Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 4b

I. Pointwise bounds on G_{LF}: case 1

We are now ready to estimate

$$G_{LF}(x,t;y) = \frac{1}{2\pi i}\oint_\Gamma e^{\lambda t} G_\lambda(x,y)\,d\lambda.$$

Case 1 ($|x-y| \gg t$)**.** In this case, we have from the trivial bound $|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}$, $\eta>0$ on $\rho(L)$, together with $e^{-\eta|x-y|+\varepsilon t} \le e^{-\eta(|x-y|+t)/2}$ the negligible error bound:

$$|G_{LF}(x,t;y)| \le Ce^{-\eta(|x-y|+t)/2}, \qquad \eta>0.$$

Likewise, $|\bar U'(x)e(y,t)| \le Ce^{-\eta(|x-y|+t)/2}$, since $|\bar U'(x)| \le Ce^{-\theta|x|}$ dominates unless $|x| \ll |y|$, in which case $|y| \gg t + |x-y|$ and $|e(y,t)| \le e^{-\theta|y|}$.

Pointwise bounds on G_{LF}: case 2

Case 2 ($|x-y| \ll t$). In this case, we move the contour Γ to the vertical line $\Re\lambda = -\eta$, $\eta > 0$, accepting residues from the pole terms of $|\bar{u}'(x)\psi(y)|$, with sum corresponding to $\bar{u}'(x)e(y,t)$ with

$$e(y,t) := \frac{1}{2}\sum_{a_j^{\pm} \lessgtr 0} \Big(\mathrm{errfn}\big(\frac{-y-a_j^{\pm}t}{\sqrt{4t}}\big) - \mathrm{errfn}\big(\frac{-y+a_j^{\pm}t}{\sqrt{4t}}\big)\Big) l_j(y)$$

up to negligible error $O(e^{-\eta(|x-y|+t)})$ by $|x-y| \ll t$.

Meanwhile, from the integral on Γ, using the crude bound $|G_\lambda(x,y)| \le Ce^{c|x-y|}|$, $c>0$ and $|x-y| \ll t$, we obtain the negligible contribution $O(e^{-\eta(|x-y|+t)/2})$, $\eta > 0$.

Combining, we have

$$|\tilde{G}(x,t;y)| = |G_{LF}(x,t;y) - \bar{u}'(x)e(y,t)| \le Ce^{-\eta(|x-y|+t)/2}, \qquad \eta > 0,$$

in either case (the former because $\bar{u}_x e$ is negligible for $|x-y| \gg t$).

Pointwise bounds on G_{LF}: case 3

Case 3 ($|x-y| \sim t \le C$)**.** In this case, we just use $|G_{LF}(x,t;y)| \le C \le C_2 e^{-\eta(|x-y|+t)/2}$, $\eta > 0$.

Likewise,

$$\bar{u}'(x)e(y,t)| \le C \le C_2 e^{-\eta(|x-y|+t)/2},$$

so that $|\tilde{G}(x,t;y)|$ is again exponentially neglibible in $|x-y|$, t.

Pointwise bounds on G_{LF}: case 4 (critical case)

Case 4 ($|x-y| \sim t \gg 1$). In this case, we move Γ to the vertical line $\Re\lambda = 0$, tangent to $\sigma_{ess}(L)$ converting the inverse Laplace to an inverse Fourier transform (with respect to t) by substitution $\lambda = ik$:

$$\frac{1}{2\pi}\int_{-\varepsilon}^{+\varepsilon} e^{ikt} G_{ik}(x,y)dk.$$

RECALL: (i) Gaussian inversion rule: Fourier inverse of $e^{-k^2\alpha}$ is $e^{-x^2/4b}/\sqrt{4\pi\alpha}$. (ii) Fourier inverse of product is convolution of inverses. (iii) Fourier inverse of P.V. (1/k) is Heaviside function.

Case 4a

4(a) (pole terms). Writing the contribution from pole terms as $\bar{u}'(x)\psi(y)$ times

$$P.V.\frac{1}{2\pi}\int_{-i\varepsilon}^{+i\varepsilon} e^{ikt}(c/k)e^{-\mu(ik)y}\,dk = \\ P.V.\frac{1}{2\pi}\int_{-i\varepsilon}^{+i\varepsilon} e^{ikt}(c/k)e^{(ik/a_j - k^2\beta_j/a_j^3 + O(k^3))y}\,dk \tag{1}$$

and discarding time-exponentially small error terms, we may write the solution as the convolution of a Heaviside function (the inverse of $1/ik$ and a Gaussian (by the Gaussian Fourier inverse formula):

$$ce^{-a_j(y+a_jt)^2/4\beta_j y},$$

giving up to exponentially negligible error

$$\bar{u}'(x)\sum_j \psi_j(y)\big(\mathrm{errfn}(y+a_jt) - \mathrm{errfn}(y-a_jt)\big).$$

Case 4b, sample term i

4(b) (scattering terms). Sample term $\phi(x)\psi(y)$ times

$$
\begin{aligned}
P.V.\frac{1}{2\pi}\int_{-i\varepsilon}^{+i\varepsilon} e^{ikt} e^{-\mu(ik)(x-y)} dk = \\
P.V.\frac{1}{2\pi}\int_{-i\varepsilon}^{+i\varepsilon} e^{ikt} e^{(ik/a_j - k^2\beta_j/a_j^3 + O(k^3))(x-y)} dk
\end{aligned}
\tag{2}
$$

and discarding time-exponentially small error terms, we may write the solution as Gaussian $ce^{-a_j(y+a_jt)^2/4\beta_j y}$, giving up to exponentially negligible error

$$c\phi(x)\psi(y)e^{(x-y-a_jt)^2/4\beta t}.$$

This occurs only for $x, y > 0$ or $x, y < 0$.

Case 4b, sample term ii

Sample term $\phi(x)\psi(y)$ times

$$P.V. \frac{1}{2\pi} \int_{-i\varepsilon}^{+i\varepsilon} e^{ikt} e^{+\mu_j(ik)x - \mu_k(ik)y} dk =$$
$$P.V. \frac{1}{2\pi} \int_{-i\varepsilon}^{+i\varepsilon} e^{ikt} e^{(ik/a_j - k^2\beta_j/a_j^3 + O(k^3))x - (ik/a_l - k^2\beta_l/a_l^3 + O(k^3))y} dk. \tag{3}$$

Identical computation gives approximate Gaussian

$$ce^{-(x/a_j - y/a_l + t)^2/(4\beta_j(-y/a_l^3 + x/a_j^3)}$$

corresponding to reflection from or transmission through, the shock layer. (This occurs for any configuration of signs of x, y.)

Case 4b, remainder terms

Remainder terms may be estimated as successive derivatives of previously obtained terms (all obeying similar but improved bounds), to arbitrary order, leaving finally

$$P.V.\frac{1}{2\pi}\int_{-i\varepsilon}^{+i\varepsilon} e^{ikt} O(k^N e^{-(k^2+O(k^3)|x-y|} dk,$$

N arbitrarily large, which may be crudely estimated as $O(t^{-(N+1)/2})$, which, on the restricted domain $|x-y| \sim t$ of interest, is order $t^{-(N-1)/2}$ in any L^p norm, hence negligible for $L^q \to L^p$ convolution bounds.

Important detail: y-derivative bounds

At this point, we have the desired bounds on $\tilde{G}$ and $\bar{u}'(x)e(y,t)$; however, we also need more delicate bounds on their y-derivatives. We can obtain these by a simple bootstrap argument appealing to *conservation form* of the equations, which is indeed the source of these improved bounds.

Namely, we observe that pole and outgoing scattering terms are the only terms not decaying in L^1, hence determine the time-asymptotic mass of the perturbation. On the other hand, by conservation form, the mass (defined as x-integral) of the initial perturbation is constant in time. As there are precisely $n-1$ outgoing scattering terms, time-asyptotically carrying mass in $r_j^{\pm}$ modes, whereas $\bar{u}_x(x)$ has integral $[U]$ in the directon of the shock jump, the amount of mass carried in each of these n directions is uniquely determined by the mass of the initial data. (Important observation of Tai-Ping Liu.)

y-derivatives and conservation

On the other hand, we have from our formulae that the amplitudes of these waves are given by integral of intial perturbation against terms $\psi(y)$. It follows that $\psi(y)$ must be identically constant in space.

Thus, y-derivatives applied to these lowest-order terms fall entirely on the Gaussian (resp. errorfunction) factor, giving an extra $t^{-1/2}$ order decay, as claimed.

Corollary: stated $L^q \to L^p$ bounds

Lemma

$G_{LF}(x,t;y) = \bar U'(x)e(y,t) + \tilde G(x,t;y)$, *with*

$$e(y,t) := \frac{1}{2}\sum_{a_j^\pm \lessgtr 0}\left(errfn\Big(\frac{-y-a_j^\pm t}{\sqrt{4t}}\Big) - errfn\Big(\frac{-y+a_j^\pm t}{\sqrt{4t}}\Big)\right) l_j^\pm, \quad (4)$$

$l_j \pm$ *constant, and, for* $1 \le q \le 2 \le p \le \infty$ *(Gaussian rates):*

$$\left\| \int_{-\infty}^{+\infty} \tilde G(x,t;y) f(y)dy \right\|_{L^p(x)} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)} \|f\|_{L^q \cap L^p},$$

$$\left\| \int_{-\infty}^{+\infty} \partial_y \tilde G(x,t;y) f(y)dy \right\|_{L^p(x)} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}} \|f\|_{L^q \cap L^p},$$

$$\left\| \int_{-\infty}^{+\infty} \partial_{x,y,t} e(x,t;y) f(y)dy \right\|_{L^p} \le (1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{(j+k)}{2}} \|f\|_{L^q}.$$

Conclusion

THIS COMPLETES THE PROOF OF LOW-FREQUENCY BOUNDS, AND NONLINEAR STABILITY.

Remarks: 1. Simplified 1D stability proof based on energy estimates and nonlinear damping (HF) and direct Henry-Monteiro derivation of resolvent kernel formulae, new in these lectures.

2. "Overcompressive" shocks (transversal connection giving ℓ-parameter family of profiles) treatable by the same argument, additional bookkeeping.

3. "Undercompressive" shocks (nontransveral connection) appear to require more detailed, pointwise nonlinear stability argument [Howard-Z, Raoofi-Z].

4. Full, pointwise bounds on G may be obtained with further effort as in [Mascia-Z], [Z:Handbook] (on reserve). For a particularly accessible treatment, see the posted scalar notes of Yingwei Liu.

CODA: Verification of the Evans condition

SMALL AMPLITUDE CASE:

A beautiful center-manifold analyis of [Majda-Pego], extended by Freistühler to Kawashia class systems, shows existence of shock profiles in the small-amplitude limit, i.e., for $U\pm$ sufficiently close to a base point U_0, for each characteristic family j that is simple at U_0, with speed $s \approx a_j(U_0)$ and jump $[U] \approx r_j(U_0)$, r_j the eigenvector associated with eigenvalue a_j of $A(U_0) := (dF/dU)(U_0)$.

Under the additional assumption of *genuine nonlinearity* [Lax]: $(da_j/dU)r_j|_{U_0} \neq 0$, spectral stability has been verified for small-amplitude profiles by the combination of an energy estimate due to [Goodman], extended by Humpherys-Z to partially parabolic Kawashima class systems (see [Humpherys-Z], on reserve), and a fundamental relation between viscous and inviscid stability having to do with the behavior of the Evans function near the origin.

The energy estimate of Goodman

The first observation is that $\lambda U = LU = (BU_x)_x - (AU)_x$ can, on the set of consistent splitting where bounded solutions are known to be exponentially decaying, integrated in x to find that $\lambda \int U = 0$. Thus, $W := \int_{-\infty}^{x} U$ is an L^2 solution of the "integrated equation:"

$$\lambda W = \tilde{L} W := BW_{xx} - AW_x \tag{5}$$

for $\lambda \in \{\Re\lambda \geq 0\} \setminus \{0\}$ if and only if U is a solution of $\lambda U = LU$. Thus, to eliminate nonzero roots, one may study the better-behaved (5) (for which the translational mode $\bar{U}_x$ has been removed, since its integral $\bar{U} - U_-$ does not decay as $x \to +\infty$).

Indication in a simple case

Consider a scalar equation $u_t + f(u)_x = u_{xx}$, satisfying genuine nonlinearity, in this setting $\Leftrightarrow$ (WLOG)Goodman $f''(u) > 0$.

The linearized equation is thus $\lambda u = -a(x)u_x + u_{xx}$, where a is monotone decreasing. Take now the real part of the complex L^2 inner product of u against the equation, to obtain:

$$\Re\lambda|u|_2^2 = \langle u, -au_x\rangle + \langle u, u_{xx},$$

or, integrating by parts:

$$\Re\lambda|u|_2^2 = \langle u, (1/2)a_x u\rangle - |u_x|_2^2 \leq 0,$$

with equality only if $u_x \equiv 0$, which implies $u \equiv 0$ by $a_x < 0$.

Remarks

Treatment of transverse modes in the system case, using exponential weights similarly as in our proof of nonlinear damping estimates, is a technically difficult aspect of Goodman's ingenious proof. In the partially parabolic (Kawashima) case, damping-type estimates introduced by Matsumura, Nishihara, and Kawashima play an important role as well.

Viscous and inviscid stability conditions

Inviscid stability, viewing linearized shock dynamics as a free boundary problem with transmission/speed condition given by the linearization of the Rankine-Hugoniot conditions $s[U] = [F(U)]$ ($[\cdot]$ denoting jump across the shock), is readily seen to be the condition that (RH) be full rank with respect to s and characteristic variables outgoing from the shock (i.e., incoming to the domain).

This in turn is easily computed to be equivalent to nonvanishing of the *Lopatinski determinant*:

$$\delta := \det \begin{pmatrix} [U] & A_- R_- & A_+ R_+ \end{pmatrix},$$

where $R_\pm$ denote the matrices whose columns correspond to outgoing eigenmodes $r_j^\pm$ of $A_\pm$.

A fundamental relation

Lemma (Gardner-Z1998,Z-Serre1999,Mascia-Z2003)

$\partial_\lambda^\ell D(0) \neq 0 \Leftrightarrow$ *transversality of the profile connection plus hyperbolic stability of the corresponding inviscid shock:* $\delta \neq 0$.

Proof. Straightforward block-determinant reduction and conservation form; see [Z:Handbook], on reserve.

Consequences: 1. As $\delta \neq 0$ evidently in the small-shock limit, and as transversality of profiles follows from the small-amplitude existence theory, the absence of nonzero eigenvalues with nonnegative real part (given by Goodman-style energy estimate) is sufficient to show small-amplitude stability.

Consequences, continued

2. Fix a left state U_- and vary U_+, s along the Hugoniot curve of solutions to (RH). Then, inviscid instability, $\delta \neq 0$ corresponds generically to passage of a viscous eigenvalue from the stable (negative real part) to the unstable (positive real part) complex half plane. In particular, the *signed* Lopatinski determinant, though indicating nothing about inviscid stability, counts parity of the number of unstable viscous roots. (!)

Concluding comments

1. (*) All results so far have analogous versions in multi-D (save for pointwise description of Green function, complicated hyperbolic characteristic surfaces).

2. At a symbol level (by a rescaling argument), many similarities with the inviscid limit problem; see [Métivier-Z], [Guès-Métivier-Williams-Z].

NEXT: Numerical evaluation of the Evans function, stable and unstable examples.