Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 2b
Asymptotically constant-coefficient operators and ILT representation of the Green distribution

\[ L = \partial_x(B(x)\partial_x + \partial_x A(x)), \text{ limits } A_\pm, B_\pm \text{ as } x \to \pm\infty. \]

**GOAL:** Express Solution operator \( S(t) \) for \( u_t = Lu \) as

\[
 u(x, t) = \int_{\mathbb{R}} G(x, t; y)u(y)dy,
\]  

(distributional sense), via **ILT formula**:

\[
 G(x, y) = P.V. \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} G_\lambda(x, y)d\lambda,
\]  

(ILT)

where \( G_\lambda(x, y) \) is resolvent kernel associated with \((\lambda - L)^{-1}\).

(Formally, (ILT) applied to \( \delta_y(x) \)...)

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Construction of the resolvent kernel: the conjugation lemma

By introducing phase variables, we can reduce resolvent equation \((\lambda - L)u = f\) to first-order equation

\[
W' - A(\lambda, x)W = F. \tag{2}
\]

Example: \(L = \partial_x^2 + A \partial_x, \ W := (u, u')^t, \ A = \begin{pmatrix} 0 & \text{Id} \\ \lambda & A \end{pmatrix}, \ F = \begin{pmatrix} 0 \\ f \end{pmatrix}.\)
The **conjugation lemma** [Métivier-Z], analog for asymptotically constant ODE to Floquet’s Lemma in the periodic case.

**Lemma**

For $A(\lambda)$ converging exponentially to $A_{\pm}(\lambda)$, there exist coordinate changes $T_{\pm}(\lambda)$ on $x \geq 0$, converging exponentially to $\text{Id}$ as $x \to \pm \infty$, such that $W = T_{\pm}Z_{\pm}$, $F = T_{\pm}\tilde{F}_{\pm}$ reduces resolvent eq. (2) to constant coefficients:

$$Z'_{\pm} - A_{\pm}(\lambda)Z_{\pm} = \tilde{F}_{\pm}.$$

Moreover, $T_{\pm}$ retains the regularity in $\lambda$ of $A$ (in this case analytic).

**Proof** (postponed): Contraction mapping/Lyapunov-Perron [Levinson] plus homological equations for conjugation of $A$ to $A_{\pm}$. 

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**Construction of Henry (Monteiro)**

**Lyapunov-Perron:** Projecting const.-coeff. equation onto eigencomponents, write:

\[
Z_+(x) = e^{A_+x} P_+ Z_+(0) + \int_0^x e^{A_+(x-y)} P_+ \tilde{F}_+(y) dy - \int_x^\infty e^{A_+(x-y)} Q_+ \tilde{F}_+(y) dy,
\]

where \(P_+, Q_+\) are eigenprojections onto stable, unstable subspaces of \(A_+\), and similarly for \(x < 0\); matching conditions:

\[
T - Z_-(0) = T_+ Z_+(0).
\]

**ASSUMPTIONS** (i) (ODE) hyperbolicity, \(\sigma(A)\) has nonvanishing real part. (ii) consistent splitting, dimensions of stable/unstable subspaces same for \(A_\pm\).

**Remark.** Domain of consistent splitting bounded by dispersion curves of \(A_\pm\), spectra of limiting constant-coeff. operators \(L_\pm\).
Solving for kernel

**Obs.** Existence of eigenvalue equivalent to solution with $\tilde{F} = 0$, i.e., nontrivial $Z(0)$ s.t. $Q_+ Z + (0), \ Q_- Z - (0) = 0$.

Setting $x = 0$ in (3), get

$$Q_+ Z_+(0) = \int_0^\infty - \int_x^\infty e^{A_+(-y)} Q_+ \tilde{F}_+(y) dy,$$

and similarly for $x < 0$. If no e-value, then this determines $Z_{\pm}(0)$, by matching conditions, bounded linear operation.

**COMBINING**, we have $W(x) = \int G_\lambda (x, y) F(y) dy$, where

$$|G_\lambda (x, y)| \leq Ce^{-\eta|x-y|},$$

exponential decay in $|x - y|$.

**Remark.** Uniqueness clear from construction (=solution).
Henry’s Theorem

From convolution bound $|f \ast g|_p \leq |f|_p |g|_1$, and evident $L^1$ bound on our bound $Ce^{-\eta|x-y|}$ for $|G_\lambda(x,y)|$, get $L^p$ boundedness of $(\lambda - L)^{-1}$ on set of consistent splitting, whenever $\lambda$ is not an eigenvalue.

**Corollary (Henry)**

$\sigma_{ess}(L)$ lies to the left of the rightmost dispersion curve of the limiting operators $L_{\pm}$ (equivalently, their curves of essential spectra).

**BUT**: for nonself-adjoint operators, can be open regions of essential spectrum where consistent splitting fails (easy examples). Thus, ILT is different from generalized Fourier decomposition...
High-frequency bounds

Previous bounds \textit{not uniform in} $\lambda$. For high-frequency (large $\lambda$) bounds, use different, semiclassical limit-type, asymptotic ODE estimates (e.g., “tracking,” or “reduction” lemmas of [Z-Howard, Handbook], etc.)

**IDEA:** For rapidly varying solutions (large $\lambda$), behaves approximately as “frozen coefficient equation,” tracks close to stable/unstable subspaces. We will omit these (for now), as we won’t actually use them in our stability analysis.

**Result:** Uniform bounds $|G_\lambda(x, y)| \leq Ce^{-\eta|x-y|}$, ind. of $\lambda$, $\Re \lambda$ sufficiently large.
Validity of ILT for Green distribution now follows by splitting the operator ILT integral as before, then observing that all terms are of form $L^k$ applied to an absolutely convergent integral in $\lambda$ and $L^p(\mathbb{R})$, hence converges to a distribution $= L^k$ applied to $L^2$ function.
We will use the more detailed Green distribution description in order to obtain our desired linearized estimates.

**NOTE:** Again, similarly as in operator-ILT, that the representation formula is only a starting point. Once we obtain bounds on the solution (which, for distribution, means on very regular test function data), these *extend to the full semigroup.*