Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 6b

Modulation of periodic wavetrains: stability and relation to hyperbolic conservation laws

CONTINUING... In the second part, we'll:

(a) present basic stability argument.

(b) relate behavior to hyperbolic-parabolic conservation laws.

I. Statement of the theorem

RECALL SETUP:

Spatially periodic traveling-wave solution $u(x,t) = \bar{u}(x - ct)$ of a reaction diffusion system

$$u_t + g(u) = Du_{xx},$$

$x, t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $g \in C^5(\mathbb{R}^n \to \mathbb{R}^n)$.

Diffusive stability conditions

D1) $\sigma(L_\xi) \subset \{\mathrm{Re}\lambda < 0\}$ for $\xi \neq 0$.
(D2) $\mathrm{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}$ and $|\xi|$ sufficiently small.
(D3) $\lambda = 0$ is a simple eigenvalue of L_0 (transversality in traveling wave ODE).

(D1)–(D3) $\Rightarrow$ analytic e-value, right/left e-functions of L_ξ bifurcating from $\lambda = 0$:

$$\begin{aligned} \lambda_*(\xi) &= -ia_j\xi + O(|\xi|) \\ q_*(\xi) &= q_*(0,\cdot) + O(|\xi|), \\ \tilde{q}_*(\xi) &= \tilde{q}_*(0,\cdot) + O(|\xi|), \end{aligned} \tag{1}$$

projector $\Pi_*(\xi) = \Pi_*(0) + O(|\xi|) = q_*(0,\cdot)\langle\tilde{q}_*(0,\cdot),\cdot\rangle$.

Statement of the theorem

Proposition (Johnson-Z10)

Let $\tilde u$ be a solution of $u_t + g(u) = Du_{xx}$. Assuming smoothness of g, (D1)–(D3), and $E_0 := \|\tilde u - \bar u\|_{L^1\cap H^2}|_{t=0}$ sufficiently small,

$$\begin{aligned} \|\tilde u - \bar u(\cdot - \psi)\|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} E_0, \\ \|(\psi_t, \psi_x)\|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} E_0, \\ \|\psi(t)\|_{L^p},\ \|\tilde u - \bar u(\cdot)\|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)} E_0, \end{aligned}$$

for some $\psi(x,t)$, $C>0$, and all $t\ge 0$, $p \ge 2$.

II. Linearized bounds: Decomposition of solution operator

Linearized solution operator for $u_t = Lu$:

$$S(t)u_0 = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} S_\xi(t) \check{u}_0(\xi, x) d\xi \qquad (2)$$

where $S_\xi(t)$ is solution operator for $u_t = L_\xi u$, $L_\xi := e^{-i\xi x} L e^{i\xi x}$.

Split as $S(t) = S^p(t) + \tilde{S}(t) = \bar{u}_x s^p(t) + \tilde{S}(t)$, where

$$s^p(t)u_0 = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \alpha(\xi) e^{\lambda_*(\xi)t} \langle \tilde{q}_*(0, \cdot, \check{u}_0(\xi, \cdot)\rangle d\xi, \qquad (3)$$

α a smooth cutoff function supported on a sufficiently small ball around the origin and identically 1 near $\xi = 0$.

(Here, we used $q_*(0) = \bar{u}_x \ldots$)

Linearized estimates

Proposition

Assuming (D1)–(D3), for $1 \le q \le 2 \le p \le \infty$, $0 \le r \le 4$,

$$
\begin{aligned}
\|s^p(t)f\|_{L^p(x)} &\le C(1+t)^{-\frac{1}{2}(1/q-1/p)}\|f\|_{L^q\cap H^1}, \\
\left\|d^r_{x,t}\nabla_{x,t}s^p(t)f\right\|_{L^p(x)} &\le C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q\cap H^1}, \\
\left\|\tilde{S}(t)f\right\|_{L^p(x)} &\le C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q\cap H^1}.
\end{aligned}
\tag{4}
$$

"Infinitesimal phase ($\sim s^p$) decays at Gaussian rate.

Phase gradients ($\sim \nabla_{x,t}s^p$) and remainder decay like derivatives of Gaussians...

Proof: s^p

Since $\Re\lambda_*(\xi) \le -\theta|\xi|^2$ on *suppt* α, Hausdorff-Young (Fourier transform version) gives:

$$\begin{aligned}
&\|\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\xi x} e^{\lambda_*(\xi)t}\langle \tilde{q}_*, \check{f}\rangle(\xi)d\xi\|_{L^p(x)} \\
&\qquad \le \|e^{-\xi^2 t}\|_{L^q(\xi}\| \|q_*\|_{L^\infty(\xi, L^p(x))}\|\check{f}\|_{L^\infty(\xi, L^1(x))} \\
&\qquad \le t^{-\frac{1}{2}(1-1/p)}\|f\|_{L^1(x)},
\end{aligned} \tag{5}$$

verifying s^p bound, $q = 1$, and similarly for $1 \le q \le 2$.

Derivative bounds

Examining $s^p(t)u_0 = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda_*(\xi)t}\langle \tilde q_*(0,\cdot,\check u_0(\xi,\cdot)\rangle d\xi$, we see that ∂_t yields additional factor of $\lambda_*(\xi)\sim|\xi|$ in the integrand, while ∂_x yields factor exactly $i\xi$. This yields an additional factor of $t^{-\frac{1}{2}}$ decay in the estimates by Hausdorff-Young, verifying the bounds claimed for $\nabla_{x,t}s^p$, $r=0$. Further derivatives only improve the bounds, completing the proof for $0\le r\le 4$

Remark: S^p includes also $\bar u_x$, hence ∂_x *does not* improve L^p estimates for the full operator S^p. This is a crucial point, and the reason we need to separate out phase s^p in our analysis.

$\tilde S$ bounds

The term $\tilde S$ splits as the sum of two terms. The first is

$$\begin{aligned}\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda_*(\xi)t}q_*(\xi,x)\langle\tilde q_*(\xi,\cdot,\check f(\xi,\cdot))\rangle d\xi \\ -\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda_*(\xi)t}q_*(0,x)\langle\tilde q_*(0,\cdot,\check f(\xi,\cdot))\rangle d\xi,\end{aligned} \tag{6}$$

which induces an additional factor $O(|\xi|)$ in the integrand by Taylor expansion, yielding the claimed bounds by Hausdorff-Young (Bloch transform version, "basic diffusive estimate").

The 2nd is the sum of $\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)S_\xi(t)(\mathrm{Id}-\Pi_*(\xi))\check f(\xi)d\xi$ and $\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))S_\xi(t)\check f(\xi)d\xi$. These satisfy time-exponential $H^s\to H^s$ decay bounds due to spectral gap of L_ξ on $\xi\in$ *suppt* α or restricted to $\mathrm{Range}\,(\mathrm{Id}-\Pi_*)$. **(END PROOF)**

III. Proof of stability: nonlinear cancellation estimate

Given a solution $\tilde{u}(x,t)$, define $v = u - \bar{u} = \tilde{u}(x + \psi(x,t)) - \bar{u}(x)$.

Lemma

$(\partial_t - L)\,(v - \bar{u}'(x)\psi) = \mathcal{N}$, *where*
$\mathcal{N} = \mathcal{O}(|(v, \psi_{x,t}, d_{x,t}\psi_{x,t}, d^2_{x,t}\psi_{x,t})|^2)$.

(Nonlinear quantification of the statement that phase shift-translational mode $\psi\bar{u}'$- is principal part of perturbation.)

Duhamel/Variation of constants gives

$$v(t) = \bar{u}'\Big(\psi + s^p(t)v_0 + \int_0^t s^p(t-s)\mathcal{N}(s)ds\Big)$$
$$+ \tilde{S}(t)v_0 + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds.$$

Defining $\psi := -s^p(t)v_0 - \int_0^t s^p(t-s)\mathcal{N}(s)ds$ cancels "bad" $\bar{u}_x$-terms, $\Rightarrow$

Integral representation

Closed system in $(v, \nabla_{x,t}\psi)$:

$$v(\cdot,t) = \tilde S(t)v_0 + \int_0^t \tilde S(t-s)\mathcal{N}(s)ds,$$
$$\nabla_{x,t}\psi(\cdot,t) = -\nabla_{x,t}s^p(t)v_0 - \int_0^t \nabla_{x,t}s^p(t-s)\mathcal{N}(s)ds,$$

where $\mathcal{N} = \mathcal{O}(|(v, \psi_{x,t}, \dots)|^2)$ and $\tilde S(t)$, $\nabla_{x,t}s^p(t)$ obey bounds of a differentiated solution operator for the heat equation.

Nonlinear iteration

The rest goes as in viscous shock case, or decay to a constant for the scalar Burgers equation $u_t - u_{xx} = (u^2)_x$. Define

$$\zeta(t) := \sup_{0\le s\le t,\, 2\le p\le\infty} \Big(|v|_{L^p}(s)(1+t)^{\frac{1}{2}(1-1/p)} + |v, \nabla_{x,t}\psi|_{H^2}(s)(1+t)^{\frac{1}{4}} + |\nabla_{x,t}\psi(s)|(1+s)^{1/2} \Big). \tag{7}$$

Lemma

For all $t \ge 0$ for which $\zeta(t)$ is finite, some $C > 0$, and $E_0 := |v_0|_{L^1\cap H^2}$ sufficiently small,

$$\zeta(t) \le C(E_0 + \zeta(t)^2). \tag{8}$$

With the established bounds on $\tilde{G}$ and e, the proof of (8) is identical to that of the shock wave (and almost identical to that of the Burgers or constant-coefficient) case.

Proof of claim

Via $L^q \to L^p$ estimates, we verify the claimed L^p bounds on $v, \nabla_{x,t}\psi$, and (since the s^p estimates permit arbitrary orders of derivatives $0 \le r \le 4$) the claimed H^2 bound on $\nabla_{x,t}\psi$. The H^2 bound on v then follows as in the shock wave case by the following *nonlinear damping estimate*, proved by a straightforward energy parabolic estimate analogous to (but much simpler than!) that of the hyperbolic-parabolic shock wave case:

Proposition

So long as $|v|_{H^2}$ remains sufficiently small, for some $\theta > 0$, C,

$$|U(t)|^2_{H^2} \le C\Big(e^{-\theta t}|v(0)|^2_{H^2} + \int_0^t e^{-\theta(t-s)}(|U|^2_{L^2} + |\nabla_{x,t}\psi|^2_{H^2})(s)\,ds\Big). \tag{9}$$

Proof of Theorem

We now appeal to short-time existence/continuous dependence in H^2 (standard second-order parabolic theory), yielding continuity in H^2 with respect to time, hence, by Sobolev embedding, of ζ, as well as short time continuability of the solution so long as ζ remains sufficiently small.

"Continuous induction:" For $E_0 < 1/4C^2$, therefore, assuming nonstrict inequality $\zeta(t) \le E_0$, we find from $\zeta(t) \le C(E_0 + \zeta(t)^2)$ that $\zeta(t) \le CE_0 + C(2CE_0)^2 < 2CE_0$, yielding strict inequality. By continuity of ζ/continuability of the solution so long as ζ remains bounded, we may conclude that $\zeta \le 2C_0E_0$ amd the solution exists for all time, with the stated bounds following by definition of ζ.

(**END PROOF OF THEOREM**)

Comments

• Compare cancellation computations in frequency domain [Schneider], analogy to $u_t - u_{xx} = u^p$, $p > 3$.)

• Renormalization method of Schneider is similar to that used by Bricmont-Kupianen for Cahn-Hilliard fronts, another problem without spectral gap. For a version of our argument in the Cahn-Hilliard case, see the work of Peter Howard.

IV. The Whitham equation: relation to hyperbolic conservation laws

Rescaling $(x, t) \to (\epsilon x, \epsilon t)$: $u_t + g(u) = \varepsilon D u_{xx}$, reduces to rapid-oscillation/small-wavelength limit, studied by Whitham, Lax-Levermore-Venakides, ... **Whitham equations** (WKB expansion) give slow-modulation approximation

$$u(x,t) \sim \bar{u}^{(k)(x,t)}(\Psi(x,t)),$$

where $\bar{u}^k$ represents family of periodic profiles indexed by wave number $k = 1/\mathrm{period}$, and

$$k_t + \omega_x = 0, \tag{10}$$

where $k = \Psi_x$, $\omega = ck = -\Psi_t$, and $c = -\Psi_t/\Psi_x$ are wave number, frequency, and wavespeed, with nonlinear dispersion relation $\omega = \omega(k)$.

Comments

Whitham equation $k_t + \omega_x = 0$ is scalar conservation law, characteristic speed $\alpha = \omega'(k)$. Linear group velocity $\sim$ linear wave propagation: *different from the phase velocity* $c = \omega(k)/k$.

Result on behavior: asymptotic convergence to second-order (diffusive order) Whitham equation (long time) [Sandstede-Scheel-Schneider-UeckerU13, Johnson-Noble-Rodrigues-Z13].
Bounded time small viscosity done by Doelman-Sandstede-Scheel-Schneider substantially earlier.

• For systems with additional conserved quantities, there are additional nearby periodic waves, and Whitham equation becomes a hyperbolic *system*, with multiple linear group velocities. Extensions by JNRZ; not accessible by renormalization.

(**END SECOND LECTURE**)

Next time

Blake Barker will speak on a) numerical Evans function analysis of spectral stability of periodic wave-trains. b) Numerical proof of stability of small-amplitude roll waves in thin film flow.

Appendix: WKB expansion, expanded...

Plugging into $u_t + g(u) = \varepsilon u_{xx}$ the approximate solution

$$u^\varepsilon(x,t) = u^0\left(x, t, \frac{\psi(x,t)}{\varepsilon}\right) + \varepsilon u^1\left(x, t, \frac{\psi(x,t)}{\varepsilon}\right) + \cdots,$$

$\epsilon \to 0$, and matching terms, gives at yields **at** ε^{-1} **order:**

$$\psi_t u^0_\theta + g(u^0) = \psi_x^2 u_{\theta\theta},$$

where θ denotes the fast variable in $u^j(x,t,\theta)$. Recalling $k = \psi_x$, $c = -\psi_t/\psi_x$, we obtain the traveling-wave profile ODE

$$-ck(u^0)' + g(u^0) = k^2(u^0)'',$$

so $u^0(x,t,\cdot) = \bar{u}^{k(x,t)}$ is a period-1 traveling-wave profile.