-x} \le \gamma$ for $0 \le x \le 1$. Convergence as $x \to -\infty$ follows by a similar, but simpler computation; see [2]. □

Lemma A.1. *For each $\gamma \ge 1$, the following identity holds for $\Re e\lambda \ge 0$:*

$$\begin{aligned} &(\Re e(\lambda) + |\Im m(\lambda)|)\int_{\mathbb{R}} \hat{v}|u|^2 - \frac{1}{2}\int_{\mathbb{R}} \hat{v}_x|u|^2 + \int_{\mathbb{R}} |u'|^2 \\ &\le \sqrt{2}\int_{\mathbb{R}} \frac{h(\hat{v})}{\hat{v}^{\gamma}}|v'||u| + \int_{\mathbb{R}} \hat{v}|u'||u|. \end{aligned} \tag{A.1}$$

Proof. We multiply (2.9b) by $\hat{v}\bar{u}$ and integrate along x. This yields

$$\lambda\int_{\mathbb{R}} \hat{v}|u|^2 + \int_{\mathbb{R}} \hat{v}u'\bar{u} + \int_{\mathbb{R}} |u'|^2 = \int_{\mathbb{R}} \frac{h(\hat{v})}{\hat{v}^{\gamma}}v'\bar{u}.$$

We get (A.1) by taking the real and imaginary parts and adding them together, and noting that $|\Re e(z)| + |\Im m(z)| \le \sqrt{2}|z|$. □

Lemma A.2. *For each $\gamma \ge 1$, the following identity holds for $\Re e\lambda \ge 0$:*

$$\int_{\mathbb{R}} |u'|^2 = 2\Re e(\lambda)^2\int_{\mathbb{R}} |v|^2 + \Re e(\lambda)\int_{\mathbb{R}} \frac{|v'|^2}{\hat{v}} + \frac{1}{2}\int_{\mathbb{R}}\left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}}\right]|v'|^2. \tag{A.2}$$

Proof. We multiply (2.9b) by $\bar{v}'$ and integrate along x. This yields

$$\lambda\int_{\mathbb{R}} u\bar{v}' + \int_{\mathbb{R}} u'\bar{v}' - \int_{\mathbb{R}} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 = \int_{\mathbb{R}} \frac{1}{\hat{v}}u''\bar{v}' = \int_{\mathbb{R}} \frac{1}{\hat{v}}(\lambda v' + v'')\bar{v}'.$$

Using (2.9a) on the right-hand side, integrating by parts, and taking the real part gives

$$\Re e\left[\lambda\int_{\mathbb{R}} u\bar{v}' + \int_{\mathbb{R}} u'\bar{v}'\right] = \int_{\mathbb{R}}\left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{\hat{v}_x}{2\hat{v}^2}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}} \frac{|v'|^2}{\hat{v}}.$$

The right hand side can be rewritten as

$$\Re e\left[\lambda\int_{\mathbb{R}} u\bar{v}' + \int_{\mathbb{R}} u'\bar{v}'\right] = \frac{1}{2}\int_{\mathbb{R}}\left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}}\frac{|v'|^2}{\hat{v}}. \tag{A.3}$$

Now we manipulate the left-hand side. Note that

$$\begin{aligned}\lambda\int_{\mathbb{R}} u\bar{v}' + \int_{\mathbb{R}} u'\bar{v}' &= (\lambda+\bar{\lambda})\int_{\mathbb{R}} u\bar{v}' - \int_{\mathbb{R}} u(\bar{\lambda}\bar{v}' + \bar{v}'') \\ &= -2\Re e(\lambda)\int_{\mathbb{R}} u'\bar{v} - \int_{\mathbb{R}} u\bar{u}'' \\ &= -2\Re e(\lambda)\int_{\mathbb{R}}(\lambda v + v')\bar{v} + \int_{\mathbb{R}}|u'|^2.\end{aligned}$$

Hence, by taking the real part we get

$$\Re e\left[\lambda\int_{\mathbb{R}} u\bar{v}' + \int_{\mathbb{R}} u'\bar{v}'\right] = \int_{\mathbb{R}}|u'|^2 - 2\Re e(\lambda)^2\int_{\mathbb{R}}|v|^2.$$

This combines with (A.3) to give (A.2). □

Lemma A.3. *For each* $\gamma \ge 1$*, for* $h(\hat{v})$ *as in* (2.8)*, we have*

$$\sup_{\hat{v}}\left|\frac{h(\hat{v})}{\hat{v}^{\gamma}}\right| = \gamma\frac{1-v_+}{1-v_+^{\gamma}} \le \gamma. \tag{A.4}$$

Proof. Defining

$$g(\hat{v}) := h(\hat{v})\hat{v}^{-\gamma} = -\hat{v} + a(\gamma-1)\hat{v}^{-\gamma} + (a+1), \tag{A.5}$$

we have $g'(\hat{v}) = -1 - a\gamma(\gamma-1)\hat{v}^{-\gamma-1} < 0$ for $0 < v_+ \le \hat{v} \le v_- = 1$, hence the maximum of g on $\hat{v} \in [v_+, v_-]$ is achieved at $\hat{v} = v_+$. Substituting (2.5) into (A.5) and simplifying yields (A.4). □

Proof of Proposition 2.2. Using Young's inequality twice on right-hand side of (A.1) together with (A.4), we get

$$\begin{aligned}&(\Re e(\lambda) + |\Im m(\lambda)|)\int_{\mathbb{R}}\hat{v}|u|^2 - \frac{1}{2}\int_{\mathbb{R}}\hat{v}_x|u|^2 + \int_{\mathbb{R}}|u'|^2 \\ &\le \sqrt{2}\int_{\mathbb{R}}\frac{h(\hat{v})}{\hat{v}^{\gamma}}|v'||u| + \int_{\mathbb{R}}\hat{v}|u'||u| \\ &\le \theta\int_{\mathbb{R}}\frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \frac{(\sqrt{2})^2}{4\theta}\int_{\mathbb{R}}\frac{h(\hat{v})}{\hat{v}^{\gamma}}\hat{v}|u|^2 + \epsilon\int_{\mathbb{R}}\hat{v}|u'|^2 + \frac{1}{4\epsilon}\int_{\mathbb{R}}\hat{v}|u|^2 \\ &< \theta\int_{\mathbb{R}}\frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \epsilon\int_{\mathbb{R}}|u'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}}\hat{v}|u|^2.\end{aligned}$$

Assuming that $0 < \epsilon < 1$ and $\theta = (1-\epsilon)/2$, this simplifies to

$$\begin{aligned}&(\Re e(\lambda) + |\Im m(\lambda)|)\int_{\mathbb{R}}\hat{v}|u|^2 + (1-\epsilon)\int_{\mathbb{R}}|u'|^2 \\ &\quad < \frac{1-\epsilon}{2}\int_{\mathbb{R}}\frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}}\hat{v}|u|^2.\end{aligned}$$

Applying (A.2) yields

$$(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}} \hat{v}|u|^2 < \left[\frac{\gamma}{1-\epsilon} + \frac{1}{4\epsilon}\right] \int_{\mathbb{R}} \hat{v}|u|^2,$$

or equivalently,

$$(\Re e(\lambda) + |\Im m(\lambda)|) < \frac{(4\gamma - 1)\epsilon - 1}{4\epsilon(1-\epsilon)}.$$

Setting $\epsilon = 1/(2\sqrt{\gamma} + 1)$ gives (2.11). □

Appendix B. Nonvanishing of D^0

As pointed out in Sect. 3.3, the limiting eigenvalue system (3.3), (3.4), together with the limiting boundary conditions derived in Sect. 3.2 may be expressed equivalently as the integrated eigenvalue problem

$$\lambda v + v' - u' = 0, \tag{B.1a}$$

$$\lambda u + u' - \frac{1-\hat{v}}{\hat{v}} v' = \frac{u''}{\hat{v}}, \tag{B.1b}$$

corresponding to a pressureless gas, $\gamma = 0$, with special boundary conditions

$$(u, u', v, v')(-\infty) = (0, 0, 0, 0), \quad (u, u', v, v')(+\infty) = (c, 0, 0, 0). \tag{B.2}$$

Motivated by this observation, we establish stability of the limiting system by a Matsumura–Nishihara-type spectral energy estimate exactly analogous to that used to prove stability for $\gamma = 1$ in [2,36].

Proof of Proposition 3.5. Multiplying (B.1b) by $\hat{v}\bar{u}/(1-\hat{v})$ and integrating on some subinterval $[a.b] \subset \mathbb{R}$, we obtain

$$\lambda \int_a^b \frac{\hat{v}}{1-\hat{v}}|u|^2 dx + \int_a^b \frac{\hat{v}}{1-\hat{v}} u'\bar{u} dx - \int_a^b v'\bar{u} dx = \int_a^b \frac{u''\bar{u}}{1-\hat{v}} dx.$$

Integrating the third and fourth terms by parts yields

$$\begin{aligned} &\lambda \int_a^b \frac{\hat{v}}{1-\hat{v}}|u|^2 dx + \int_a^b \left[\frac{\hat{v}}{1-\hat{v}} + \left(\frac{1}{1-\hat{v}}\right)'\right] u'\bar{u} dx \\ &\quad + \int_a^b \frac{|u'|^2}{1-\hat{v}} dx + \int_a^b v(\overline{\lambda v + v'}) dx \\ &= \left[v\bar{u} + \frac{u'\bar{u}}{1-\hat{v}}\right]\Big|_a^b. \end{aligned}$$

Taking the real part, we have

$$\begin{aligned} &\Re e(\lambda) \int_a^b \left(\frac{\hat{v}}{1-\hat{v}}|u|^2 + |v|^2\right) dx + \int_a^b g(\hat{v})|u|^2 dx + \int_a^b \frac{|u'|^2}{1-\hat{v}} dx \\ &= \Re e\left[v\bar{u} + \frac{u'\bar{u}}{1-\hat{v}} - \frac{1}{2}\left[\frac{\hat{v}}{1-\hat{v}} + \left(\frac{1}{1-\hat{v}}\right)'\right]|u|^2 - \frac{|v|^2}{2}\right]\Big|_a^b, \end{aligned} \tag{B.3}$$

where

$$g(\hat{v}) = -\frac{1}{2}\left[\left(\frac{\hat{v}}{1-\hat{v}}\right)' + \left(\frac{1}{1-\hat{v}}\right)''\right].$$

Note that

$$\frac{d}{dx}\left(\frac{1}{1-\hat{v}}\right) = -\frac{(1-\hat{v})'}{(1-\hat{v})^2} = \frac{\hat{v}_x}{(1-\hat{v})^2} = \frac{\hat{v}(\hat{v}-1)}{(1-\hat{v})^2} = -\frac{\hat{v}}{1-\hat{v}}.$$

Thus, $g(\hat{v}) \equiv 0$ and the third term on the right-hand side vanishes, leaving

$$\Re e(\lambda)\int_a^b \left(\frac{\hat{v}}{1-\hat{v}}|u|^2 + |v|^2\right)dx + \int_a^b \frac{|u'|^2}{1-\hat{v}}dx$$
$$= \left[\Re e(v\bar{u}) + \frac{\Re e(u'\bar{u})}{1-\hat{v}} - \frac{|v|^2}{2}\right]\Big|_a^b .$$

We show next that the right-hand side goes to zero in the limit as $a \to -\infty$ and $b \to \infty$. By Proposition 4.5, the behavior of u, v near $\pm\infty$ is governed by the limiting constant–coefficient systems $W' = A^0_\pm(\lambda)W$, where $W = (u, v, v')^T$ and $A^0_\pm = A^0(\pm\infty, \lambda)$. In particular, solutions W asymptotic to $(1, 0, 0)$ at $x = +\infty$ decay exponentially in (u', v, v') and are bounded in coordinate u as $x \to +\infty$. Observing that $1 - \hat{v} \to 1$ as $x \to +\infty$, we thus see immediately that the boundary contribution at b vanishes as $b \to +\infty$.

The situation at $-\infty$ is more delicate, since the denominator $1-\hat{v}$ of the second term goes to zero at rate e^x as $x \to -\infty$, the rate of convergence of the limiting profile $\hat{v}$. By inspection, the limiting coefficient matrix

$$A^0_- = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda & \lambda & 1-\lambda \end{pmatrix}, \tag{B.4}$$

has eigenvalues

$$\mu = -\lambda,\ \frac{1 \pm \sqrt{1+4\lambda}}{2},$$

hence for $\Re e\lambda \geq 0$ the unique decaying mode at $x = +\infty$ is the unstable eigenvector corresponding to $\mu = \frac{1+\sqrt{1+4\lambda}}{2}$, with growth rate

$$\Re e\left(\frac{1+\sqrt{1+4\lambda}}{2}\right) = \frac{1}{2} + \frac{1}{2}\Re e\sqrt{1+4\lambda} > 1/2.$$

Thus, $|u|, |u'|, |v'|, |v| \leq Ce^{(1+\epsilon)x/2}$ as $x \to -\infty$, $\epsilon > 0$, and in particular

$$\left|\frac{\Re e(u'\bar{u})}{1-\hat{v}}\right| \leq Ce^{(1+\epsilon)x}/e^x \leq Ce^{\epsilon x} \to 0$$

as $x \to -\infty$. It follows that the boundary contribution at $x = a$ vanishes also as $a \to -\infty$, hence, in the limit as $a \to -\infty$, $b \to +\infty$,

$$\Re e(\lambda)\int_{-\infty}^{+\infty} \left(\frac{\hat{v}}{1-\hat{v}}|u|^2 + |v|^2\right)dx + \int_{-\infty}^{+\infty} \frac{|u'|^2}{1-\hat{v}}dx = 0. \tag{B.5}$$

But, for $\Re e\lambda \geq 0$, this implies $u' \equiv 0$, or $u \equiv$ constant, which, by $u(-\infty) = 0$, implies $u \equiv 0$. This reduces (B.1a) to $v' = \lambda v$, yielding the explicit solution $v = Ce^{\lambda x}$. By $v(\pm\infty) = 0$, therefore, $v \equiv 0$ for $\Re e\lambda \geq 0$. It follows that there are no nontrivial solutions of (B.1), (B.2) for $\Re e\lambda \geq 0$. □

Remark B.1. The above energy estimate is equivalent to multiplying the system by the special symmetrizer $\begin{pmatrix} 1 & 0 \\ 0 & \hat{v}/(1-\hat{v}) \end{pmatrix}$, then taking the L^2 inner product with $(v, u)^T$. The analog of the high-frequency estimates of Appendix A would be obtained using the alternative symmetrizer $\begin{pmatrix} 1-\hat{v} & 0 \\ 0 & \hat{v} \end{pmatrix}$ optimized for its effect on second-order derivative term $u''/\hat{v}$. This may clarify somewhat the strategy of the energy estimates used in [2,36].

Appendix C. Quantitative Conjugation Estimates

Consider a general first-order system

$$W' = A(x, \lambda)W. \tag{C.1}$$

Proposition C.1 (Quantitative Gap Lemma [13,50]). *Let V^+ and μ^+ be an eigenvector and associated eigenvalue of $A_+(\lambda)$ and suppose that there exist complementary generalized eigenprojections (i.e., A-invariant projections) P and Q such that*

$$\begin{aligned} |Pe^{(A_+-\mu^+)x}| &\leq C_1 e^{-\hat{\eta}x} \quad x \leq 0, \\ |Qe^{(A_+-\mu^+)x}| &\leq C_1 e^{-\hat{\eta}x} \quad x \geq 0, \\ |(A-A_+)(x)| &\leq C_2 e^{-\eta x} \quad x \geq 0, \end{aligned} \tag{C.2}$$

with $0 \leq \hat{\eta} < \eta$. Then, there exists a solution $W = e^{\mu^+ x}V(x, \lambda)$ of (C.1) with

$$\frac{|V(x,\lambda) - V^+(\lambda)|}{|V^+(\lambda)|} \leq \frac{C_1 C_2 e^{-\eta x}}{(\eta - \hat{\eta})(1-\epsilon)} \quad \text{for } x \geq L \tag{C.3}$$

provided $(\eta - \hat{\eta})^{-1} C_1 C_2 e^{-\eta L} \leq \epsilon < 1$.

Proof. Writing $V' = (A_+ - \mu^+)V + (A - A_+)V$ and imposing the limiting behavior $V(+\infty, \lambda) = V^+$, we seek a solution in the form $V = TV$,

$$\begin{aligned} TV(x) := V^+ &- \int_x^{+\infty} Pe^{(A_+-\mu^+)(x-y)}(A-A_+)V(y)dy \\ &+ \int_L^x Qe^{(A_+-\mu^+)(x-y)}(A-A_+)V(y)dy, \end{aligned}$$

from which the result follows by a straightforward Contraction Mapping argument, using (C.2) to compute that

$$\begin{aligned}|TV_1 - TV_2|(x) &= \Big| -\int_x^{+\infty} Pe^{(A_+-\mu^+)(x-y)}(A-A_+)(V_1-V_2)(y)dy \\ &\quad + \int_L^x Qe^{(A_+-\mu^+)(x-y)}(A-A_+)(V_1-V_2)(y)dy \Big| \\ &\le C_1C_2\int_L^{+\infty} e^{-\hat\eta(x-y)}e^{-\eta y}dy\|V_1-V_2\|_{L^\infty[L,+\infty)} \\ &= \frac{C_1C_2e^{-\hat\eta x}e^{-(\eta-\hat\eta)L}}{\eta-\hat\eta}\|V_1-V_2\|_{L^\infty[L,+\infty)},\end{aligned}$$

and thus $\|TV_1 - TV_2\|_{L^\infty[L,+\infty)} \le \frac{C_1C_2e^{-\eta L}}{\eta-\hat\eta}\|V_1-V_2\|_{L^\infty[L,+\infty)}$. □

Corollary C.2. *Let V^+ and μ^+ be an eigenvector and associated eigenvalue of $A_+(\lambda)$, where A_+ is $n\times n$ with at most k eigenvalues of real part $< \Re\mu^+$ and*

$$\max|(A_+-\mu)_{ij}| \le C_0; \qquad |(A-A_+)(x)| \le C_2e^{-\eta x} \quad x\ge 0, \tag{C.4}$$

$0<\hat\eta<\eta$*. Then, there exists a solution $W = e^{\mu^+x}V(x,\lambda)$ of (C.1) with*

$$\frac{|V(x,\lambda)-V^+(\lambda)|}{|V^+(\lambda)|} \le \frac{16nn!(C_0)^nC_2e^{-\hat\eta x}}{\delta^n(\eta-\hat\eta)(1-\epsilon)} \quad \textit{for } x\ge L, \tag{C.5}$$

$\delta := \frac{\eta-\hat\eta}{2k+2}$, *provided* $\frac{16nn!(C_0)^nC_2e^{-\hat\eta L}}{\delta^n(\eta-\hat\eta)} \le \epsilon < 1$.

Proof. Without loss of generality, take $\mu\equiv 0$. Dividing $[-\eta,-\hat\eta]$ into $k+1$ equal subintervals, we find by the pigeonhole principle that at least one subinterval contains the real part of no eigenvalue of A_+. Denoting the midpoint of this interval by $-\tilde\eta > \hat\eta$, we have

$$\min|\Re e\sigma(A_+)-\tilde\eta| \ge \delta := \frac{\eta-\hat\eta}{2k+2}. \tag{C.6}$$

Defining P to be the total eigenprojection of A_+ associated with eigenvalues of real part greater than $\hat\eta$ and Q the total eigenprojection associated with eigenvalues of real part less than $\hat\eta$, and estimating Pe^{A_+x}, Qe^{A_+x} using the inverse Laplace transform representation

$$e^{A_+x} = \frac{1}{2\pi i}\oint_\Gamma e^{zx}(z-A_+)^{-1}dz,$$

with Γ chosen to be a rectangle of side $4nC_0$ centered about the real axis, with one vertical side passing through $\Re e\lambda \equiv -\tilde\eta$ and the other respectively lying respectively to the right and to the left, and estimating

$$|(\lambda - A_+)^{-1}| \le n!C_0^{n-1}\delta^{-n}$$

crudely by Kramer's rule, we obtain (C.2) with $C_1 = 16nn!C_0^n\delta^{-n}$, whence the result follows by Proposition C.1. □

Corollary C.3 (Quantitative Conjugation Lemma). *Proposition 4.5 holds for* $0 < \epsilon < 1$ *with*

$$C(L, \hat{\eta}, \eta, \max|(M_+)_{ij}|, \dim M_+) = \frac{16nn!(C_0)^n C_2 e^{-\hat{\eta}x}}{\delta^n(\eta - \hat{\eta})(1-\epsilon)},$$

$n := (\dim M_+)^2$, $k := \frac{(\dim M_+)^2 - \dim M}{2}$, *when* $\frac{16nn!(C_0)^n C_2 e^{-\hat{\eta}L}}{\delta^n(\eta-\hat{\eta})} \le \epsilon$.

Proof. Writing the homological equation expressing conjugacy of variable- and constant-coefficient systems following [37], we have

$$P' = M_+P - PM_+ + \Theta M.$$

Considering this as an asymptotically constant-coefficient system on the n^2-dimensional vector space of matrices P, noting that the linear operator $\mathcal{M}_+P := M_+P - PM_+$, as a Sylvester matrix, has at least n zero eigenvalues and equal numbers of stable and unstable eigenvalues, we see that the number of its stable eigenvalues is not more than $k := \frac{n^2-n}{2}$, whence the result follows by Corollary C.2. □

Appendix D. An Illuminating Example

Consider Burgers' equation, $u_t + (u^2)_x = u_{xx}$, and the family of stationary viscous shock solutions

$$\hat{u}^\epsilon(x) := -\epsilon\tanh(\epsilon x/2), \qquad \lim_{z\to\pm\infty} \hat{u}^\epsilon(z) = \mp\epsilon \tag{D.1}$$

of amplitude $|u_+ - u_-| = 2\epsilon$ going to zero as $\epsilon \to 0$.

The linearized eigenvalue equation $u'' = (\hat{u}^\epsilon u)' + \lambda u$ about $\bar{u}^\epsilon$, expressed in the integrated variable $w(x) := \int_{-\infty}^x u(y)dy$, appears as

$$w'' = \hat{u}^\epsilon w' + \lambda w. \tag{D.2}$$

This reduces by the linearized Hopf–Cole transformation $w = \operatorname{sech}(\epsilon x/2)z$ to the constant-coefficient linear oscillator equation $z'' = (\lambda + \epsilon^2/4)z$, yielding exact solutions

$$w^\pm(x,\lambda) = \operatorname{sech}(\epsilon x/2)e^{\mp\sqrt{\epsilon^2/4+\lambda}x} \tag{D.3}$$

decaying, respectively, as $x \to \pm\infty$, with asymptotic behavior

$$\mathcal{W}^\pm(x,\lambda) \sim e^{\mu_\pm(\lambda)x}\mathcal{V}_\pm(\lambda), \tag{D.4}$$

where $\mu_\pm(\lambda) := \mp(\epsilon/2 + \sqrt{\epsilon^2/4+\lambda}$ and $\mathcal{V}_\pm := (1, \mu_\pm(\lambda))^T$ are the eigenvalues and eigenvectors of the limiting constant-coefficient equations at plus and minus spatial infinity written as a first-order system, $\mathcal{W}_\pm := (w, w')^T_\pm$.

Defining an Evans function $\mathcal{D}(\lambda) := \det(\mathcal{W}^-, \mathcal{W}^+)_{|x=0}$, we may compute explicitly

$$\mathcal{D}(\lambda) = -2\sqrt{\epsilon^2/4+\lambda}. \tag{D.5}$$

However, this is not "the" Evans function

$$D(\lambda) := \det(W^-, W^+)_{|x=0}$$

specified in Sect. 2.4, which is constructed, rather, from a special basis

$$W_\pm = c_\pm(\lambda)\mathcal{W}^\pm \sim e^{\mu_\pm x} V^\pm,$$

where $V_\pm = c_\pm(\lambda)\mathcal{V}^\pm$ are "Kato" eigenvectors determined uniquely (up to a constant factor independent of λ) by the property that there exist corresponding left eigenvectors $\tilde{V}^\pm$ such that

$$(\tilde{V} \cdot V)^\pm \equiv \text{constant}, \quad (\tilde{V} \cdot \dot{V})^\pm \equiv 0, \tag{D.6}$$

where " $\dot{}$ " denotes $d/d\lambda$; see [26,13,22] for further discussion.

Computing dual eigenvectors $\tilde{\mathcal{V}}^\pm = (\lambda + \mu^2)^{-1}(\lambda, \mu_\pm)$ satisfying $(\tilde{\mathcal{V}} \cdot \mathcal{V})^\pm \equiv 1$, and setting $V^\pm = c_\pm \mathcal{V}^\pm$, $\tilde{V}^\pm = \mathcal{V}^\pm / c_\pm$, we find after a brief calculation that (D.6) is equivalent to the complex ODE,

$$\dot{c}_\pm = -\left(\frac{\tilde{V} \cdot \dot{V}}{\tilde{V} \cdot V}\right)^\pm c_\pm = -\left(\frac{\dot{\mu}}{2\mu - \epsilon}\right)_\pm c_\pm, \tag{D.7}$$

which may be solved by exponentiation, yielding the general solution

$$c_\pm(\lambda) = C(\epsilon^2/4 + \lambda)^{-1/4}. \tag{D.8}$$

Initializing at a fixed nonzero point, without loss of generality $c_\pm(1) = 1$,[5] and noting that $D_\epsilon(\lambda) = c_- c_+ \mathcal{D}_\epsilon(\lambda)$, we thus obtain the remarkable formula

$$D_\epsilon(\lambda) \equiv -2\sqrt{\epsilon^2/4 + 1}. \tag{D.9}$$

That is, with the "Kato" normalization, the Evans function associated with a Burgers shock is not only stable (nonvanishing), but *identically constant*. Moreover, in the weak shock limit, $\epsilon \to 0$, $D_\epsilon(\lambda)$ converges uniformly to the constant function $D_0(\lambda) \equiv -2$.[6]

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[5] In the numerics of Sect. 6, we initialize always at $\lambda = 10$.

[6] Not to be confused with the limiting Evans function D^0 in the strong shock limit $v_+ \to 0$.

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