Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 6b
Modulation of periodic wavetrains: stability and relation to hyperbolic conservation laws

CONTINUING... In the second part, we’ll:

(a) present basic stability argument.

(b) relate behavior to hyperbolic-parabolic conservation laws.
RECALL SETUP:

Spatially periodic traveling-wave solution $u(x, t) = \bar{u}(x - ct)$ of a reaction diffusion system

$$u_t + g(u) = D u_{xx},$$

$x, t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $g \in C^5(\mathbb{R}^n \to \mathbb{R}^n)$.
Diffusive stability conditions

D1) $\sigma(L_\xi) \subset \{\text{Re}\lambda < 0\}$ for $\xi \neq 0$.
(D2) $\text{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}$ and $|\xi|$ sufficiently small.
(D3) $\lambda = 0$ is a simple eigenvalue of $L_0$ (transversality in traveling wave ODE).

(D1)–(D3) $\Rightarrow$ analytic e-value, right/left e-functions of $L_\xi$

bifurcating from $\lambda = 0$:

$$\lambda_*(\xi) = -ia_j\xi + O(|\xi|)$$
$$q_*(\xi) = q_*(0, \cdot) + O(|\xi|), \quad (1)$$
$$\tilde{q}_*(\xi) = \tilde{q}_*(0, \cdot) + O(|\xi|),$$

projector $\Pi_*(\xi) = \Pi_*(0) + O(|\xi|) = q_*(0, \cdot)\langle \tilde{q}_*(0, \cdot), \cdot \rangle$. 
Statement of the theorem

Proposition (Johnson-Z10)

Let \( \tilde{u} \) be a solution of \( u_t + g(u) = Du_{xx} \). Assuming smoothness of \( g \), (D1)–(D3), and \( E_0 := \| \tilde{u} - \bar{u} \|_{L^1 \cap H^2} \big|_{t=0} \) sufficiently small,

\[
\begin{align*}
\| \tilde{u} - \bar{u}(\cdot - \psi) \|_{L^p(t)} &\leq C(1 + t)^{-\frac{1}{2}(1-1/p)} E_0, \\
\| (\psi_t, \psi_x) \|_{L^p} &\leq C(1 + t)^{-\frac{1}{2}(1-1/p)} E_0, \\
\| \psi(t) \|_{L^p} &\quad, \| \tilde{u} - \bar{u}(\cdot) \|_{L^p(t)} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)} E_0,
\end{align*}
\]

for some \( \psi(x, t), C > 0 \), and all \( t \geq 0, p \geq 2 \).
II. Linearized bounds: Decomposition of solution operator

Linearized solution operator for $u_t = Lu$:

$$ S(t)u_0 = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} S_\xi(t)\tilde{u}_0(\xi, x) d\xi $$

(2)

where $S_\xi(t)$ is solution operator for $u_t = L_\xi u$, $L_\xi := e^{-i\xi x}Le^{i\xi x}$.

Split as $S(t) = S^p(t) + \tilde{S}(t) = \tilde{u}_x s^p(t) + \tilde{S}(t)$, where

$$ s^p(t)u_0 = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi)e^{\lambda*(\xi)t} \langle \tilde{q}_*(0, \cdot, \tilde{u}_0(\xi, \cdot)) \rangle d\xi, $$

(3)

$\alpha$ a smooth cutoff function supported on a sufficiently small ball around the origin and identically 1 near $\xi = 0$.

(Here, we used $q_*(0) = \tilde{u}_x$...)
Proposition

Assuming (D1)–(D3), for $1 \leq q \leq 2 \leq p \leq \infty$, $0 \leq r \leq 4$,

\[
\|s^p(t)f\|_{L^p(x)} \leq C(1 + t)^{-\frac{1}{2}(1/q-1/p)}\|f\|_{L^q \cap H^1},
\]

\[
\|d_{x,t}^r \nabla_x t s^p(t)f\|_{L^p(x)} \leq C(1 + t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q \cap H^1}, \tag{4}
\]

\[
\|\tilde{S}(t)f\|_{L^p(x)} \leq C(1 + t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q \cap H^1}.
\]

“Infinitesimal phase ($\sim s^p$) decays at Gaussian rate.

Phase gradients ($\sim \nabla_x t s^p$) and remainder decay like derivatives of Gaussians...
Proof: $s^p$

Since $\Re \lambda_*(\xi) \leq -\theta |\xi|^2$ on $\text{suppt} \, \alpha$, Hausdorff-Young (Fourier transform version) gives:

\[
\| \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\xi x} e^{\lambda_*(\xi)t} \langle \tilde{q}_*, \tilde{f} \rangle(\xi) d\xi \|_{L_p(x)} \leq \| e^{-\xi^2 t} \|_{L^q(\xi)} \| q_* \|_{L^\infty(\xi, L^p(x))} \| \tilde{f} \|_{L^\infty(\xi, L^1(x))}
\]

\[
\leq t^{-\frac{1}{2}(1-1/p)} \| f \|_{L^1(x)},
\]

verifying $s^p$ bound, $q = 1$, and similarly for $1 \leq q \leq 2$. 
Derivative bounds

Examining $s^p(t)u_0 = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda_*(\xi)t} \langle \tilde{q}_*(0, \cdot, \tilde{u}_0(\xi, \cdot)) \rangle d\xi$, we see that $\partial_t$ yields additional factor of $\lambda_*(\xi) \sim |\xi|$ in the integrand, while $\partial_x$ yields factor exactly $i\xi$. This yields an additional factor of $t^{-\frac{1}{2}}$ decay in the estimates by Hausdorff-Young, verifying the bounds claimed for $\nabla_x, t s^p$, $r = 0$. Further derivatives only improve the bounds, completing the proof for $0 \leq r \leq 4$.

Remark: $S^p$ includes also $\bar{u}_x$, hence $\partial_x$ does not improve $L^p$ estimates for the full operator $S^p$. This is a crucial point, and the reason we need to separate out phase $s^p$ in our analysis.
The term $\tilde{S}$ splits as the sum of two terms. The first is

$$\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda_*(\xi)t} q_*(\xi, x) \langle \tilde{q}_*(\xi, \cdot, \tilde{f}(\xi, \cdot)) \rangle d\xi$$

$$- \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda_*(\xi)t} q_*(0, x) \langle \tilde{q}_*(0, \cdot, \tilde{f}(\xi, \cdot)) \rangle d\xi,$$

which induces an additional factor $O(|\xi|)$ in the integrand by Taylor expansion, yielding the claimed bounds by Hausdorff-Young (Bloch transform version, “basic diffusive estimate”).

The 2nd is the sum of

$$\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) S_\xi(t)(\text{Id} - \Pi_*(\xi)) \tilde{f}(\xi) d\xi$$

and

$$\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} (1 - \alpha(\xi)) S_\xi(t) \tilde{f}(\xi) d\xi.$$ These satisfy time-exponential $H^s \to H^s$ decay bounds due to spectral gap of $L_\xi$ on $\xi \in \text{supp} \alpha$ or restricted to $\text{Range} (\text{Id} - \Pi_*)$. (END PROOF)
III. Proof of stability: nonlinear cancellation estimate

Given a solution $\tilde{u}(x, t)$, define $v = u - \bar{u} = \tilde{u}(x + \psi(x, t)) - \bar{u}(x)$.

Lemma

$$(\partial_t - L)(v - \bar{u}'(x)\psi) = N,$$

where

$$N = \mathcal{O}(|(v, \psi_{x,t}, d_{x,t}\psi_{x,t}, d_{x,t}^2\psi_{x,t})|^2).$$

(Nonlinear quantification of the statement that phase shift-translational mode $\psi\bar{u}'$- is principal part of perturbation.)

Duhamel/Variation of constants gives

$$v(t) = \bar{u}'\left(\psi + s^P(t)v_0 + \int_0^t s^P(t - s)\mathcal{N}(s)ds\right)$$

$$+ \tilde{S}(t)v_0 + \int_0^t \tilde{S}(t - s)\mathcal{N}(s)ds.$$

Defining $\psi := -s^P(t)v_0 - \int_0^t s^P(t - s)\mathcal{N}(s)ds$ cancels “bad” $\bar{u}_x$-terms, $\Rightarrow$
Integral representation

Closed system in $(v, \nabla_x t \psi)$:

\[ v(\cdot, t) = \tilde{S}(t)v_0 + \int_0^t \tilde{S}(t - s)\mathcal{N}(s)ds, \]

\[ \nabla_x t \psi(\cdot, t) = -\nabla_x t s^p(t)v_0 - \int_0^t \nabla_x t s^p(t - s)\mathcal{N}(s)ds, \]

where $\mathcal{N} = \mathcal{O}(|(v, \psi_x, t, \ldots)|^2)$ and $\tilde{S}(t)$, $\nabla_x t s^p(t)$ obey bounds of a differentiated solution operator for the heat equation.
Nonlinear iteration

The rest goes as in viscous shock case, or decay to a constant for the scalar Burgers equation \( u_t - u_{xx} = (u^2)_x \). Define

\[
\zeta(t) := \sup_{0 \leq s \leq t, 2 \leq p \leq \infty} \left( |v|_{L^p(s)}(1 + t)^{1/2(1-1/p)} + |v, \nabla_x, t \psi|_{H^2(s)}(1 + t)^{1/4} \right) \\
+ |\nabla_x, t \psi(s)|(1 + s)^{1/2}.
\]

(7)

Lemma

For all \( t \geq 0 \) for which \( \zeta(t) \) is finite, some \( C > 0 \), and \( E_0 := |v_0|_{L^1 \cap H^2} \) sufficiently small,

\[
\zeta(t) \leq C(E_0 + \zeta(t)^2).
\]

(8)

With the established bounds on \( \tilde{G} \) and \( e \), the proof of (8) is identical to that of the shock wave (and almost identical to that of the Burgers or constant-coefficient) case.
Proof of claim

Via $L^q \to L^p$ estimates, we verify the claimed $L^p$ bounds on $v, \nabla_{x,t} \psi$, and (since the $s^p$ estimates permit arbitrary orders of derivatives $0 \leq r \leq 4$) the claimed $H^2$ bound on $\nabla_{x,t} \psi$. The $H^2$ bound on $v$ then follows as in the shock wave case by the following \textit{nonlinear damping estimate}, proved by a straightforward energy parabolic estimate analogous to (but much simpler than!) that of the hyperbolic-parabolic shock wave case:

\textbf{Proposition}

\textit{So long as $|v|_{H^2}$ remains sufficiently small, for some $\theta > 0$, $C$,}

$$|U(t)|^2_{H^2} \leq C \left( e^{-\theta t} |v(0)|^2_{H^2} + \int_0^t e^{-\theta (t-s)} (|U|^2_{L^2} + |\nabla_{x,t} \psi|^2_{H^2}) (s) \, ds \right).$$

(9)
We now appeal to short-time existence/continuous dependence in $H^2$ (standard second-order parabolic theory), yielding continuity in $H^2$ with respect to time, hence, by Sobolev embedding, of $\zeta$, as well as short time continuability of the solution so long as $\zeta$ remains sufficiently small.

“Continuous induction:” For $E_0 < 1/4C^2$, therefore, assuming nonstrict inequality $\zeta(t) \leq E_0$, we find from $\zeta(t) \leq C(E_0 + \zeta(t)^2)$ that $\zeta(t) \leq CE_0 + C(2CE_0)^2 < 2CE_0$, yielding strict inequality. By continuity of $\zeta$/continuability of the solution so long as $\zeta$ remains bounded, we may conclude that $\zeta \leq 2C_0E_0$ and the solution exists for all time, with the stated bounds following by definition of $\zeta$.

(END PROOF OF THEOREM)
• Compare cancellation computations in frequency domain [Schneider], analogy to $u_t - u_{xx} = u^p$, $p > 3$.

• Renormalization method of Schneider is similar to that used by Bricmont-Kupianen for Cahn-Hilliard fronts, another problem without spectral gap. For a version of our argument in the Cahn-Hilliard case, see the work of Peter Howard.
IV. The Whitham equation: relation to hyperbolic conservation laws

Rescaling \((x, t) \rightarrow (\epsilon x, \epsilon t)\): \(u_t + g(u) = \epsilon Du_{xx}\), reduces to rapid-oscillation/small-wavelength limit, studied by Whitham, Lax-Levermore-Venakides, ... Whitham equations (WKB expansion) give slow-modulation approximation

\[ u(x, t) \sim \bar{u}^{(k)}(x, t)(\Psi(x, t)), \]

where \(\bar{u}^{k}\) represents family of periodic profiles indexed by wave number \(k = 1/\text{period}\), and

\[ k_t + \omega_x = 0, \tag{10} \]

where \(k = \Psi_x\), \(\omega = ck = -\Psi_t\), and \(c = -\Psi_t/\Psi_x\) are wave number, frequency, and wavespeed, with nonlinear dispersion relation \(\omega = \omega(k)\).
Whitham equation $k_t + \omega_x = 0$ is scalar conservation law, characteristic speed $\alpha = \omega'(k)$. Linear group velocity $\sim$ linear wave propagation: different from the phase velocity $c = \omega(k)/k$.

Result on behavior: asymptotic convergence to second-order (diffusive order) Whitham equation (long time) [Sandstede-Scheel-Schneider-UeckerU13, Johnson-Noble-Rodrigues-Z13]. Bounded time small viscosity done by Doelman-Sandstede-Scheel-Schneider substantially earlier.

- For systems with additional conserved quantities, there are additional nearby periodic waves, and Whitham equation becomes a hyperbolic system, with multiple linear group velocities. Extensions by JNRZ; not accessible by renormalization.

(END SECOND LECTURE)
Next time

Blake Barker will speak on a) numerical Evans function analysis of spectral stability of periodic wave-trains. b) Numerical proof of stability of small-amplitude roll waves in thin film flow.
Appendix: WKB expansion, expanded...

Plugging into \( u_t + g(u) = \varepsilon u_{xx} \) the approximate solution

\[
 u^\varepsilon(x, t) = u^0 \left( x, t, \frac{\psi(x, t)}{\varepsilon} \right) + \varepsilon u^1 \left( x, t, \frac{\psi(x, t)}{\varepsilon} \right) + \cdots,
\]

\( \varepsilon \to 0 \), and matching terms, gives at yields at \( \varepsilon^{-1} \) order:

\[
 \psi_t u_\theta^0 + g(u^0) = \psi_x^2 u_{\theta\theta},
\]

where \( \theta \) denotes the fast variable in \( u^j(x, t, \theta) \). Recalling \( k = \psi_x \), \( c = -\psi_t/\psi_x \), we obtain the traveling-wave profile ODE

\[
 -ck(u^0)' + g(u^0) = k^2 (u^0)'',
\]

so \( u^0(x, t, \cdot) = \overline{u}^k(x, t) \) is a period-1 traveling-wave profile.