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Stability of detonation profiles in the ZND limit

Kevin Zumbrun*

Abstract

Confirming a conjecture of Lyng–Raoofi–Texier–Zumbrun, we show that stability of strong detonation waves in the ZND, or small-viscosity, limit is equivalent to stability of the limiting ZND detonation together with stability of the viscous profile associated with the component Neumann shock. More, on bounded frequencies the nonstable eigenvalues of the viscous detonation wave converge to those of the limiting ZND detonation, while on frequencies of order one over viscosity, they converge to one over viscosity times those of the associated viscous Neumann shock. This yields immediately a number of examples of instability and Hopf bifurcation of reacting Navier–Stokes detonations through the extensive numerical studies of ZND stability in the detonation literature.

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Contents

* Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.

1. Introduction

In one-dimensional, Lagrangian coordinates, the reactive Navier–Stokes (rNS) equations modeling reacting flow for a one-step reaction may be written in abstract form as

$$\begin{aligned} u_t + f(u)_x &= \varepsilon(B(u)u_x)_x + kq\phi(u)z, \\ z_t &= \varepsilon(C(u,z)z_x)_x - k\phi(u)z, \end{aligned} \tag{1.1}$$

where u, f, $q \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, z, k, C, $\phi \in \mathbb{R}^1$, and k, $\varepsilon > 0$ [Z1,LyZ1,LyZ2,LRTZ,TZ4]. Here, u comprises the gas-dynamical variables of specific volume, particle velocity, and total energy; z measures mass fraction of unburned reactant or, more generally, “progress” of a single reaction involving multiple reactants [FD,LyZ1]; $\phi(u)$ is an “ignition function”, monotone increasing in temperature, and usually assumed for fixed density to be zero below a certain ignition temperature and positive above; q comprises quantities produced in reaction, in particular heat released; and k corresponds to reaction rate. Coefficients B and C model transport effects of, respectively, viscosity and heat conduction, and species diffusion, and ε measures relative size of transport vs. reaction coefficients, typically quite small.

A *right-going viscous strong detonation wave* is a smooth traveling-wave solution

$$(u,z)(x,t) = (\bar{u},\bar{z})(x-st), \quad \lim_{x\to\pm\infty}(\bar{u},\bar{z})(x) = (u_\pm, z_\pm) \tag{1.2}$$

of solutions of (1.1) with speed $s > 0$ connecting a burned state on the left to an unburned state on the right,

$$z_- = 0, \quad z_+ = 1, \tag{1.3}$$

with necessarily

$$\phi(u_-) > 0, \quad \phi(u_+) = \phi'(u_+) = 0, \tag{1.4}$$

and satisfying the extreme Lax characteristic conditions

$$a_1^- < \cdots < a_{n-1}^- < s < a_n^-, \quad a_1^+ < \cdots < a_n^+ < s, \tag{1.5}$$

where $a_j^\pm$ denote the eigenvalues of $df(u_\pm)$, ordered by increasing real part. Left-going viscous detonation waves satisfy symmetric conditions obtained by reflection, $x \to -x$.

Multi-step reactions may be modeled by the same equations with vectorial reaction variable $z \in \mathbb{R}^m$, and coefficients q, C, ϕ, k modified accordingly. Likewise, the functions f, ϕ, B may be modified to depend, more realistically, also on z, reflecting the different chemical makeup of the gas after reaction, with no essential change at a mathematical level. For further discussion, see, e.g., [CF, FD, GS, Z1, LyZ1, LyZ2, LRTZ, TZ4, HuZ2].

A standard simplification in detonation theory is to neglect the small constant ε and consider instead the formal $\varepsilon = 0$ limit, or Zeldovich–von Neumann-Doering (ZND) model

$$\begin{aligned} u_t + f(u)_x &= kq\phi(u)z, \\ z_t &= -k\phi(u)z. \end{aligned} \tag{1.6}$$

Indeed, there is by now a tremendous body of literature on this model; see for example [CF, FD, FW, Er1, Er2, LS, BMR, AT, AlT, KS] and references therein. The corresponding object to a viscous detonation wave for the (ZND) model is a *right-going strong ZND detonation* $\bar{u}^0$ of form (1.2)–(1.4) satisfying (1.6), smooth except at a single shock discontinuity at (without loss of generality) $x = 0$, known as a "Neumann shock" [CF, M, GS, LyZ1, LyZ2], where u jumps from u_* to u_+ as x crosses zero from left to right, with $z \equiv 1$. We have the intuitive picture [CF] of a shock, or "reaction spike", compressing a quiescent mixture and heating it to ignition point, followed by a slow "reaction tail" in which the reaction proceeds until all reactant is burned, while, meanwhile, u varies from u_* to u_-.

A ZND detonation profile is determined implicitly [CF, HuZ2] by the property, obtained by integrating the traveling-wave ODE

$$\begin{aligned} -su' + f(u)' &= kq\phi(u)z, \\ -sz' &= -k\phi(u)z \end{aligned} \tag{1.7}$$

and adding q times the second equation to the first, that $-s(u + qz) + f(u) \equiv$ constant, which, together with $z = 1$ for $x \geq 0$ and $z(-\infty) = 0$ implies that

$$-su_+ + f(u_+) = -su_* + f(u_*) = -s(u_- - q) + f(u_-), \tag{1.8}$$

giving a unique u_- and profile $\bar{u}^0$, $x \leq 0$, for each Neumann shock (u_*, u_+) of speed s, so long as $df(u) - sI$ remains invertible for all $0 \leq z \leq 1$ along the curve determined by

$$-s(u + qz) + f(u) \equiv -s(u_+) + f(u_+). \tag{1.9}$$

For, solving (1.9) for $u = u(z)$ by the Implicit Function Theorem then yields the profile on $x \leq 0$ by solution of the second equation $z' = (-k/s)\phi(u(z))z$, a scalar equation with nonvanishing righthand side, so long as u remains in the region for which $\phi(u) > 0$.

A natural question is the relation between the formally limiting (ZND) equations and the behavior of the full (rNS) equations as $\varepsilon \to 0$. At the level of existence of detonation profiles, this was investigated by Majda [M] for a simplified model with u, $z \in \mathbb{R}^1$ and $B \equiv 1$, $C \equiv 0$ using direct, planar phase portrait analysis,[1] and extended to the physical (rNS) model by Gardner [G] using Conley index techniques and Gasser–Szmolyan [GS] by geometric singular perturbation theory. More recently, Williams [W] has revisited the (rNS) existence problem using more quantitative singular perturbation methods, generating detailed matched asymptotic expansions to all orders. In each case, with varying levels of detail, the result is that for each strong detonation profile $(\bar{u}^0, \bar{z}^0)$ of the

[1] See [L] for a treatment of the related case u, $z \in \mathbb{R}^1$, $B \equiv 1$, $C \equiv 1$.

(ZND) model with physical choice of f, there exists a family $(\bar u^\varepsilon, \bar z^\varepsilon)$ of strong detonation profiles converging away from $x=0$ as $\varepsilon \to 0$ to $(\bar u^0, \bar z^0)$, and near $x=0$ to a viscous shock profile for the associated Neumann shock, in microscopic variables $\tilde x = x/\varepsilon$.

In the present paper, using singular perturbation/asymptotic Evans function techniques developed in [PZ, HLZ, CHNZ, BHZ, HLyZ1, HLyZ2, OZ, Z3], we investigate the *stability* of profiles $(\bar u^\varepsilon, \bar z^\varepsilon)$ in the ZND limit $\varepsilon \to 0$ for a class of models (1.1) including both the Majda model[2] studied in [M, L, RV, LyZ2, LRTZ] and the physical (rNS) equations studied in [G, GS, W, Z1, LyZ1, JLW, TZ4]. Our conclusion, confirming a conjecture of [LRTZ], is that (linear and nonlinear) *stability in the ZND limit is equivalent to viscous stability of the component Neumann shock profile together with hyperbolic stability of the associated ZND detonation.*

Together with the results of [HLyZ1] verifying viscous stability of ideal gas shocks, this gives a rigorous connection between nonlinear viscous stability of (rNS) detonations and linearized spectral (or "normal modes") stability of the associated (ZND) detonations. This yields immediately a number of numerical (rNS) stability and bifurcation results through the extensive numerical (ZND) literature on spectral stability; see, for example, Propositions C.2 and C.3, Appendix C. Specialized to the Majda model, it recovers a variant of the sole previous analytical result, due to Roquejoffre and Vila [RV], via the explicit (ZND) computations of [JY]; see Corollary B.2, Appendix B.

1.1. Assumptions

Loosely following [Z1, Z2, MaZ3, MaZ4, LRTZ, TZ4], we make the assumptions:

(H0) f, B, ϕ, $C \in C^2$.

(H1) The eigenvalues of $df(u)$ are real, distinct, and different from s, for all u near the image of ZND profile $\bar u^0$, in particular for $u = u_-, u_*, u_+$.

(H2) $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, with $\Re\sigma b$, $\Re\sigma C \geq \theta > 0$, and $df = \begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix}$, df_{11} and df_{12} constant, with the eigenvalues of df_{11} real, semisimple, and of one sign relative to s, for all u under consideration (i.e., near a given detonation profile).

(H3) $\Re\sigma\Big(-i\xi df(u) - \xi^2 B(u)\Big) \leq \frac{-\theta\xi^2}{1+\xi^2}$, $\theta > 0$, for all $\xi \in \mathbb{R}$, and for all u near the image of ZND profile $\bar u^0$, in particular for $u = u_-, u_*, u_+$.

For the nonlinear analysis of [TZ4] (only), we require also:

(H4) There exists a smooth block-diagonal symmetric positive definite symmetrizer $S(u) = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ such that $S_1 df_{11}$ is symmetric and $\Re S_2 b > 0$ for all u under consideration, and $Sdf(u_\pm)$ are symmetric.

Remark 1.1. By block upper-triangular structure, we obtain from (H3) also

$$\Re\sigma\Big(-i\xi \begin{pmatrix} df(u) & 0 \\ 0 & 0 \end{pmatrix} - \xi^2 \begin{pmatrix} B(u) & 0 \\ 0 & C(u,z) \end{pmatrix} + \begin{pmatrix} 0 & qk\phi(u) \\ 0 & -k\phi(u) \end{pmatrix} \Big) \leq \frac{-\theta\xi^2}{1+\xi^2}, \tag{1.10}$$

$\theta > 0$, for all $\xi \in \mathbb{R}$, and for all u near the image of ZND profile $\bar u^0$, in particular for $u = u_-, u_*, u_+$, an assumption in the nonlinear stability/bifurcation analysis of [TZ4].

[2] Strictly speaking, a variant [L] with nonzero z-diffusion $C > 0$; however, the extension to the original Majda model is straightforward, substituting the weighted norm analysis of Sattinger [Sa] for the pointwise analysis of [LRTZ] in order to conclude nonlinear stability.

Remark 1.2. Under the strengthened assumption (H4), condition (H3) is equivalent to the more easily-verified genuine coupling condition of Kawashima [Kaw]:

(GC) There exists no eigenvector of $df(u_\pm)$ lying in $\ker B(u_\pm)$.

In turn, (H4) follows from existence of a convex viscosity-compatible entropy for the nonreacting gas-dynamical system $u_t + f(u)_x = \varepsilon(B(u)u_x)_x$ resulting when $z \equiv 0$. See [Z2, Kaw] for further discussion and verification for ideal gas dynamics and MHD.

Regarding connecting profiles, we make the further assumptions:

(P1) There exists a ZND profile $u(x,t) = \bar u^0(x - st)$ of (1.6), smooth for $x \gtrless 0$, with $\bar u^0(0^- = u_*)$ and $\bar u^0(0^+ = u_+)$, that is transversal in the sense that $df(\bar u^0) - sI$ is invertible for all x, so that the profile is locally unique by (1.8).

(P2) There exists a viscous Neumann shock profile

$$u(x,t) = \hat u\Big(\frac{x-st}{\varepsilon}\Big), \qquad \lim_{x\to-\infty} \hat u(x) = u_*, \quad \lim_{x\to+\infty} \hat u(x) = u_+ \tag{1.11}$$

of the associated nonreacting Navier-Stokes equations $u_t + f(u)_x = \varepsilon(B(u)u_x)_x$, i.e., a connection between u_*, u_+ of the traveling-wave ODE $B(\hat u)\hat u' = f(\hat u) - f(u_+) - s(\hat u - u_+)$, that is transversal in the sense that $df_{11}(\hat u) - s$ (constant by assumption (H2)) is invertible.[3]

(P3) For $\delta > 0$ fixed and $\varepsilon > 0$ sufficiently small, there exist viscous detonation profiles $\bar u^\varepsilon$ of (1.1), (1.2) satisfying for some C, $\theta > 0$, and $0 \le j \le 1$, $0 \le k \le 2$,

$$|\partial_x^k\big((\bar u^\varepsilon, \bar z^\varepsilon) - (\bar u^0, \bar z^0)\big)|(x) \le C\varepsilon e^{-\theta|x|} \quad \text{for } x \le -\delta, \tag{1.12}$$

$$|\partial_x^k\big((\bar u^\varepsilon, \bar z^\varepsilon)(x) - (\hat u(x/\varepsilon), 1)\big)| \le C\varepsilon + C\varepsilon^{1-k}e^{-\theta|x|/\varepsilon} \quad \text{for } -\delta \le x \le 0, \tag{1.13}$$

and

$$|\partial_\varepsilon^j\partial_{\tilde x}^k\big((\bar u^\varepsilon, \bar z^\varepsilon)(\varepsilon\tilde x) - (\hat u(\tilde x), 1)\big)| \le C\varepsilon^{\max\{1-j,0\}}e^{-\theta\tilde x} \quad \text{for } \tilde x \ge 0 \tag{1.14}$$

with $(\bar u^\varepsilon, \bar z^\varepsilon)$ a C^1 function from ε to $L^\infty(\tilde x \ge 0)$; in particular, $|\partial_x^k\big((\bar u^\varepsilon, \bar z^\varepsilon)(x) - (\hat u(x/\varepsilon), 1)\big)| \le C\varepsilon^{1-k}e^{-\theta|x|/\varepsilon}$ for $x \ge 0$ and

$$|\partial_\varepsilon^j\partial_{\tilde x}^k\big((\bar u^\varepsilon, \bar z^\varepsilon)(\varepsilon\tilde x) - (\hat u(\varepsilon\tilde x), 1)\big)| \le C(M)\varepsilon^{\max\{1-j,0\}}e^{-\theta\tilde x} \quad \text{for } \tilde x \ge -M \quad (\text{fixed}).^4 \tag{1.15}$$

Example 1.1. The physical single-species reactive compressible Navier–Stokes equations, in Lagrangian coordinates, are [Ch, TZ4]

$$\begin{cases} \partial_t\tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = \partial_x(\nu\tau^{-1}\partial_x u), \\ \partial_t E + \partial_x(pu) = \partial_x\big(\kappa\tau^{-1}\partial_x T + \nu\tau^{-1}u\partial_x u\big) + qk\phi(T)z, \\ \partial_t z = \partial_x(d\tau^{-2}\partial_x z) - k\phi(T)z, \end{cases} \tag{1.16}$$

where $\tau > 0$ denotes specific volume, u velocity, $E = e + \frac12 u^2$ specific gas-dynamical energy (i.e., the part of the total specific energy not including specific chemical potential energy qz) $e > 0$ specific

[3] This implies that the traveling wave ODE is nondegenerate type; the profile is then necessarily transversal by the extreme shock assumption (1.5) [MaZ3].

[4] The extension to $[-M, 0]$ follows by smooth dependence of solutions of ODE, noting that $\varepsilon \to 0$ is a regular perturbation in coordinates $\tilde x := x/\varepsilon$.

internal energy, and $0 \le z \le 1$ mass fraction of the reactant. Here, $\nu > 0$ is a viscosity coefficient, $\kappa > 0$ and $d > 0$ are respectively coefficients of heat conduction and species diffusion, $k > 0$ represents the rate of the reaction, and q is the heat release parameter, with $q > 0$ corresponding to an exothermic reaction and $q < 0$ to an endothermic reaction. Finally, $T = T(\tau, e, z) > 0$ represents temperature and $p = p(\tau, e, z)$ pressure.

Under the standard assumptions of a reaction-independent ideal gas equation of state, $p = \Gamma\tau^{-1}e$, $T = c^{-1}e$, where $c > 0$ is the specific heat constant and Γ is the Gruneisen constant, and a smooth ignition function ϕ vanishing identically for $T \le T_i$ and strictly positive for $T > T_i$, it is shown in [MaZ3, MaZ4, TZ4] that each of (H0)–(H4) are satisfied. Conditions (H3)–(H4) are most easily verified by writing the third equation of (1.1) (subtracting u times the second equation to eliminate several terms) in quasilinear form $\partial_t e + p\partial_x u = \partial_x\big(\kappa\tau^{-1}\partial_x T\big) + qk\phi(T)z + \nu\tau^{-1}\partial_x u^2$ and viewing the gas-dynamical part of (1.1), modulo error term $\nu\tau^{-1}\partial_x u^2$, as a system

$$\begin{pmatrix}\tau\\ u\\ e\end{pmatrix}_t + \begin{pmatrix}0 & -1 & 0\\ -\Gamma\tau^{-2}e & 0 & \tau^{-1}\Gamma\\ 0 & \tau^{-1}\Gamma e & 0\end{pmatrix}\begin{pmatrix}\tau\\ u\\ e\end{pmatrix}_x = \partial_x\begin{pmatrix}0 & 0 & 0\\ 0 & \nu\tau^{-1} & 0\\ 0 & 0 & \kappa\tau^{-1}c^{-1}\end{pmatrix}\begin{pmatrix}\tau\\ u\\ e\end{pmatrix}_x \tag{1.17}$$

in (τ, u, e), with symmetrizer $S = \mathrm{diag}\,\{1, \tau^2/\Gamma e, \tau^2/\Gamma e^2\}$. (Here, we are using the standard fact that (H3)–(H4) are invariant under coordinate changes preserving block structure (H2).) Alternatively, they may be deduced using Remark 1.2 from the well-known existence of a viscosity-compatible entropy [Kaw, MaZ4, Z2]. Likewise, (P1)–(P3) have been verified in [W] in this case; see in particular Theorem 5.3 and Corollary 5.5 of [W].[5] For a verification of (P1) by explicit computation, see (C.6)–(C.9), Appendix C. For (P2), see also [Gi].

Remark 1.3. We expect that (P3) can be shown by an argument like that of [W] to be a general consequence of (H0)–(H3) and (P1)–(P2) and not an independent assumption. The striking fact, (1.15), that variation in the profile is confined to a narrow $O(\varepsilon)$ layer for $x \ge 0$ may be understood by the fact that, in $\tilde{x} = x/\varepsilon$ coordinates, the profile ODE near $(u_+, 1)$ does not involve ε, but is essentially the equation for a gas-dynamical shock profile, adjoined with a decoupled z-equation.

Remark 1.4. In (H1), it is enough that eigenvalues be semisimple. In the present, spectral, analysis, we use only that eigenvalues are real and distinct from s (used to separate decaying and growing slow modes in the analysis of Section 6.2.2). In the linearized and nonlinear stability analysis for viscous detonations, one may relax the strict hyperbolicity assumption of [TZ4] to semisimplicity, as discussed for the viscous shock case in [MaZ4, Z1].

1.2. Main results

Recall that, associated with the linearized eigenvalue problems for ZND and rNS detonations are the *Evans–Lopatinski determinant* D_{ZND} and the *Evans determinant* D^{ε}_{rNS}, each analytic on

[5] Bounds (1.14) may also be obtained directly by the observation that these are just standard estimates on smooth dependence of the (entirely "fast") stable manifold at $(u_+, 1)$ of the viscous profile ODE in $\tilde{x}$-coordinates once there is established (either by the direct estimates of [W] or by Fenichel estimates as in [F, GS]) the smooth dependence in ε of the (partly "slow") singular unstable manifold at $(u_-, 0)$ and, as shown in Thm. 5.3, [W], the transversality of their intersection at $\varepsilon = 0$, giving (1.14) for all $0 \le j$ such that $f, B, \phi \in C^{j+1}$. Indeed, the situation is simpler as the stable manifold is locally independent of ε; see Remark 1.3. Slightly weaker bounds, with the same rates of convergence but to an equilibrium depending smoothly on ε, follow from transversality at $\varepsilon = 0$ of the unstable manifold at $(u_-, 0)$ with the stable-center manifold at $(u_+, 1)$ as shown in [GS], the asserted exponential decay following from the observation that the center manifold is simply a union of equilibria. At the expense of further effort/bookkeeping, these would also suffice for our analysis.

$\Re\lambda \geq -\eta < 0$, with zeros corresponding to normal modes of the respective linear problems; see Sections 2 and 3 for precise definitions. Likewise, there is an Evans determinant D_{NS} associated with the linearized eigenvalue problem for the associated viscous Neumann shock of the nonreacting Navier–Stokes equations (NS); see Section 6.

Weak Evans–Lopatinski stability of ZND detonations is defined as nonvanishing of D_{ZND} on $\Re\lambda > 0$ and strong Evans–Lopatinski stability as nonvanishing on $\Re\lambda \geq 0$ except for a simple zero at $\lambda = 0$ [Er1, Er2, Z1, JLW]. Similarly, weak Evans stability of rNS detonations is defined as nonvanishing of D^{ε}_{rNS} on $\Re\lambda > 0$ and strong Evans stability as nonvanishing on $\Re\lambda \geq 0$ except for a simple zero at $\lambda = 0$ [Z1, LyZ1, LyZ2, JLW, LRTZ, TZ4]. Likewise, weak Evans stability of the associated viscous Neumann shock is defined as nonvanishing of D_{NS} on $\Re\lambda > 0$ and strong stability as nonvanishing on $\Re\lambda \geq 0$ except for a simple zero at $\lambda = 0$: equivalently, nonvanishing of $\frac{D_{NS}(\lambda)}{\lambda}$ on $\Re\lambda \geq 0$ [MaZ3, Z1, Z2, Z3]. Failure of weak stability is alternatively denoted as strong instability.

The following result established in [LRTZ, TZ4] equates strong Evans stability with linear and nonlinear stability of rNS detonations. A corresponding result holds for the component viscous Neumann shock [MaZ4, Z2, R, HR, HRZ, RZ, Z4].

Proposition 1.1 ([TZ4]). *For any fixed $\varepsilon > 0$, given (H0)–(H4), a viscous detonation profile* (1.2) *of* (1.1) *is $L^1 \cap L^p \to L^p$ linearly orbitally stable for some $p \geq 1$ if and only if it is strongly Evans stable, in which case it is $L^1 \cap H^3 \to L^p \cap H^3$ nonlinearly asymptotically orbitally stable for all $p > 1$ and orbitally stable for $p = 1$, in the sense that for any solution $(\tilde u, \tilde z)$ of* (1.1) *with initial data $(\tilde u_0, \tilde z_0)$ with $|(\tilde u_0, \tilde z_0) - (\bar u, \bar z)|_{L^1 \cap H^3}$ sufficiently small, there exists $\alpha(\cdot) \in W^{1,\infty}$ such that*

$$
\begin{aligned}
|(\tilde u, \tilde z)(\cdot, t) - (\bar u, \bar z)(\cdot - st - \alpha(t))|_{L^p} &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}|(\tilde u_0, \tilde z_0) - (\bar u, \bar z)|_{L^1 \cap H^3},\\
|\alpha(t)| &\leq C|(\tilde u_0, \tilde z_0) - (\bar u, \bar z)|_{L^1 \cap H^3},\\
|\dot\alpha(t)| &\leq C(1+t)^{-\frac{1}{2}}|(\tilde u_0, \tilde z_0) - (\bar u, \bar z)|_{L^1 \cap H^3}
\end{aligned}
\tag{1.18}
$$

for all $p \geq 1$, where $C > 0$ is a uniform constant. Moreover, strong Evans instability implies nonlinear instability (defined as failure of bounded stability) from $L^1 \cap H^3 \to L^1 \cap H^3$.

Proof. This was established in Theorem 1.12, [TZ4], for the case described in example 1.1. However, the proof relied only on (H0)–(H4) and the consequent estimate (1.10), hence extends also to the general case.

Our main theorem is the following result linking Evans stability of (rNS) profiles $(\bar u^{\varepsilon}, \bar z^{\varepsilon})$ in the limit as $\varepsilon \to 0$ with Evans–Lopatinski stability of the limiting (ZND) profile $(\bar u^0, \bar z^0)$.

Theorem 1.1. *Assuming (H0)–(H3), (P1)–(P3), weak Evans stability of $(\bar u^{\varepsilon}, \bar z^{\varepsilon})$ for all $\varepsilon > 0$ sufficiently small implies weak Evans stability of the viscous profile of the component Neumann shock together with weak Lopatinski stability of the limiting ZND detonation $(\bar u^0, \bar z^0)$, while strong Evans stability of $(\bar u^{\varepsilon}, \bar z^{\varepsilon})$ for all $\varepsilon > 0$ sufficiently small is implied by strong Evans stability of the viscous profile of the component Neumann shock together with strong Lopatinski stability of the limiting ZND detonation $(\bar u^0, \bar z^0)$.*

More precisely, (i) For $\varepsilon > 0$ and $\eta > 0$ sufficiently small, there are no zeros of D_{rNS} on $\Re\lambda \geq -\eta$ for $|\lambda| \geq C/\varepsilon$, C sufficiently large. (ii) For $C \leq |\lambda| \leq C/\varepsilon$, C sufficiently large, on $\Re\lambda > -\eta$ for η, $\varepsilon > 0$ sufficiently small, ε times each zero of D^{ε}_{rNS} converges to a zero of $\frac{D_{NS}(\lambda)}{\lambda}$ on $\Re\lambda \geq 0$; moreover, each zero of D_{NS} on $\Re\lambda > 0$ is the limit of ε times a zero of D_{rNS} on $\Re\lambda > 0$, for $C \leq |\lambda| \leq C/\varepsilon$. (iii) For $|\lambda| \leq C_0$, C_0 arbitrary, on $\Re\lambda \geq -\eta < 0$, the zeros of D^{ε}_{rNS} converge in location and multiplicity as $\varepsilon \to 0$ to the zeros of D_{ZND}, for any sufficiently small $\eta > 0$ such that D_{ZND} does not vanish for $\Re\lambda = -\eta$ and $|\lambda| \leq C_0$.

Proof. Assertions (i)–(iii) are established in Proposition 7.1, Corollary 6.1, and Corollary 5.1, whence the remaining assertions follow by definition of weak and strong stability of the various waves.

Remark 1.5. Assuming Evans stability of the associated viscous Neumann shock, we recover from (ii)–(iii), taking $C_0 \to \infty$ and $\varepsilon \to 0$, the somewhat delicate result of [Er1, Er2] that D_{ZND} does not vanish on $\Re\lambda \geq 0$ for $|\lambda|$ sufficiently large.

Remark 1.6. Recall [JLW] that, for fixed ε, low-frequency Evans stability, defined as strong Evans stability for $|\lambda| \leq c_0$, some $c_0 > 0$, is equivalent for either ZND or rNS detonations to the simpler condition of "Chapman–Jouget" stability; see [JLW] for further details. Thus, low-frequency stability of ZND waves is necessary for Evans stability of rNS detonations, along with weak Evans–Lopatinski stability as stated in Theorem 1.1.

Corollary 1.1. *Given (H0)–(H4), (P1)–(P3), the family of viscous detonation profiles* $\{(\bar u^\varepsilon, \bar z^\varepsilon\}$ *of* (1.1) *are linearly and nonlinearly stable for each fixed* $0 < \varepsilon \leq \varepsilon_0$*, for some* ε_0 *sufficiently small, if the associated viscous Neumann shock and ZND profiles are strongly Evans (resp. Lopatinski) stable, and are linearly and nonlinearly unstable for each fixed* $0 < \varepsilon \leq \varepsilon_0$*, some* ε_0 *sufficiently small, if either the viscous Neumann shock or ZND profile is strongly Evans (resp. Evans-Lopatinski) unstable.*

1.3. Discussion and open problems

Together, Proposition 1.1 and Theorem 1.1 give a rigorous connection between Evans–Lopatinski stability of (ZND) detonations and nonlinear stability of nearby (rNS) detonations for $\varepsilon > 0$ sufficiently small, as expressed in Corollary 1.1, giving a satisfying mathematical validation of physical conclusions made through the extensive ZND stability studies in detonation literature. By contrast, to our knowledge there is no analog of Proposition 1.1 for the (ZND) equations themselves, and indeed the physical meaning of Evans–Lopatinski stability in that context in terms of nonlinear stability or well-posedness is unclear; see [CJLW] for further discussion.

More, convergence of zeros of D^ε_{rNS} to those of D_{ZND} implies that finer phenomena such as Hopf bifurcation are inherited in the ZND limit as well as stability. This is perhaps more important, as it is well-known that ZND detonations are frequently unstable, bifurcating to pulsating and cellular fronts [FD, FW, Er1, Er2, LS, AT, BN, CH, BMR, AlT, KS]. See [TZ2, TZ3, TZ4, SS, BeSZ] for rigorous nonlinear results on Hopf bifurcation analogous to the stability result of Theorem 1.1 in the context of (rNS), under the assumption that a single complex conjugate pair of zeros of the Evans function crosses the imaginary axis with nonzero speed as a bifurcation parameter is varied. To our knowledge, no such result is available in the more sensitive case of (ZND) (though see the formal analyses of [Er5, BMR]).

We emphasize that we do not assert uniform nonlinear stability estimates in the (ZND) limit as $\epsilon \to 0$; note that $\varepsilon > 0$ is fixed in Theorem 1.1. This fact is not just a weakness of the current paper or [TZ4], but rather the whole point of the present analysis.

For, such a uniform stability result, if true, would strongly suggest (and perhaps even imply) a corresponding nonlinear stability result for the (ZND) equations themselves, which result does not exist in the literature, and whose truth has yet to be established. Even linearized long time estimates are not known to our knowledge for (ZND). Only the spectral stability, or "normal modes" problem has been studied in the (ZND) literature.

This spectral stability problem has been intensively studied since the 60's and work of Erpenbeck [Er1, Er2, Er3, Er4, Er5], so 45 years or so, with no rigorous linearized or nonlinear stability theory to go with it. So, it is not really clear what kind of stability is implied at the linearized or nonlinear level by these spectral stability investigations, or what is the precise link to behavior in an actual gas.

That is the problem this paper answers together with [TZ4]: for an ideal gas equation of state, or any equation of state for which the viscous shock is known to be stable, and any fixed, small enough viscosity, *spectral (ZND) stability* is equivalent to spectral (rNS) stability, which by [TZ4] *is equivalent to linearized and nonlinear (rNS) stability.* This closes the loop and gives a rigorous nonlinear consequence to the vast amount of spectral and other numerical data already collected over the years. We believe this to be quite significant at a foundational/philosophical level.

The delicacy of the (ZND) theory suggests that the basin of attraction of a viscous detonation may be rather small as $\varepsilon \to 0$, and this stability therefore may not be so robust. But, the link to (rNS) stability at least introduces a "scale" of stability and gives a starting point for further investigations, whereas up to now the question of nonlinear stability of ZND detonations was a blank wall. In the physical situation, of course, ε is always nonzero and fixed, even though small.

An important open problem at the (ZND) level is the rigorous verification of spectral instability for any single case. Though the numerical evidence is overwhelming, and despite the important development of formal arguments in the high-activation energy and Newtonian limits [BN, CH, S1, S2], there is to our knowledge as yet no completely analytical verification of spectral instability or bifurcation of detonation waves in (ZND). The necessary ingredients for such a result appear to be already present in the formal arguments of [BN, CH, S1], and this would be very interesting to carry through to the point of rigorous proof. With the results of this paper and [TZ4], this would yield also corresponding spectral, linear, and nonlinear results at the level of (rNS) in the limit of small viscosity, giving a complete analytical verification of the instability phenomena observed in experiment. We discuss these issues further in Appendix C.3.

Existence, stability, and bifurcation of detonations away from the ZND limit are other important open problems. Such general situations appear to require numerical investigation, as is standard in the combustion literature even for the simpler (ZND) model; see, e.g., [LS, HuZ2] and references therein. Treatment of the ZND limit in multi-dimensions is another important open problem. In particular, the implications for nearby (rNS) profiles of high-frequency ZND instabilities pointed out in [Er3] (specifically, turning point instabilities near glancing modes of the associated gas-dynamical system, associated with "trapped acoustic waves" [Er3, BM]) is an intriguing mathematical puzzle; see [CJLW] for further discussion.

We note, finally, that one-dimensional stability in the ZND limit was previously established in [RV] for detonation profiles of Majda's model. Our results both recover (a variant of) and illuminate this prior result, since the associated Neumann shock profile, because it is scalar, is in this case always stable (see, e.g., [Sa]), as is the limiting ZND detonation (see [JY], or Appendix B). Related singular perturbation results for systems of conservation laws may be found in [PZ, HLZ, CHNZ, HLyZ1, HLyZ2, BHZ] and (for multi-wave patterns) in [OZ, Z3]. In particular, we rely heavily on the basic methods of analysis developed in [PZ], [HLZ], [Z3], and [BHZ].

2. The Evans–Lopatinski determinant for (ZND)

We begin by recalling the linearized stability theory for ZND detonations following [Er1, Z1, JLW, HuZ2]. Shifting to coordinates $\tilde{x} = x - st$ moving with the background Neumann shock, write (1.6) as

$$W_t + F(W)_x = R(W), \tag{2.1}$$

where

$$W := \begin{pmatrix} u \\ z \end{pmatrix}, \quad F := \begin{pmatrix} f(u) - su \\ -sz \end{pmatrix}, \quad R := \begin{pmatrix} qkz\phi(u) \\ -kz\phi(u) \end{pmatrix}. \tag{2.2}$$

To investigate solutions in the vicinity of a discontinuous detonation profile, we postulate existence of a single shock discontinuity at location $X(t)$, and reduce to a fixed-boundary problem by the change of variables $x \to x - X(t)$. In the new coordinates, the problem becomes

$$W_t + (F(W) - X'(t)W)_x = R(W), \quad x \neq 0, \tag{2.3}$$

with jump condition

$$X'(t)[W] - [F(W)] = 0, \tag{2.4}$$

$[h(x,t)] := h(0^+,t) - h(0^-,t)$ as usual denoting jump across the discontinuity at $x = 0$.

2.1. Linearization

In moving coordinates, $\bar W^0$ is a standing detonation, hence $(\bar W^0, \bar X) = (\bar W^0, 0)$ is a steady solution of (2.3)–(2.4). Linearizing (2.3)–(2.4) about $(\bar W^0, 0)$, we obtain the *linearized equations*

$$(W_t - X'(t)(\bar W^0)'(x)) + (AW)_x = EW, \tag{2.5}$$

$$X'(t)[\bar W^0] - [AW] = 0, \quad x = 0, \tag{2.6}$$

where

$$A := (\partial/\partial W)F, \quad E := (\partial/\partial W)R. \tag{2.7}$$

2.2. Reduction to homogeneous form

As pointed out in [JLW], it is convenient for the stability analysis to eliminate the front from the interior equation (2.5). Therefore, we reverse the original transformation to linear order by the change of dependent variables

$$W \to W - X(t)(\bar W^0)'(x), \tag{2.8}$$

motivated by the calculation $W(x - X(t), t)) - W(x,t) \sim -X(t)W_x(x,t) \sim -X(t)(\bar W^0)'(x)$, approximating to linear order the original, nonlinear transformation. Substituting (2.8) in (2.5)–(2.6), and noting that x-differentiation of the steady profile equation $F(\bar W^0)_x = R(\bar W^0)$ gives $(A(\bar W^0)(\bar W^0)'(x))_x = E(\bar W^0)(\bar W^0)'(x)$, we obtain modified, *homogeneous* interior equations

$$W_t + (AW)_x = EW \tag{2.9}$$

agreeing with those that would be obtained by a naive calculation without consideration of the front, together with the modified jump condition

$$X'(t)[\bar W^0] - [A\big(W + X(t)(\bar W^0)'\big)] = 0 \tag{2.10}$$

correctly accounting for front dynamics.

2.3. The stability determinant

Seeking normal mode solutions $W(x,t) = e^{\lambda t}W(x)$, $X(t) = e^{\lambda t}X$, W bounded, of the linearized equations (2.9)–(2.10), we are led to the generalized eigenvalue equations

$$(AW)' = (-\lambda I + E)W, \quad x \neq 0, \tag{2.11}$$

$$X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [AW] = 0,$$

where "$\prime$" denotes d/dx. or, setting $Z := AW$, to

$$Z' = GZ, \quad x \neq 0, \tag{2.12}$$

$$X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [Z] = 0, \tag{2.13}$$

with

$$G := (-\lambda I + E)A^{-1}, \tag{2.14}$$

where we are implicitly using the fact that A is invertible, by (P1) and $s > 0$.

Lemma 2.1 ([Er1,Er2,JLW]). *On $\mathbb{R}\lambda > 0$, the limiting $(n+1)\times(n+1)$ coefficient matrices $G_\pm := \lim_{z\to\pm\infty} G(z)$ have unstable subspaces of fixed rank: full rank $n+1$ for G_+ and rank n for G_-. Moreover, these subspaces extend analytically to $\mathbb{R}\lambda \leq -\eta < 0$.*

Proof. Straightforward calculation using upper-triangular form of $G_\pm$ [Er1,Er2,Z1,JLW].

Corollary 2.1 ([Z1,JLW]). *On $\mathbb{R}\lambda > 0$, the only bounded solution of (2.12) for $x > 0$ is the trivial solution $W \equiv 0$. For $x < 0$, the bounded solutions consist of an (n)-dimensional manifold* $\mathrm{Span}\{Z_1^+, \dots, Z_n^+\}(\lambda, x)$ *of exponentially decaying solutions, analytic in λ and tangent as $x \to -\infty$ to the subspace of exponentially decaying solutions of the limiting, constant-coefficient equations $Z' = G_- Z$; moreover, this manifold extends analytically to $\mathbb{R}\lambda \leq -\eta < 0$.*

Proof. The first observation is immediate, using the fact that G is constant for $x > 0$. The second follows from standard asymptotic ODE theory, using the conjugation lemma of [MeZ1] (see Lemma A.1, Appendix A) together with the fact that G decays exponentially to its end state as $x \to -\infty$.

Definition 1. We define the *Evans–Lopatinski determinant*

$$\begin{aligned} D_{ZND}(\lambda) &:= \det\left(Z_1^-(\lambda,0), \cdots, Z_n^-(\lambda,0), \lambda[\bar{W}^0] - [A(\bar{W}^0)']\right) \\ &= \det\left(Z_1^-(\lambda,0), \cdots, Z_n^-(\lambda,0), \lambda[\bar{W}^0] + A(\bar{W}^0)'(0^-)\right), \end{aligned} \tag{2.15}$$

where $Z_j^-(\lambda,x)$ are as in Corollary 2.1.

The function D_{ZND} is exactly the *stability function* derived in a different form by Erpenbeck [Er1,Er2]. The formulation (2.15) is of the standard form arising in the simpler context of (nonreactive) shock stability [Er4]. Evidently (by (2.13) combined with Corollary 2.1), λ is a generalized eigenvalue/normal mode for $\mathbb{R}\lambda \geq 0$ if and only if $D_{ZND}(\lambda) = 0$.

Remark 2.1. From the traveling-wave ODE, $A(\bar{W}^0)' := dF(\bar{W}^0)(\bar{W}^0)' = R(\bar{W}^0)$, yielding the simpler form

$$D_{ZND}(\lambda) = \det\left(Z_1^-(\lambda,0), \cdots, Z_n^-(\lambda,0), \lambda[\bar{W}^0] + R(\bar{W}^0)(0^-)\right). \tag{2.16}$$

3. The Evans determinant for (rNS)

3.1. Linearization

Linearizing (1.1) about $(\bar u^\varepsilon, \bar z^\varepsilon)$ in moving coordinates yields linearized eigenvalue equations

$$\lambda W + (\tilde A W)_x = EW + (\varepsilon \tilde B W_x)_x, \tag{3.1}$$

where $W = \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, $W_2 = \begin{pmatrix} u_2 \\ z \end{pmatrix}$, $\bar W^\varepsilon = \begin{pmatrix} \bar u^\varepsilon \\ \bar z^\varepsilon \end{pmatrix}$, $\tilde B = \begin{pmatrix} 0 & 0 \\ 0 & \tilde b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & C \end{pmatrix} (\bar W^\varepsilon)$, and

$$\tilde A v := dF(\bar W^\varepsilon) v - \varepsilon (d\tilde B(\bar W^\varepsilon) v) \bar W^\varepsilon_x, \qquad (\tilde A_{11}, \tilde A_{12}) = (df_{11} - s, (df_{12}, 0)) \equiv \text{constant}, \tag{3.2}$$

E, F as in (2.9).

3.2. Expression as a first-order system

Setting $\tilde x = x/\varepsilon$ and

$$\mathcal{W} = \begin{pmatrix} Y \\ W_2 \end{pmatrix} := \begin{pmatrix} \tilde A W - \varepsilon \tilde B W_x \\ W_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad W_2 := \begin{pmatrix} u_2 \\ z \end{pmatrix},$$

we may write (3.1) as a first-order system

$$\dot{\mathcal{W}} = \mathcal{G}^\varepsilon(\lambda, \tilde x) \mathcal{W}, \tag{3.3}$$

where

$$\mathcal{G}^\varepsilon := \begin{pmatrix} \varepsilon(E_{11} - \lambda)\tilde A_{11}^{-1} & 0 & \varepsilon(E_{12} - (E_{11} - \lambda)\tilde A_{11}^{-1} \tilde A_{12}) \\ \varepsilon E_{21} & 0 & \varepsilon((E_{22} - \lambda) - E_{21} A_{11}^{-1} A_{12}) \\ \tilde b^{-1} \tilde A_{21} \tilde A_{11}^{-1} & -\tilde b^{-1} & \tilde b^{-1}(\tilde A_{22} - \tilde A_{21} \tilde A_{11}^{-1} \tilde A_{12}) \end{pmatrix} \tag{3.4}$$

and $\dot{}$ denotes $d/d\tilde x$. Here, we are using implicitly the facts that $\tilde A_{11}$ and $\tilde b$ are invertible, by (3.2), (P2), and (H2).

We have the following analogs of Lemma 2.1 and Corollary 2.1, which follow in essentially the same way; see [LRTZ, TZ4] for further details.

Lemma 3.1 ([Z1, TZ4]). *On $\mathbb{R}\lambda > 0$, the limiting $(n+1+r) \times (n+1+r)$ coefficient matrices $\mathcal{G}_\pm := \lim_{z \to \pm\infty} \mathcal{G}(z)$, $r = \dim u_2 + 1$, have stable subspaces of fixed, equal rank r. Moreover, these subspaces extend analytically to $\mathbb{R}\lambda \le -\eta < 0$.*

Corollary 3.1 ([Z1, TZ4]). *On $\mathbb{R}\lambda > 0$, the bounded solutions of (3.1) on $x \le 0$ consist of an $(n+1)$-dimensional manifold $\mathrm{Span}\{\mathcal{W}_1^-, \dots, \mathcal{W}_{n+1}^-\}(\lambda, x)$ of exponentially decaying solutions in the backward x direction, analytic in λ and tangent as $x \to -\infty$ to the subspace of exponentially decaying solutions in the backward x direction of the limiting, constant-coefficient equations $\mathcal{W}' = \mathcal{G}_- \mathcal{W}$; the bounded solutions of (3.1) on $x \ge 0$ consist of an $(r-1)$-dimensional manifold $\mathrm{Span}\{\mathcal{W}_{n+2}^+, \dots, \mathcal{W}_{n+1+r}^+\}(\lambda, x)$ of exponentially decaying solutions, analytic in λ and tangent as $x \to +\infty$ to the subspace of exponentially decaying solutions of the limiting, constant-coefficient equations $\mathcal{W}' = \mathcal{G}_+ \mathcal{W}$; Moreover, these manifolds extend analytically to $\mathbb{R}\lambda \le -\eta < 0$.*

3.3. The stability determinant

Definition 1. We define the *Evans function*

$$D^{\varepsilon}_{rNS}(\lambda) := \det\left(\mathcal{W}_1^-, \cdots, \mathcal{W}_{n+1}^-, \mathcal{W}_{n+2}^+, \cdots, \mathcal{W}_{n+1+r}^+\right)(\lambda, 0), \tag{3.5}$$

where $\mathcal{W}_j^{\pm}(\lambda, x)$ are as in Corollary 3.1.

4. Fast vs. slow coordinates

Note that the coordinate transformation to stretched, or "fast" variables

$$(\tilde{x}, \tilde{t}, \tilde{\lambda}) = (x/\varepsilon, t/\varepsilon, \varepsilon\lambda), \tag{4.1}$$

changes equations (1.1) to

$$\begin{aligned} u_{\tilde{t}} + f(u)_{\tilde{x}} &= (B(u)u_{\tilde{x}})_{\tilde{x}} + \varepsilon kq\phi(u)z, \\ z_{\tilde{t}} &= (C(u,z)z_{\tilde{x}})_{\tilde{x}} - \varepsilon k\phi(u)z, \end{aligned} \tag{4.2}$$

shifting the small parameter ε from diffusion to reaction terms. The computations to follow may be thought of as alternating between (original) slow variables and fast variables in our analysis, as convenient for different regions in λ and x. In particular, first-order equations (3.3)–(3.4) may be recognized as the first-order linearization of fast equations (4.2).

5. Region I: $|\lambda| \le C$

We first study the critical "ZND" region $|\lambda| \le C$, where behavior of the (rNS) Evans function D_{rNS} is governed by that of the Evans–Lopatinski determinant D_{ZND}. Setting $M >> 1$ to be a large constant to be determined later, we study separately the zones $\tilde{x} = x/\varepsilon \le -M$ and $\tilde{x} = x/\varepsilon \ge -M$, on which the profile $\bar{W}^{\varepsilon}$ is dominated respectively by the (ZND) profile $\bar{W}^0$ and the viscous shock profile $\hat{W}$, as described in (P3).

5.1. "Slow", or "reaction" zone, $\tilde{x} \le -M$

Note that $\tilde{A}$, hence $N := \tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})$ is invertible for $x \le -M\varepsilon$, $M >> 1$, by (3.2) and (P1)–(P2). Setting $\mathcal{W} = T\mathcal{Z}$, where $T := \begin{pmatrix} I & 0 \\ -N^{-1}\ell & I \end{pmatrix}$ and $\ell := (\tilde{b}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}, -\tilde{b}^{-1})$,

$$-N^{-1}\ell = -(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})^{-1}(\tilde{A}_{21}\tilde{A}_{11}^{-1}, -I),$$

and noting that $\dot{N}$, $\dot{\ell} \sim \dot{\bar{W}}^{\varepsilon}$ on $\tilde{x} \le -M$, by (P3), we transform (3.3)–(3.4) to

$$\dot{\mathcal{Z}} = \mathcal{H}^{\varepsilon}\mathcal{Z}, \tag{5.1}$$

$$\mathcal{H}^{\varepsilon} = T^{-1}\mathcal{G}^{\varepsilon}T - T^{-1}\dot{T} = \begin{pmatrix} \varepsilon(E-\lambda)\tilde{A}^{-1} & \varepsilon m \\ 0 & N + O(\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ O(\varepsilon + |\dot{\bar{W}}^{\varepsilon}|) & 0 \end{pmatrix}, \tag{5.2}$$

where $m := (E - \lambda)\begin{pmatrix} -\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ I \end{pmatrix}$, and terms $O(\cdot)$ are smooth in x and analytic in λ. Here, N, by (1.5) and the relation between viscous and inviscid shock structure [MaZ3], has one positive eigenvalue and $r-1$ negative eigenvalues, where $r = \dim u_2 + 1$.

By a further transformation $\mathcal{Z} = S\mathcal{V}$, $S = \begin{pmatrix} I & -\varepsilon N^{-1} m \\ 0 & I \end{pmatrix}$, we transform to $\dot{\mathcal{V}} = \mathcal{K}^\varepsilon \mathcal{V}$, where

$$\mathcal{K}^\varepsilon = S^{-1}\mathcal{H}^\varepsilon S - S^{-1}\dot{S} = \begin{pmatrix} \varepsilon(E-\lambda)\tilde{A}^{-1} & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2 + \varepsilon|\dot{\bar{W}}^\varepsilon|) & O(\varepsilon^2 + \varepsilon|\dot{\bar{W}}^\varepsilon|) \\ \\ O(\varepsilon + |\dot{\bar{W}}^\varepsilon|) & O(\varepsilon + |\dot{\bar{W}}^\varepsilon|) \end{pmatrix}. \tag{5.3}$$

By a further transformation $\begin{pmatrix} I & 0 \\ 0 & r \end{pmatrix}$ if necessary, we may take without loss of generality N block-diagonal, $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$, with $\Re N_1 \ge \eta > 0$ and $\Re N_2 \le -\eta < 0$, both with a uniform spectral gap from block $\varepsilon(E-\lambda)\tilde{A}^{-1} \sim \varepsilon$.

Applying the asymptotic ODE results of Lemma A.4 with $\eta \sim 1$, $\delta = O(\varepsilon + |\dot{\bar{W}}^\varepsilon|)$, and $|\Theta| \le C$, and of Remark A.7.1 with $\eta \sim 1$, $\delta := \varepsilon(\varepsilon + |\dot{\bar{W}}^\varepsilon|)$, $|\Theta_{12}| + |\Theta_{11}| \le C$, $|\Theta_{21}| + |\Theta_{22}| \le C/\varepsilon$, we find that there is a change of coordinates $\mathcal{Z} = S\mathcal{U}$, $\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 \\ \begin{pmatrix} \mathcal{U}_2 \\ \mathcal{U}_3 \end{pmatrix} \end{pmatrix}$, defined on $\tilde{x} \le -M$,

$$S^{-1} = \begin{pmatrix} I & -\Phi_1 \\ -\Phi_2 & I \end{pmatrix} = \begin{pmatrix} I & \varepsilon O(\varepsilon + |\dot{\bar{W}}^\varepsilon|) \\ O(\varepsilon + |\dot{\bar{W}}^\varepsilon|) & I \end{pmatrix} = \begin{pmatrix} I & o(\varepsilon) & o(\varepsilon) \\ o(1) & I & o(1) \\ o(1) & o(1) & I \end{pmatrix} \tag{5.4}$$

close to the identity converting (5.1) into three decoupled equations

$$\begin{aligned} &\dot{\mathcal{Z}}_1 = \varepsilon(E-\lambda)\tilde{A}^{-1}\mathcal{Z}_1 + O(\varepsilon^2 + \varepsilon|\dot{\bar{W}}^\varepsilon|)\mathcal{Z}_1; \qquad \mathcal{U}_2 \equiv 0, \mathcal{U}_3 \equiv 0, \\ &\dot{\mathcal{Z}}_2 = (N_1 + o(1))\mathcal{Z}_2; \qquad \mathcal{U}_1 \equiv 0, \mathcal{U}_3 \equiv 0, \\ &\dot{\mathcal{Z}}_3 = (N_2 + o(1))\mathcal{Z}_3; \qquad \mathcal{U}_1 \equiv 0, \mathcal{U}_2 \equiv 0. \end{aligned} \tag{5.5}$$

Focusing on the "slow" $\mathcal{Z}_1$ mode, changing coordinates from $\tilde{x}$ back to $x = \varepsilon\tilde{x}$, we obtain

$$\mathcal{Z}_1' = (E-\lambda)\tilde{A}^{-1}\mathcal{Z}_1 + O(\varepsilon + |\dot{\bar{W}}^\varepsilon(x/\varepsilon)|)\mathcal{Z}_1, \tag{5.6}$$

where "$'$" denotes d/dx, and all coefficients converge uniformly as $Ce^{-\eta|x|}$, $\eta > 0$ to their limits as $x \to -\infty$. Applying the convergence lemma, Lemma A.2 (Appendix A.2), together with Remark A.2, (A.16), on $x \le -M\varepsilon$ with $\delta(\varepsilon) = \varepsilon + \sup_{x \le -M\varepsilon} |\dot{\bar{W}}^\varepsilon(x/\varepsilon)| = O(e^{-\eta M})$, we find that there is a further coordinate change $\mathcal{Z}_1 = PZ$, $P = I + O(\varepsilon + e^{-\eta M}) = I + o(1)$ for $|x| \le 2M\varepsilon$, taking the decaying/growing modes of (5.6) to those of

$$Z' = (E-\lambda)\tilde{A}^{-1}Z,$$

which may be recognized as exactly the interior equation (2.12), (2.14) associated with the Evans–Lopatinski development.

Tracing back through our coordinate transformations, we find that the n slowly-decaying modes $\mathcal{W}_1^-, \dots \mathcal{W}_n^-$ as $\tilde{x} \to -\infty$ are given at $\tilde{x} = -M$ by

$$\mathcal{W}_j^- = (I + o(1)) \begin{pmatrix} Z_j^- \\ * \end{pmatrix}, \quad j = 1, \dots, n, \tag{5.7}$$

where Z_j^- are as in (2.15), while the single fast-decaying mode as $\tilde x \to -\infty$ is given at $\tilde x = -M$ by

$$\mathcal{W}_{n+1}^- = c(-M)\begin{pmatrix} o(1)\varepsilon v \\ (I+o(1))v \end{pmatrix}, \tag{5.8}$$

where $c(\tilde x)$ is exponentially decaying as $\tilde x \to -\infty$ and v lies in the unique stable eigendirection of $N(-M)$, $N := \tilde b^{-1}(\tilde A_{22} - \tilde A_{21}\tilde A_{11}^{-1}\tilde A_{12})$. Here, $o(1)$ depends on M, going to zero as $M \to \infty$ and $\varepsilon \to 0$ at the same time.

Remark 5.1. At $\varepsilon = 0$, the single fast-decaying mode reduces to the translational zero eigenfunction of the associated viscous Neumann shock (necessarily fast-decaying), and so we may refine (5.8) to

$$\mathcal{W}_{n+1}^- = \begin{pmatrix} o(1)\varepsilon \dot{\bar W}_2^0 \\ (I+o(1))\dot{\bar W}_2^0 \end{pmatrix}, \qquad \dot{\bar W}_2^0 = \begin{pmatrix} \dot{\hat u}_2 \\ \dot{\hat z} \end{pmatrix}. \tag{5.9}$$

5.2. *"Fast", or "Neumann shock" zone, $\tilde x \geq -M$*

Applying the convergence lemma, Lemma A.2, together with Remark A.2, (A.16), on $\tilde x \geq -M$ with $\delta(\varepsilon) = \varepsilon$, using the asymptotics of (P3), on $|\tilde x| \leq M$ we find that there is a change of coordinates $\hat P_+^\varepsilon$ with $\hat P_+^\varepsilon = I + O(\varepsilon)$ for $|\tilde x| \leq M$, such that $\mathcal{W} := \hat P_+^\varepsilon \mathcal{V}$ converts (3.3)–(3.4) to the same equations with $\varepsilon \equiv 0$, i.e., the shock eigenvalue system at $\lambda = 0$, hence, by inspection, the decaying modes $\mathcal{V}_{n+2}^+, \dots, \mathcal{V}_{n+1+r}^+$ at $+\infty$, necessarily fast-decaying, by the Lax characteristic assumption (1.5) (see [MaZ3]), are by inspection (see [Z1] for similar calculations) of form

$$\mathcal{V}_j^+ = \begin{pmatrix} 0 \\ v_j(x) \end{pmatrix}, \tag{5.10}$$

v_j independent, hence for $|\tilde x| \leq M$

$$\mathcal{W}_j^+ = (I + O(\varepsilon))\begin{pmatrix} 0 \\ v_j \end{pmatrix}, \qquad j = n+2, n+1+r. \tag{5.11}$$

Here the constant $O(\cdot)$ depends on (growing exponentially with) the fixed constant M, since we are shifting $\tilde x \to \tilde x + M$ in order to apply the convergence lemma. Evaluating at at $\tilde x = -M$, we have

$$\mathcal{W}_j^+(-M) = (I + O(\varepsilon))\begin{pmatrix} 0 \\ v_j(-M) \end{pmatrix}, \qquad j = n+2, \dots, n+1+r. \tag{5.12}$$

5.2.1. *Variation in ε*

At this point, gathering information, and noting, by Abel's formula, that the Wronskian $D_{rNS}(\lambda)$ evaluated at $\tilde x = 0$ is equal to a nonzero constant $\psi = e^{\int_{-M}^0 \mathrm{Tr}\, \mathcal{G}^\varepsilon d\tilde x}$ times the same Wronskian evaluated at the point $\tilde x = -M$ at which we have information about solutions from both sides, we have that $D_{rNS}(\lambda)$ is proportional by a nonzero constant, analytic in λ, to

$$\begin{aligned} \det\Bigg((I+o(1))\begin{pmatrix} Z_1^- \\ * \end{pmatrix}, \dots, (I+o(1))\begin{pmatrix} Z_n^- \\ * \end{pmatrix}, (I+o(1)\varepsilon)\begin{pmatrix} 0 \\ v_{n+1} \end{pmatrix}, \\ (I+O(\varepsilon))\begin{pmatrix} 0 \\ v_{n+2} \end{pmatrix}, \dots, (I+O(\varepsilon))\begin{pmatrix} 0 \\ v_{n+1+r} \end{pmatrix}\Bigg) \\ = O(\varepsilon), \end{aligned} \tag{5.13}$$

since up to $O(\varepsilon)$ there are only n nonzero entries Z_j^- in the $(n+1)$-dimensional Y block. Here, the constants $O(\varepsilon)$ and $o(1)$ by construction are analytic in λ as well.

This reflects the fact that at $\varepsilon = 0$ there is a solution $W = \partial_{\tilde{x}} \hat{W} = \begin{pmatrix} \partial_{\tilde{x}} \hat{u} \\ 0 \end{pmatrix}$ decaying at both $\pm\infty$ of (3.1), corresponding to the translational eigenmode of the linearized equations about the associated viscous Neumann shock; see Remark 5.1. To extract the next-order behavior, we compute the first variation of this special mode with respect to ε at $\varepsilon = 0$, by an argument similar to those used in [GZ, ZS, LyZ1, LyZ2] to compute the first variation with respect to λ at $\lambda = 0$.

Specifically, recalling that $v_{n+2}, \ldots, v_{n+1+r}$ are a basis in $\mathcal{C}^r$, we may by a change of basis arrange without loss of generality that $v_{n+1} = v_{n+2}$, or, equivalently, that

$$\mathcal{W}_{n+1}^-|_{\varepsilon=0} = \mathcal{W}_{n+2}^+|_{\varepsilon=0} = \begin{pmatrix} 0 \\ \partial_x \hat{W}_2 \end{pmatrix}.$$

Performing a column operation subtracting the $(n+2)$nd column from the $(n+1)$st column to cancel the entry v_{n+1} in determinant (5.13), we thus obtain

$$\begin{aligned} D_{rNS}^{\varepsilon}(\lambda) = \psi(\lambda, \varepsilon)(1 + o(1)) \det \begin{pmatrix} v_{n+2} & \ldots & v_{n+1+r} \end{pmatrix} \\ \times \det \begin{pmatrix} Z_1^- & \ldots & Z_n^- & -\varepsilon Y_\varepsilon|_{\varepsilon=0} + o(1) \end{pmatrix}, \end{aligned} \tag{5.14}$$

hence

$$\frac{D_{rNS}^{\varepsilon}}{\varepsilon \Psi(\lambda, \varepsilon; M)} = \det \begin{pmatrix} Z_1^- & \ldots & Z_n^- & -Y_\varepsilon|_{\varepsilon=0} \end{pmatrix} + o(1), \tag{5.15}$$

as $\varepsilon \to 0$, $o(1) \to 0$ as $M \to \infty$, where ψ and $\Psi := \psi \det \begin{pmatrix} v_{n+2} & \ldots & v_{n+1+r} \end{pmatrix}$ are nonvanishing factors analytic in λ, and

$$Y_\varepsilon := \partial_\varepsilon Y^* = \partial_\varepsilon \mathcal{W}^*, \tag{5.16}$$

where $\mathcal{W}^*(\varepsilon, \lambda, x)$ is the (necessarily fast-) decaying solution of (3.1) at $x \to +\infty$ defined by

$$\mathcal{W}^* = \hat{P}_+^{\varepsilon}(\lambda, x) \begin{pmatrix} 0 \\ v_{n+2} \end{pmatrix}, \tag{5.17}$$

where $\hat{P}_+^{\varepsilon}(\lambda, x)$ is the conjugating transformation described above, and $W^*|_{\varepsilon=0} = \partial_{\tilde{x}} \hat{W} = \begin{pmatrix} \partial_{\tilde{x}} \hat{u} \\ 0 \end{pmatrix}$. Here, we are using in an essential way assumption (P3), which yields bounds (A.17) with $p = \varepsilon$ sufficient to conclude C^1 regularity in ε of $\hat{P}_+^{\varepsilon}$ by Lemma A.3.

Writing (3.1) in $\tilde{x}$ coordinates as $\varepsilon(E - \lambda)W = (\tilde{A}^{\varepsilon} W - \tilde{B}^{\varepsilon} W_{\tilde{x}})_{\tilde{x}} = \dot{Y}$, differentiating with respect to ε, and setting $\varepsilon = 0$, we obtain the variational equations $(E - \lambda)W^*|_{\varepsilon=0} = \dot{Y}_\varepsilon|_{\varepsilon=0}$, or

$$\dot{Y}_\varepsilon|_{\varepsilon=0} = (E - \lambda)\hat{W}_{\tilde{x}}. \tag{5.18}$$

Recalling (see (2.7)) that $E = dR(W)$, so that $E|_{\varepsilon=0} = dR(\hat{W})$, we find that $E\hat{W}_{\tilde{x}} = R(\hat{W})_{\tilde{x}}$ is a perfect derivative. Integrating (5.18) in $\tilde{x}$ from $\tilde{x} = -M$ to $\tilde{x} = +\infty$ thus yields

$$\begin{aligned} -Y_\varepsilon|_{\varepsilon=0}(-M) &= \lambda \hat{W}|_{-M}^{+\infty} + R(\hat{W}(-M)) \\ &= \lambda[\bar{W}0] + R(\bar{W}^0(0^-)) + o(1), \end{aligned} \tag{5.19}$$

where $\bar{W}^0$ as in (P1) denotes the associated ZND profile, $[\cdot]$ the jump in values across its Neumann shock discontinuity, and $o(1) \to 0$ as $M \to \infty$.

5.3. Convergence to D_{ZND}

Proposition 5.1. *Assuming (H0)–(H3), (P1)–(P3), for $|\lambda| \le C$ and $\Re\lambda \ge -\eta$, $\eta > 0$ sufficiently small, $\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\tilde\Psi(\lambda,\varepsilon)}$ converges uniformly as $\varepsilon \to 0$ to $D_{ZND}(\lambda)$, where $\tilde\Psi(\cdot,\cdot)$ is a nonvanishing factor that is analytic in λ.*

Proof. Choosing monotone sequences M_j (increasing) and ε_j (decreasing) such that $o(1) \le 1/j$ in (5.19) and (5.15), define $\tilde\Psi(\lambda,\varepsilon)$ to be equal to the function $\Psi(\lambda,\varepsilon; M_j)$ in (5.15), where j is the maximum integer such that $\varepsilon \le \varepsilon_j$. Then, $\tilde\Psi$ is analytic in λ by construction, and, combining (2.16), (5.15), (5.19), and the definition of ε_j, M_j, we have

$$\Big|\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\tilde\Psi(\lambda,\varepsilon)} - D_{ZND}(\lambda)\Big| \le C/j \to 0 \quad \text{as } \varepsilon \to 0.$$

Corollary 5.1. *Assuming (H0)–(H3), (P1)–(P3), for $|\lambda| \le C$ and $\Re\lambda \ge -\eta$, $\eta > 0$, the set of zeros of D^ε_{rNS} converges as $\varepsilon \to 0$ to the set of zeros of D^ε_{rNS}, for any sufficiently small $\eta > 0$ such that D_{ZND} does not vanish for $\Re\lambda = -\eta$ and $|\lambda| \le C$.*

Proof. Noting that zeros of D^ε_{rNS} agree with zeros of $\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\tilde\Psi(\lambda,\varepsilon)}$, we obtain the result by properties of uniform limits of analytic functions.

6. Region II: $C/\varepsilon \ge |\lambda| \ge C >> 1$

We next consider the "Neumann shock" region $C\varepsilon \le \tilde\lambda \le C$, $C >> 1$, in which behavior of D_{rNS} is dominated by that of the Evans function D_{NS} of the associated viscous Neumann shock. Here, D_{NS} is defined similarly as in Definition 1 of D_{rNS} as

$$D^\varepsilon_N(\tilde\lambda) := \det\left(\hat{\mathcal{W}}^-_1, \cdots, \hat{\mathcal{W}}^-_{n+1}, \hat{\mathcal{W}}^+_{n+2}, \cdots, \hat{\mathcal{W}}^+_{n+1+r}\right)(\tilde\lambda, 0), \tag{6.1}$$

where $\hat{\mathcal{W}}^\pm_j$ are decaying modes at $\pm\infty$ of the linearized eigenvalue equation

$$\tilde\lambda W + (\hat A W)_{\tilde x} = (\hat B W_{\tilde x})_{\tilde x} \tag{6.2}$$

about $\hat W$, written as a first order system

$$\dot{\hat{\mathcal{W}}} = \hat{\mathcal{G}}(\tilde\lambda,\tilde x)\hat{\mathcal{W}}, \qquad \hat{\mathcal{G}} := \begin{pmatrix} -\tilde\lambda\hat A_{11}^{-1} & 0 & \tilde\lambda\hat A_{11}^{-1}\tilde A_{12} \\ 0 & 0 & -\tilde\lambda \\ \hat b^{-1}\hat A_{21}\hat A_{11}^{-1} & -\hat b^{-1} & \hat b^{-1}(\hat A_{22} - \hat A_{21}\hat A_{11}^{-1}\hat A_{12}) \end{pmatrix}, \tag{6.3}$$

with $\hat{\mathcal{W}} = \begin{pmatrix} \hat Y \\ \hat W \end{pmatrix} = \begin{pmatrix} \hat A\hat W - \hat B\dot{\hat W} \\ \hat W \end{pmatrix}$, $\hat B := \tilde B(\hat W)$, and $\hat A v := dF(\hat W)v - (d\tilde B(\hat W)v)\bar W_{\tilde x}$, where F is as in (2.9). For further details, see, e.g., [Z1,Z2].

6.1. Fast zone $\tilde{x} \ge -M$

Noting that $\tilde{\lambda} = \varepsilon\lambda$ is bounded by assumption, we have by (3.4) and (P3)

$$\mathcal{G}^\varepsilon := \begin{pmatrix} \varepsilon E_{11} - \tilde{\lambda}\tilde{A}_{11}^{-1} & 0 & \varepsilon(E_{12} - E\tilde{A}_{11}^{-1}\tilde{A}_{12}) + \tilde{\lambda}\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ \varepsilon E_{21} & 0 & \varepsilon((E_{22} - E_{21}A_{11}^{-1}A_{12}) - \tilde{\lambda} \\ \tilde{b}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1} & -\tilde{b}^{-1} & \tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}) \end{pmatrix} = \hat{\mathcal{G}} + O(\varepsilon e^{-\eta|x|}), \tag{6.4}$$

where $\hat{\mathcal{G}}$ is bounded and converges at uniform exponential rate to its limits at $\pm\infty$. Applying the convergence lemma, Lemma A.2, for $\tilde{x} \ge -M$, $\delta(\varepsilon) = \varepsilon$, together with Remark A.2, (A.14), similarly as in Section 5.2 but now treating $\tilde{\lambda}$ as a fixed parameter, we find that there is a change of coordinates $\hat{P}_+^\varepsilon$ with $\hat{P}_+^\varepsilon = I + O(\varepsilon)$ for $\tilde{x} \ge -M$, such that $\mathcal{W} := \hat{P}_+^\varepsilon\hat{\mathcal{W}}$ converts (3.3)–(3.4) to the viscous shock system (6.3). Evaluating at $\tilde{x} = -M$, we thus have $\mathcal{W}_j^+(\lambda, -M) = (I + O(\varepsilon))\hat{\mathcal{W}}_j^+(\tilde{\lambda}, -M)$, or, by the assumption that $|\tilde{\lambda}| >> \varepsilon$,

$$\mathcal{W}_j^+(\lambda, -M) = \big(I + o(\tilde{\lambda})\big)\,\hat{\mathcal{W}}_j^+(\tilde{\lambda}, -M), \qquad j = n+2, \dots, n+1+r. \tag{6.5}$$

6.2. Slow zone $\tilde{x} \le -M$

6.2.1. Case a. $C/\varepsilon \ge |\lambda| \ge 1/C\varepsilon$, $C > 0$ arbitrary

We first treat the easier case $1/C\varepsilon \le |\lambda| \le C/\varepsilon$, or $C^{-1} \le |\tilde{\lambda}| \le C$, for arbitrary $C > 0$. From (H3), it follows that $\hat{\mathcal{G}}(\lambda, -\infty)$ has no pure imaginary eigenvalues for $\Re\tilde{\lambda} \ge 0$ and $\lambda \ne 0$. By continuity, the same holds for $\mathcal{G}^\varepsilon(\tilde{\lambda}, \tilde{x})$ for $\tilde{x} \le -M$, $\Re\tilde{\lambda} \ge 0$ and $\tilde{\lambda}$ bounded and bounded away from zero.

By the assumption $1/C \le |\tilde{\lambda}| \le C$, therefore, the stable and unstable subspaces of $\mathcal{G}^\varepsilon$ have a uniform spectral gap for all $\tilde{x} \le -M$, hence, by standard matrix perturbation theory [K, ZH, Z5, GMWZ5], there exist smooth transformations $T(\mathcal{G})$ such that

$$T^{-1}\mathcal{G}T = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \qquad \Re M_1 \ge \eta > 0, \quad \Re M_2 \le -\eta < 0.$$

Making the change of variables $\mathcal{W} = TZ$, we thus obtain

$$\dot{Z} = (T^{-1}\mathcal{G}T - T^{-1}\dot{T})Z = \Big(\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} + o(1)\Big)Z$$

for $\tilde{x} \le -M$. Applying the tracking lemma, Lemma A.4, we find that the manifold of solutions of (3.3)–(3.4) decaying at $-\infty$ is within angle $o(1)$ of the unstable subspace of $\mathcal{G}^\varepsilon(\tilde{\lambda}, \tilde{x})$ at each $\tilde{x} \le -M$, in particular for $\tilde{x} = -M$.

But, by the same reasoning, the manifold of decaying solutions of (6.3) at $\tilde{x} = -M$ is also within angle $o(1)$ of the unstable subspace of $\hat{\mathcal{G}}$, which, by continuity in ε/uniform spectral gap is within angle $O(\varepsilon) = o(1) = o(\tilde{\lambda})$ of the unstable subspace of $\mathcal{G}^\varepsilon$. From this, and (6.5), it follows that, up to a normalizing factor $\hat{\Psi}(\varepsilon, \tilde{\lambda})$ that may be taken analytic in $\tilde{\lambda}$,

$$\frac{D_{rNS}(\lambda)}{\hat{\Psi}(M, \varepsilon, \lambda)} = D_{NS}(\tilde{\lambda}) + o(\tilde{\lambda}) \tag{6.6}$$

on $\Re\lambda > -\eta$, $C^{-1} \le |\tilde{\lambda}| \le C$, uniformly as $M \to \infty$, and $\varepsilon \to 0$, for $C > 0$ arbitrary.

6.2.2. Case b. $1/C\varepsilon \geq |\lambda| \geq C >> 1$

Finally, we treat the more delicate case $C \leq |\lambda| \leq 1/C\varepsilon$, or $C\varepsilon \leq |\tilde\lambda| \leq C^{-1}$, with $C >> 1$. Proceeding as in Section 5.1 by the series of coordinate transformations (5.1)–(5.6), but taking account of the different order of $\tilde\lambda$ in this case, in particular, noting that εm in (5.2) is now $O(\tilde\lambda)$ and not $O(\varepsilon)$ as before, we obtain

$$\mathcal{K}^\varepsilon = \begin{pmatrix} (\varepsilon E - \tilde\lambda)\tilde A^{-1} + \tilde\lambda^2 \beta & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} O(\varepsilon|\tilde\lambda| + |\tilde\lambda||\dot{\tilde W}^\varepsilon|) & O(\varepsilon|\tilde\lambda| + |\tilde\lambda||\dot{\tilde W}^\varepsilon|) \\ \\ O(\varepsilon + |\dot{\tilde W}^\varepsilon|) & O(\varepsilon + |\dot{\tilde W}^\varepsilon|) \end{pmatrix} \tag{6.7}$$

in place of (5.3), where β is the "frozen-coefficient corrector" obtained by dropping terms involving $\tilde x$-derivatives of the transformations involved,

$$S^{-1} = \begin{pmatrix} I & -\Phi_1 \\ -\Phi_2 & I \end{pmatrix} = \begin{pmatrix} I & |\tilde\lambda| O(\varepsilon + |\dot{\tilde W}^\varepsilon|) \\ O(\varepsilon + |\dot{\tilde W}^\varepsilon|) & I \end{pmatrix} = \begin{pmatrix} I & o(|\tilde\lambda|) & o(|\tilde\lambda|) \\ o(1) & I & o(1) \\ o(1) & o(1) & I \end{pmatrix} \tag{6.8}$$

for $\tilde x \geq -M$ in place of (5.4), and

$$\mathcal{W}^-_{n+1} = c(-M) \begin{pmatrix} o(1)|\tilde\lambda| v \\ (I + o(1))v \end{pmatrix} \tag{6.9}$$

in place of (5.8), where v lies in the unique stable eigendirection of $N(-M)$, with

$$N := \tilde b^{-1}(\tilde A_{22} - \tilde A_{21}\tilde A_{11}^{-1}\tilde A_{12}).$$

By the same reasoning, applied to the viscous shock equations (6.3), we have

$$\hat{\mathcal{W}}^-_{n+1} = c(-M) \begin{pmatrix} o(1)|\tilde\lambda| \hat v \\ (I + o(1))\hat v \end{pmatrix},$$

where $\hat v$ lies in the same unique eigendirection. and $\hat{\mathcal{W}}_{n+1}$ is the single fast-decaying mode of (6.3) at $-\infty$. Comparing, we thus have, for some normalizing factor $\psi(M, \varepsilon, \lambda)$, analytic in λ,

$$\mathcal{W}^-_{n+1}(-M) = \big(I + o(|\tilde\lambda|)\big)\psi(M, \varepsilon, \tilde\lambda)\hat{\mathcal{W}}^-_{n+1}(-M). \tag{6.10}$$

We now turn to the description of the remaining, slow-decaying, modes $\mathcal{W}_1^-, \dots \mathcal{W}_n^-$. Focusing on the region $\tilde x \geq -C_2|\log|\tilde\lambda||$, $C_2 >> 1$, we find that the slow, first, equation of (5.5) becomes

$$\begin{aligned} \dot{\mathcal{Z}}_1 &= \Big((\varepsilon E - \tilde\lambda)\tilde A^{-1} + \beta\tilde\lambda^2\Big)\mathcal{Z}_1 + O(\varepsilon + |\dot{\tilde W}^\varepsilon(x/\varepsilon)|)|\tilde\lambda|\mathcal{Z}_1 \\ &= \Big((\varepsilon E - \tilde\lambda)\tilde A^{-1} + \beta\tilde\lambda^2\Big)\mathcal{Z}_1 + o(1)|\tilde\lambda|^2\mathcal{Z}_1, \end{aligned} \tag{6.11}$$

where $\big((\varepsilon E - \tilde\lambda)\tilde A^{-1} + \beta\tilde\lambda^2\big)$ agrees to second order in $\tilde\lambda << 1$ with the restriction of $\mathcal{G}$ to its slow subspace, which, recall, is the direct sum of n slow-decaying modes as $\tilde x \to -\infty$ and a single slow-growing mode as $\tilde x \to -\infty$. Here, $o(1) \to 0$ as $C \to \infty$ and $\varepsilon \to 0$.

Computing explicitly, and noting that $\bar z \sim e^{-\theta\varepsilon|\tilde x|}$, we have

$$\varepsilon E = \varepsilon \begin{pmatrix} qkd\phi(\bar u^\varepsilon)\bar z & qk\phi(\bar u) \\ -kd\phi(\bar u^\varepsilon)\bar z & -k\phi(\bar u) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & qk\phi(\bar u) \\ 0 & -k\phi(\bar u) \end{pmatrix} + \alpha(x, \tilde\lambda),$$

where $\alpha = O(\varepsilon e^{-\theta\varepsilon|\tilde x|})$ is both $o(\tilde\lambda)$ and uniformly bounded in $L^1(\tilde x)$. Thus, up to $O(\alpha)+o(|\tilde\lambda|^2)$ the eigenvalues of $\big((\varepsilon E-\tilde\lambda)\tilde A^{-1}+\beta\tilde\lambda^2\big)$ agree with the growth rates μ_j of slow modes $e^{\mu_j\tilde x}W_j$, $W_j=$ constant, of the nonreacting frozen-coefficient operator

$$\tilde\lambda I + \begin{pmatrix} df(u)-sI & 0\\ 0 & -s\end{pmatrix}\partial_{\tilde x} + \begin{pmatrix} B(u) & 0\\ 0 & C(u,z)\end{pmatrix}\partial_{\tilde x}^2 + \begin{pmatrix} 0 & qk\phi(u)\\ 0 & -k\phi(u)\end{pmatrix} \tag{6.12}$$

obtained by neglecting $O(\partial_{\tilde x}\bar W^\varepsilon)$ derivative terms and setting $\bar z$ to zero.

A standard low-frequency matrix perturbation computation [Z1,LyZ1,LyZ2,TZ4,LRTZ] shows that the modes W_j of (6.12) are analytic in $\tilde\lambda$ and ε, lying approximately in the eigendirections of $\tilde A$, with the associated growing (i.e., negative real part) eigenvalue μ_{n+1} separated in modulus by order $|\tilde\lambda|$ from the decaying ones;[6] moreover, by (1.10), the growing eigenvalue is separated in real part by order $\sim|\tilde\lambda|^2$ from associated decaying (positive real part) eigenvalues.

From this, and the fact that the entire coefficient matrix is order $\tilde\lambda$, it follows that there exists a smooth matrix-valued function $Q(\bar W^\varepsilon)$ with

$$Q^{-1}\big((\varepsilon E-\tilde\lambda)\tilde A^{-1}+\beta\tilde\lambda^2\big)Q = \begin{pmatrix} m_1+\alpha_1 & 0\\ 0 & m_2+\alpha_2\end{pmatrix},$$

$$\Re m_1 \ge \check\eta|\tilde\lambda|^2>0,\quad \Re m_2\le -\check\eta|\tilde\lambda|^2<0,\quad |\alpha_j|_{L^1(\tilde x)}\le C,\qquad \check\eta=\text{constant}. \tag{6.13}$$

Making the change of coordinates $\mathcal Z_1 = Qz$, and noting that $Q'=O(\dot{\bar W}^\varepsilon)$, we thus obtain

$$\dot z = \Big(Q^{-1}\big((\varepsilon E-\tilde\lambda)\tilde A^{-1}+\beta\tilde\lambda^2\big)Q - Q^{-1}\dot Q\Big)z = \begin{pmatrix} m_1 & 0\\ 0 & m_2\end{pmatrix}z + \big(o(|\tilde\lambda|^2)+O(\dot{\bar W}^\varepsilon)\big)z,$$

with gap condition (6.13).

Since $|\dot{\bar W}^\varepsilon| = O(\varepsilon)=o(\tilde\lambda)$, and m_j are spectrally separated by modulus $\sim|\tilde\lambda|$, there is a further smooth coordinate change $z=\mathcal R y$ with $\mathcal R = I+o(1)$ converting the equations to

$$\dot y = \begin{pmatrix} m_1+\check\alpha_1 & 0\\ 0 & m_2+\check\alpha_2\end{pmatrix}z + \big(o(|\tilde\lambda|^2)+\beta\big)z,$$

$\check\alpha_j = \alpha_j + O(\dot{\bar W}^\varepsilon)$, $\beta = o(\dot{\bar W}^\varepsilon)\big)$, where $|\check\alpha_j|_{L^1}\le C$, $|\beta|_{L^1(\tilde x)}|=o(1)$, and m_j satisfy (6.13). That is, we have an equation of form (A.29) with $\delta = o(|\tilde\lambda|^2)$ and $\eta\sim|\tilde\lambda|^2$, so that $\delta/\eta = o(1)$.

Applying the tracking lemma, Lemma A.4, as generalized in (A.30), Remark A.8, at $\tilde x = -C|\log|\tilde\lambda||$, and untangling coordinate changes, we thus find that the slow-decaying modes $\mathcal W_j^-$, $j=1,\dots,n$ lie within angle $o(1)$ of the stable subspace of $\mathcal G(-C|\log|\tilde\lambda||)$, which in turn lies within angle $o(1)$ of the stable subspace of $\hat{\mathcal G}(\tilde\lambda,-\infty)$ and (by a repetition of the same argument), analytic in λ:multiples of the slow-decaying modes $\hat{\mathcal W}_j^-$, $j=1,\dots,n$, at $\tilde x=-M$.

Finally, going back to the original equation (6.11), and noting that $\dot{\mathcal Z}_1 = O(|\tilde\lambda|)\mathcal Z_1$, we find that the change in $\mathcal Z_1$ in evolving from $\tilde x=-C|\log|\tilde\lambda||$ to $\tilde x=-M$ is order

$$\big(e^{|\tilde\lambda||\log|\tilde\lambda||}-1\big)|\mathcal Z_1(-C|\log|\tilde\lambda||)| \sim |\tilde\lambda||\log|\tilde\lambda|||\mathcal Z_1(-C|\log|\tilde\lambda||)| = o(1)|\mathcal Z_1(-C|\log|\tilde\lambda||)|,$$

hence at $\tilde X=-M$ also the slow-decaying modes $\mathcal W_j^-$, $j=1,\dots,n$ lie within angle $o(1)$ of analytic in λ multiples of the slow-decaying modes $\hat{\mathcal W}_j^-$ at $\tilde x=-M$. Collecting facts, we have

$$\mathcal W_j^-(-M) = (I+o(1))\psi_j(M,C_2,\varepsilon,\tilde\lambda)\hat{\mathcal W}_j^-(-M),\qquad j=1,\dots,n, \tag{6.14}$$

[6] This does not require strict hyperbolicity of df, but only $\det(df-s)\neq 0$; see Remark 1.4.

where ψ_j are nonvanishing and analytic in $\tilde\lambda$. (Here, we are using also the fact that, by (6.7)–(6.8), the manifold of all slow modes, both growing and decaying, stays angle $o(1)$ close to the slow subspace of $\mathcal{G}(\tilde\lambda, \tilde x)$ for all $\tilde x \geq -M$.)

Finally, collecting estimates (6.5), (6.10), and (6.14), we see that fast modes $\mathcal{W}_j^{\pm}$, up to nonvanishing analytic factor ψ_j, are given by $(I + o(\tilde\lambda))$ times the correspong fast modes $\hat W_j^{\pm}$ of the associated shock stability problem, while slow modes $\mathcal{W}_j^-$ are are given by $(I + o(1))$ times the correspong slow modes $\hat W_j^-$. Recalling, similarly as in estimate (5.13), that at $\tilde\lambda = 0$, the fast mode $\hat{\mathcal{W}}_{n+1}^-$ is a linear combination of the fast modes $\hat{\mathcal{W}}_j^+$, so that at $\tilde\lambda$ it remains within angle $O(\tilde\lambda)$ of their combination, we find, applying a column operation cancelling $\hat{\mathcal{W}}_{n+1}^-$ to order $\tilde\lambda$ and factoring out $\tilde\lambda$ from that column, that we reduce the (rNS) determinant (3.5) to

$$\tilde\lambda(1 + o(1))(\Pi_j \hat\psi_j) D_{NS}(\tilde\lambda). \tag{6.15}$$

The key observation here is that because only fast modes are involved in the vanishing of D_{NS} at $\tilde\lambda = 0$, *only fast modes must be estimated to the sharper relative error* $o(\tilde\lambda)$ in order to obtain the result (6.15), with $o(1)$ tolerance sufficing for slow modes.

Thus, we obtain again an estimate $\frac{D_{rNS}(\lambda)}{\hat\Psi(M,C,\varepsilon,\lambda)} = D_{NS}(\tilde\lambda) + o(\tilde\lambda)$ on $\Re\lambda > -\eta$, $C\varepsilon \leq |\tilde\lambda| \leq C^{-1}$, where $o(1) \to 0$ uniformly as $M \to \infty$, $C \to \infty$, and $\varepsilon \to 0$, with $\hat\Psi := \Pi_j \hat\psi_j$ nonvanishing and analytic in $\tilde\lambda$. Note that we are using here the assumption that $C >> 1$, which was not needed in case a.

6.3. Convergence to D_{NS}

Proposition 6.1. *Assuming (H0)–(H3), (P1)–(P3), For $C/\varepsilon \geq |\lambda| \geq C >> 1$, $\Re\lambda \geq 0$, $\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\lambda\check\Psi(\lambda,\varepsilon)}$ converges uniformly as $\varepsilon \to 0$ to $\frac{D_{NS}(\varepsilon\lambda)}{\varepsilon\lambda}$, where $\check\Psi(\cdot,\cdot)$ is a nonvanishing factor that is analytic in λ.*

Proof. From (6.6), (6.15), we have in either case a or b that

$$\frac{D_{rNS}(\lambda)}{\hat\Psi(M, C, \varepsilon, \lambda)} = D_{NS}(\varepsilon\lambda) + o(1)\varepsilon\lambda \tag{6.16}$$

on $\Re\lambda > -\eta$, uniformly as $\varepsilon \to 0$. where $o(1) \to 0$ uniformly as $M \to \infty$, $C \to \infty$, and $\varepsilon \to 0$. Choosing monotone increasing sequences C_j, M_j and a monotone decreasing sequence ε_j such that $o(1) \leq 1/j$ in (6.16), define $\check\Psi(\lambda,\varepsilon)$ to be equal to the function $\hat\Psi(\lambda,\varepsilon)$ in (6.16) that is associated with C_j, M_j, where j is the maximum integer such that $\varepsilon \leq \varepsilon_j$. Then, $\hat\Psi$ is analytic in λ by construction, and, combining (6.16) with the definition of ε_j, M_j, C_j, we have $\left|\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\lambda\hat\Psi(\lambda,\varepsilon)} - \frac{D_{NS}(\varepsilon\lambda)}{\varepsilon\lambda}\right| \leq C/j \to 0$ as $\varepsilon \to 0$, giving the result.

Corollary 6.1. *Assuming (H0)–(H3), (P1)–(P3), for $C \leq |\lambda| \leq C/\varepsilon$, C sufficiently large, on $\Re\lambda > -\eta$ for η, $\varepsilon > 0$ sufficiently small, ε times each zero of D^ε_{rNS} converges to a zero of $\frac{D_{NS}(\tilde\lambda)}{\tilde\lambda}$ on $\Re\tilde\lambda \geq 0$; moreover, each zero of D_{NS} on $\Re\tilde\lambda > 0$ is the limit of ε times a zero of D_{rNS} on $\Re\lambda > 0$, for $C \leq |\lambda| \leq C/\varepsilon$.*

Proof. Noting that zeros of D^ε_{rNS} agree with zeros of $\frac{D^\varepsilon_{rNS}(\lambda)}{\varepsilon\lambda\hat\Psi(\lambda,\varepsilon)}$, we obtain the result by Proposition 6.1 and properties of uniform limits of analytic functions.

7. Region III: $|\lambda| \geq C/\varepsilon$, $C >> 1$

Finally, we consider the straightforward "hyperbolic–parabolic" region $|\lambda| \geq C/\varepsilon$, $C >> 1$, or $|\tilde\lambda| >> 1$, on which zeros of D_{rNS} are prohibited stable by basic hyperbolic–parabolic structure/well-posedness of the underlying problem (1.1).

Proposition 7.1. *D_{rNS} does not vanish for $|\lambda| \geq C/\varepsilon$, $C >> 1$, $\Re\lambda \geq 0$.*

Proof. In fast coordinates $\tilde x$, $|\tilde\lambda| >> 1$, this follows by the same high-frequency analysis used in [MaZ3] to treat the viscous shock case, based on the tracking/reduction lemma, Lemma A.4. See Propositon 5.2, [MaZ3], or Proposition 4.33, [Z2].

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Appendix A. Asymptotic ODE theory

Appendix A.1. The conjugation lemma

Consider a general first-order system

$$W' = A^p(x,\lambda)W \tag{A.1}$$

with asymptotic limits $A^p_\pm$ as $x \to \pm\infty$, where $p \in \mathbb{R}^m$ denote model parameters.

Lemma A.1 ([MeZ1,PZ]). *Suppose for fixed $\theta > 0$ and $C > 0$ that*

$$|A^p - A^p_\pm|(x,\lambda) \leq Ce^{-\theta|x|} \tag{A.2}$$

for $x \gtrless 0$ uniformly for (λ, p) in a neighborhood of (λ_0), p_0 and that A varies analytically in λ and continuously in p as a function into $L^\infty(x)$. Then, there exist in a neighborhood of (λ_0, p_0) invertible linear transformations $P^p_+(x,\lambda) = I + \Theta^p_+(x,\lambda)$ and $P^p_-(x,\lambda) = I + \Theta^p_-(x,\lambda)$ defined on $x \geq 0$ and $x \leq 0$, respectively, analytic in λ and continuous in p as functions into $L^\infty[0,\pm\infty)$, such that

$$|\Theta^p_\pm| \leq C_1 e^{-\bar\theta|x|} \quad \text{for } x \gtrless 0, \tag{A.3}$$

for any $0 < \bar\theta < \theta$, some $C_1 = C_1(\bar\theta, \theta) > 0$, and the change of coordinates $W =: P^p_\pm Z$ reduces (A.1) *to the constant-coefficient limiting systems*

$$Z' = A^p_\pm Z \quad \text{for } x \gtrless 0. \tag{A.4}$$

Proof. The conjugators $P^p_\pm$ are constructed by a fixed point argument [MeZ1] as the solution of an integral equation corresponding to the homological equation

$$P' = A^p P - P A^p_\pm. \tag{A.5}$$

The exponential decay (A.2) is needed to make the integral equation contractive with respect to $L^\infty[M,+\infty)$ (resp. $L^\infty(-\infty,-M]$) for M sufficiently large. Continuity of $P_\pm$ with respect to p (resp. analyticity with respect to λ) then follows by continuous (resp. analytic) dependence on parameters of fixed point solutions. Here, we are using the fact that (A.2) implies that the integral operator defined as an integral on $[-M,+\infty)$ (resp. $L^\infty(-\infty,-M]$) can be expressed as a uniform limit of truncated integrals on finite domains, each continuous (resp. analytic) with respect to p, in order to obtain continuity (resp. analyticity) of the integral operator with respect to p. See also [GZ, ZH, MeZ1, PZ, GMWZ5].

Remark A.1. In the case that A is block diagonal or triangular, the conjugators $P_\pm$ may evidently be taken block diagonal or triangular as well, by carrying out the same fixed-point argument on the invariant subspace of (A.5) consisting of matrices with this special form. This can be of use in problems with multiple scales; see, for example, Thm 1.16, [BHZ].

Definition 1 (Abstract Evans function). Suppose that on the interior of a set Ω in λ, p, the dimensions of the stable and unstable subspaces of $A^p_\pm(\lambda)$ remain constant, and agree at $\pm\infty$ ("consistent splitting" [AGJ]), and that these subspaces have analytic bases $R_j^\pm$ extending continuously to boundary points of Ω. Then, the Evans function is defined analytically in λ and continuously in p on Ω^{int} and continuously in (λ,p) on $\bar\Omega$ as

$$D^p(\lambda) := \det(P^+R_1^+, P^+R_2^+, P^-R_1^-, P^-R_2^-)|_{x=0}, \tag{A.6}$$

where $P^p_\pm$ are as in Lemma A.1.

Appendix A.2. The convergence lemma

Consider a family of first-order equations

$$W' = A^p(x,\lambda)W \tag{A.7}$$

indexed by a parameter p, and satisfying exponential convergence condition (A.2) uniformly in p. Suppose further that, for some $\delta(p)\to 0$ as $p\to 0$,

$$|(A^p - A^p_\pm) - (A^0 - A^0_\pm)| \le C\delta(p)e^{-\theta|x|}, \qquad \theta > 0 \tag{A.8}$$

and

$$|(A^p - A^0)_\pm)| \le C\delta(p). \tag{A.9}$$

Lemma A.2 ([PZ,BHZ]). *Assuming* (A.2) *and* (A.8)–(A.9)*, for $|p|$ sufficiently small, there exist invertible linear transformations $P^p_+(x,\lambda) = I + \Theta^p_+(x,\lambda)$ and $P^0_-(x,\lambda) = I + \Theta^p_-(x,\lambda)$ defined on $x\ge 0$ and $x \le 0$, respectively, analytic in λ as functions into $L^\infty[0,\pm\infty)$, such that*

$$|(P^p - P^0)_\pm| \le C_1\delta(p)e^{-\bar\theta|x|} \quad \text{for } x \gtrless 0, \tag{A.10}$$

for any $0<\bar\theta<\theta$, some $C_1 = C_1(\bar\theta,\theta)>0$, and the change of coordinates $W =: P^p_\pm Z$ reduces (A.7) *to the constant-coefficient limiting systems*

$$Z' = A^p_\pm(\lambda)Z \quad \text{for } x \gtrless 0. \tag{A.11}$$

Proof. Applying the conjugating transformation $W \to (P^0_+)^{-1}W$ for the $p=0$ equations, we may reduce to the case that A^0 is constant, and $P^0_+\equiv I$, noting that the estimate (A.8) persists under well-conditioned coordinate changes $W = QZ$, $Q(\pm\infty) = I$, transforming to

$$\begin{aligned} &|\big(Q^{-1}A^pQ - Q^{-1}Q' - A^p_\pm\big) - \big(Q^{-1}A^0Q - Q^{-1}Q' - A^0_\pm\big)| \\ &\quad \le |Q\big((A^p - A^p_\pm) - (A^0 - A^0_\pm)\big)Q^{-1}| + |Q^{-1}(A^p - A^0)_\pm Q - (A^p - A^0)_\pm|, \end{aligned} \tag{A.12}$$

where

$$|Q^{-1}(A^p - A^0)_\pm Q - (A^p - A^0)_\pm| = O(|Q-I|)|(A^p - A^0)_\pm| = O(e^{-\theta|x|})\delta(p). \tag{A.13}$$

In this case, (A.8) becomes just

$$|A^p - A^p_\pm| \le C_1\delta(p)e^{-\theta|x},$$

and we obtain directly from the conjugation lemma, Lemma A.1, the estimate

$$|P^p_+ - P^0_+| = |P^p_+ - I| \le CC_1\delta(p)e^{-\bar\theta|x|}$$

for $x>0$, and similarly for $x<0$, verifying the result.

Remark A.2. In the case $A^p_\pm \equiv$ constant, or, equivalently, for which (A.8) is replaced by $|A^p - A^0| \leq C_1\delta(p)e^{-\theta|x}$, we find that the change of coordinates $W = \tilde{P}^p_\pm Z$, $\tilde{P}^p_\pm := (P^0)^{-1}_\pm P^p_\pm$, converts (A.7) to $Z' = A^0 Z$, where $\tilde{P}^p_\pm = I + \tilde{\Theta}^p_\pm$ with

$$|\tilde{\Theta}^p_\pm| \leq CC_1\delta(p)e^{-\bar{\theta}|x|} \quad \text{for } x \gtrless 0. \tag{A.14}$$

That is, we may conjugate not only to constant-coefficient equations, but also to exponentially convergent variable-coefficient equations, with sharp rate (A.14).

In the general case $A^p\pm \neq A^0\pm$, we may still conjugate (A.7) to $Z' = A^0 Z$ by the change of coordinates $W = \hat{P}^p_\pm Z$, $\hat{P}^p_\pm := (P^0)^{-1}_\pm Q_\pm P^p_\pm$, where $Q_\pm$ defined by

$$Q' = A^p_\pm Q - QA^0_\pm \tag{A.15}$$

conjugates the constant-coefficient equation $Y' = A^p \pm Y$ to $X' = A^0_\pm X$, obtaining bounds

$$\hat{P}^p_\pm = I + \tilde{\Theta}^p_\pm, \qquad |\tilde{\Theta}^p_\pm| \leq CC_1\delta(p) \quad \text{for } |x| \leq C \tag{A.16}$$

valid for finite values of x. Moreover (by prescribing at the origin $x = 0$) we may arrange that this change of coordinates take stable subspace to stable subspace, unstable subspace to unstable subspace, or any smoothly varying subspace to its image at $p = 0$. However, in general, $Q_\pm$ grow without bound as $x \to \pm\infty$.[7]

Remark A.3. As observed in [PZ], provided that the stable/unstable subspaces of A^p_+/A^p_- converge to those of A^0_+/A^0_-, as typically holds given (A.9)– in particular, this holds by standard matrix perturbation theory [K] if the stable and unstable eigenvalues of $A^0_\pm$ are spectrally separated– (A.10) gives immediately uniform convergence of the Evans functions D^p to D^0 on compact sets of Ω, by definition (1).

Appendix A.3. Smooth dependence on parameters

In the case $\delta(p) = |p|$, the quantitative bound (A.10) strengthens the basic conjugation lemma from continuous dependence in a parameter to Lipshitz dependence. We note the following C^r analog, which appears to be rather sharp.

Lemma A.3. *Assuming that A^p are C^r in p as functions into $L^\infty[0, \pm\infty)$, with*

$$|\partial^j_p(A^p - A^p_\pm)| \leq Ce^{-\theta|x|}, \qquad x \gtrless 0, \qquad 0 \leq j \leq r, \tag{A.17}$$

the transformations $P^p_\pm$ of Lemma A.2 are C^r in p as functions into $L^\infty[0, \pm\infty)$, with

$$|\partial^j_p \Theta^p_\pm| \leq C_1 e^{-\bar{\theta}|x|} \quad \text{for } x \gtrless 0, \tag{A.18}$$

for any $0 < \bar{\theta} < \theta$, some $C_1 = C_1(\bar{\theta}, \theta) > 0$, and all $0 \leq j \leq r$.

Proof. Similarly as in the proof of Lemma A.1, (A.17) ensures that the contraction mapping is C^r with respect to p, whence the result follows by uniform convergence of iterates.

It is worth noting in the case of analytic dependence on p that the basic uniform decay estimate (A.2) already implies (A.17) by standard interior estimates.

Remark A.4. Note that (A.17) is what one obtains for coefficients depending on x through a transversal connection $\bar{u}^p(x)$ connecting hyperbolic rest points of a C^r (in u and p) ODE. Thus, (A.18) holds quite generally along with smooth dependence of Evans function (A.6).

Remark A.5. On the region of consistent splitting, smooth dependence of the Evans function follows from the weaker condition $|\partial^j_p(A^p - A^p_\pm)|_{L^1[0,\pm\infty)} < +\infty$; see, e.g., [Co, ZH].

[7] Indeed, (A.16) could equally be obtained by smooth dependence on parameters of solutions of ODE.

Appendix A.4. The tracking lemma

Consider an approximately block-diagonal system

$$W' = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}(x,p)W + \delta(x,p)\Theta(x,p)W, \tag{A.19}$$

where Θ is a uniformly bounded matrix, $\delta(x)$ scalar, and p a vector of parameters, satisfying a pointwise spectral gap condition

$$\min\sigma(\Re M_1^p) - \max\sigma(\Re M_2^p) \ge \eta(x) \text{ for all } x. \tag{A.20}$$

(Here as usual $\Re N := (1/2)(N+N^*)$ denotes the "real", or symmetric part of N.) Then, we have the following *tracking/reduction lemma* of [MaZ3,PZ].

Lemma A.4 ([MaZ3,PZ]). *Consider a system* (A.19) *under the gap assumption* (A.20)*, with Θ^p uniformly bounded and $\eta \in L^1_{\rm loc}$. If $\sup(\delta/\eta)(x)$ is sufficiently small, then there exist (unique) linear transformations $\Phi_1(x,p)$ and $\Phi_2(x,p)$, possessing the same regularity with respect to p as do coefficients M_j and $\delta\Theta$, for which the graphs $\{(Z_1, \Phi_2 Z_1)\}$ and $\{(\Phi_1 Z_2, Z_2)\}$ are invariant under the flow of* (A.19)*, and satisfy*

$$\sup|\Phi_1|,\, \sup|\Phi_2| \le C\sup(\delta/\eta) \tag{A.21}$$

and

$$|\Phi_1^p(x)| \le C\int_x^{+\infty} e^{\int_y^x \eta(z)dz}\delta(y)dy, \qquad |\Phi_1^p(x)| \le C\int_{-\infty}^x e^{\int_y^x -\eta(z)dz}\delta(y)dy. \tag{A.22}$$

Proof. By the change of coordinates $x \to \tilde x$, $\delta \to \tilde\delta := \delta/\eta$ with $d\tilde x/dx = \eta(x)$, we may reduce to the case $\eta \equiv$ constant $= 1$ treated in [MaZ3]. Dropping tildes and setting $\Phi_2 := \psi_2\psi_1^{-1}$, where $(\psi_1^t, \psi_2^t)^t$ satisfies (A.19), we find after a brief calculation that Φ_2 satisfies

$$\Phi_2' = (M_2\Phi_2 - \Phi_2 M_1) + \delta Q(\Phi_2), \tag{A.23}$$

where Q is the quadratic matrix polynomial $Q(\Phi) := \Theta_{21} + \Theta_{22}\Phi - \Phi\Theta_{11} + \Phi\Theta_{12}\Phi$. Viewed as a vector equation, this has the form

$$\Phi_2' = \mathcal{M}\Phi_2 + \delta Q(\Phi_2), \tag{A.24}$$

with linear operator $\mathcal{M}\Phi := M_2\Phi - \Phi M_1$. Note that a basis of solutions of the decoupled equation $\Phi' = \mathcal{M}\Phi$ may be obtained as the tensor product $\Phi = \phi\tilde\phi^*$ of bases of solutions of $\phi' = M_2\phi$ and $\tilde\phi' = -M_1^*\tilde\phi$, whence we obtain from (A.20)

$$e^{\mathcal{M}z} \le Ce^{-\eta z}, \quad \text{for } z > 0, \tag{A.25}$$

or uniform exponentially decay in the forward direction.

Thus, assuming only that Φ_2 is bounded at $-\infty$, we obtain by Duhamel's principle the integral fixed-point equation

$$\Phi_2(x) = \mathcal{T}\Phi_2(x) := \int_{-\infty}^x e^{\mathcal{M}(x-y)}\delta(y)Q(\Phi_2)(y)\,dy. \tag{A.26}$$

Using (A.25), we find that $\mathcal{T}$ is a contraction of order $O(\delta/\eta)$, hence (A.26) determines a unique solution for δ/η sufficiently small, which, moreover, is order δ/η as claimed. Finally, substituting $Q(\Phi) = O(1+|\Phi|^2) = O(1)$ in (A.26), we obtain

$$|\Phi_2(x)| \le C\int_{-\infty}^x e^{\eta(x-y)}\delta(y)\,dy$$

in $\tilde{x}$ coordinates, or, in the original x-coordinates, (A.22). A symmetric argument establishes existence of Φ_1 with the asserted bounds. Regularity with respect to parameters is inherited as usual through the fixed-point construction via the Implicit Function Theorem.

Remark A.6. For η constant and δ decaying at exponential rate strictly slower that $e^{-\eta x}$ as $x \to +\infty$, we find from (A.22) that $\Phi_2(x)$ decays like δ/η as $x \to +\infty$, while if $\delta(x)$ merely decays monotonically as $x \to -\infty$, we find that $\Phi_2(x)$ decays like (δ/η) as $x \to -\infty$, and symmetrically for Φ_1.

Remark A.7. 1. A closer look at the proof of Lemma A.4 shows that, in the approximately block lower-triangular case, $\delta\Theta_{21}$ not necessarily small, there exists a block-triangularizing transformation $\Phi_1 = O(\sup|\delta/\eta|) << 1$, under the much less restrictive conditions

$$\sup\Big(|\delta/\eta|(|\Theta_{11}| + |\Theta_{22}|)\Big) < 1 \text{ and } \sup(|\delta/\eta||\Theta_{21}|) << \frac{1}{\sup|\delta/\eta|}.$$

2. Similarly, in the standard, approximately block-diagonal case, an examination of the proof shows that bounds (A.21) may be sharpened to

$$\begin{aligned} \sup|\Phi_1| &\le C\sup(\delta/\eta)\Big(\sup|\Theta_{12}| + \sup(\delta/\eta)\sup(|\Theta_{11}| + |\Theta_{22}|) + \sup(\delta/\eta)^2\sup|\Theta_{21}|\Big),\\ \sup|\Phi_2| &\le C\sup(\delta/\eta)\Big(\sup|\Theta_{21}| + \sup(\delta/\eta)\sup(|\Theta_{11}| + |\Theta_{22}|) + \sup(\delta/\eta)^2\sup|\Theta_{12}|\Big). \end{aligned} \tag{A.27}$$

Remark A.8. An important observation of [MaZ3, PZ] is that hypothesis (A.20) of Lemma A.4 may be weakened to

$$\min\sigma(\Re M_1^p) - \max\sigma(\Re M_2^p) \ge \eta(x) + \alpha(x,p) \tag{A.28}$$

with no change in the conclusions, for any α satisfying a uniform L^1 bound $|\alpha(\cdot,p)|_{L^1} \le C_1$. (Substitute $e^{\mathcal{M}x} \le Ce^{C_1}e^{-\eta z}$ for (A.25), with no other change in the proof.) More generally, (A.19) may be replaced in Lemma A.4 by

$$W' = \begin{pmatrix} M_1 + \alpha_1 & 0 \\ 0 & M_2 + \alpha_2 \end{pmatrix}(x,p)W + (\delta\Theta + \beta)(x,p)W, \tag{A.29}$$

where $|\alpha_j|_{L^1}$ are bounded and $|\beta|_{L^1}$ is sufficiently small, with the conclusion that

$$\sup|\Phi_1|, \sup|\Phi_2| \le C(\sup(\delta/\eta) + |\beta|_{L^1}). \tag{A.30}$$

(The additional term $\int_{-\infty}^x e^{\mathcal{M}(x-y)}Q_\beta(\Phi_2)(y)\,dy$, $Q_\beta(\Phi_2) := \beta_{21} + \beta_{22}\Phi - \Phi\beta_{11} + \Phi\beta_{12}\Phi$, now appearing in the righthand side of (A.26) is contractive for $|\beta|_{L^1}$ small.) These allow us to neglect commutator terms in some of the more delicate applications of tracking: for example, the high-frequency analysis of [MaZ3], or the analysis of case IIb in Section 6.2.2.

Appendix B. Example: Majda's model with step-type ignition function

Consider strong detonation profiles of the generalized Majda model

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= \epsilon(b(u)u_x)_x + qk\phi(u)z,\\ z_t &= \epsilon(c(u,z)z_x)_x - k\phi(u)z, \end{aligned} \tag{B.1}$$

$u, z, \phi, q, k \in \mathbb{R}^1$, $q \ge 0$, $k > 0$, with a (possibly smooth) "step–type" ignition function

$$\phi(u) \equiv 0 \text{ for } u < u_i \text{ and } \phi(u) \equiv 1 \text{ for } u > u^i\ , \tag{B.2}$$

assuming

$$z_- = 0,\ z_+ = 1, \quad u_- > u^i > u_i > u_+. \tag{B.3}$$

By the invariances of (B.1), we may take without loss of generality $u_* = 2$, $u_+ = 0$, $s = 1$.

Appendix B.1. ZND Profiles, $\varepsilon = 0$

Under (B.2)–(B.3), ZND profiles may be determined explicitly [JY]. Solving (1.8), we obtain

$$u_- = 1 + \sqrt{1-2q}, \quad 0 \le q \le \frac{1}{2}, \quad 0 < k < +\infty.$$

The ZND profile equations may be written, adding q times the second equation to the first, as $-s(\overline{u} + q\overline{z})' + \left(\frac{\overline{u}^2}{2}\right)' = 0$ and $-s\overline{z}' + k\phi(\overline{u})\overline{z} = 0$. Integrating the first equation from -∞ to x and solving the resulting quadratic (1.9), we obtain $\overline{u}(x) = 1 + \sqrt{1-2q(1-\overline{z}(x))}$. Since $\phi(\overline{u}) = 1$ by (B.2)–(B.3), we may solve the second equation for $\bar{z}(x) = e^{kx}$, obtaining

$$(\overline{z}, \bar{u})(x) = (e^{kx}, 1 + \sqrt{1-2q(1-e^{kx})}). \tag{B.4}$$

Appendix B.2. Evans–Lopatinski stability of ZND profiles

Proposition B.1 ([JY]) *Under assumptions* (B.2)–(B.3),*, strong ZND detonation solutions of* (B.1)*, $\varepsilon = 0$, are strongly Evans–Lopatinski stable for any q, k, u_-, s.*

Proof. (Following [JY]) Using $(\partial/\partial u)\phi(\bar{u}) \equiv 0$, a consequence of assumption (B.2), we find that the linearized eigenvalue equation (2.14) about a ZND profile takes the triangular form $Z_1' = -\frac{\lambda}{\bar{u}-1}Z_1 - qkZ_2$ and $Z_2' = (k+\lambda)Z_2$, yielding with (B.4) the decaying solution

$$(Z_1, Z_2)(x) = \Big(- \int_{-\infty}^{x} e^{-\int_y^0 \frac{\lambda}{\sqrt{1-2q(1-e^{ks})}}ds} qke^{(k+\lambda)y}dy,\ e^{(k+\lambda)x}\Big),$$

and thus, integrating by parts,

$$\begin{aligned} Z_1(\lambda, 0) &= -\int_{-\infty}^{0} e^{-\int_y^0 P(s)ds} qke^{(k+\lambda)y}dy \\ &= -e^{-\int_y^0 P(s)ds}\frac{qk}{k+\lambda}e^{(k+\lambda)y}\Big|_{-\infty}^{0} + \int_{-\infty}^{0} e^{-\int_y^0 P(s)ds}P(y)\frac{qk}{k+\lambda}e^{(k+\lambda)y}dy \\ &= -\frac{qk}{k+\lambda}\left(1 - \lambda\Psi\right), \end{aligned}$$

and $Z_2(\lambda, 0) = 1$. where

$$\Psi := \int_{-\infty}^{0} e^{-\int_y^0 \frac{\lambda}{\sqrt{1-2q(1-e^{ks})}}ds} \frac{e^{(k+\lambda)y}}{\sqrt{1-2q(1-e^{ky})}}dy, \qquad P(\xi) := \frac{\lambda}{\sqrt{1-2q(1-e^{k\xi})}}. \tag{B.5}$$

Computing $\lambda[\bar{W}] + R(\bar{W}(0^-)) = (-2\lambda + qk, -k)^T$, where $R(\bar{W}) := (qk\phi(\bar{u}), -k\phi(\bar{u}))^T$ as in (2.2), we thus have, up to a nonvanishing analytic factor,

$$D_{ZND}(\lambda) = \det(Z^-(\lambda, 0),\ \lambda[\bar{W}] + R(\bar{W}(0^-)) = \big(2\lambda + (2-q-qk\Psi)\big)\left(\frac{\lambda}{k+\lambda}\right). \tag{B.6}$$

Noting, for $\Re\lambda \ge 0$, that

$$\begin{aligned} |\Psi| &\le \int_{-\infty}^{0} e^{-\int_y^0 \Re P(s)ds}\frac{e^{(k+\Re\lambda)y}}{\sqrt{1-2q(1-e^{ky})}}dy \le \int_{-\infty}^{0} \frac{e^{ky}}{\sqrt{1-2q(1-e^{ky})}}dy \\ &= \frac{1}{qk}\int_{-\infty}^{0} \frac{d}{dy}\Big(\sqrt{1-2q(1-e^{ky})}\Big)dy = \frac{1-\sqrt{1-2q}}{qk}, \end{aligned}$$

we have $\Re(2-q-qk\Psi) \ge 2-q-qk|\Psi| \ge 1-q+\sqrt{1-2q} > 0$ by $0 \le q \le \frac{1}{2}$. In particular, $2-q-qk\Psi \ne 0$ for $\Re\lambda \ge 0$, verifying strong Evans–Lopatinski stability by (B.6).

Appendix B.3. Nonlinear stability of viscous detonation profiles

Corollary B.2 *Under* (B.2)–(B.3)*, viscous strong detonation solutions of* (B.1) *are strongly Evans (hence nonlinearly) stable for* $b, c > 0$ *and* $0 < \varepsilon \le \varepsilon(q, k, u_-, s)$ *sufficiently small.*

Proof. Assumptions (P1)–(P3) on profile structure follow by a singular perturbation analysis similar to but much simpler than that of [W] in the full (rNS) case [RV], whence the result follows by Theorem 1.1 together with Propositions 1.1 and B.1.

Corollary B.2 was established in [RV] for the case $b \equiv 1$, $c \equiv 0$. It is an interesting question whether corresponding results hold for Majda's model for general ignition functions

$$\phi(u) = 0 \text{ for } u < u_i \text{ and } \phi(u) > 0 \text{ for } u > u_i. \tag{B.7}$$

Appendix C. Example: Ideal gas with Arrhenius ignition function

Consider right-going strong detonations of the (ZND) equations

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t E + \partial_x (pu) = qk\phi(T)z, \\ \partial_t z = -k\phi(T)z, \end{cases} \tag{C.1}$$

with ideal gas equation of state and Arrhenius-type ignition function,

$$p = \Gamma \tau^{-1} e, \quad T = c^{-1} e, \quad \phi(T) = e^{\frac{-\mathcal{E}}{T}}, \tag{C.2}$$

where $E = e + u^2/2$ is specific (gas-dynamical) energy, $c > 0$ is the specific heat constant, $\Gamma > 0$ is the Gruneisen constant, k and q are reaction rate and heat release coefficients, and $\mathcal{E}$ is activation energy.

Appendix C.1. Rescaled ZND profiles

Using the invariances of (C.1) to rescale

$$(x, t, s, \tau, u, T) \to \Big(\frac{\tau_+ sx}{L}, \frac{\tau_+ s^2 t}{L}, 1, \frac{\tau}{\tau_+}, \frac{u}{\tau_+ s}, \frac{T}{\tau_+^2 s^2}\Big), \quad (z, q, k, \mathcal{E}) \to \Big(z, \frac{q}{\tau_+^2 s^2}, \frac{Lk}{\tau_+ s^2}, \frac{\mathcal{E}}{\tau_+^2 s^2}\Big),$$

we may take without loss of generality $s = 1$, $\tau_+ = 1$ and (by translation invariance in u), $u_+ = 0$, leaving $1 > \tau_* > 0$ as the parameter determining the Neumann shock. This may be recognized as the rescaling introduced in [HLyZ1] to study stability of viscous shocks in the infinite-amplitude limit.

From Eqs. (2.29)–(2.35), [HLyZ1], we find that $\frac{\Gamma}{\Gamma+2} := \tau_{min} \le \tau_* \le 1$, $u_* = 1 - \tau_*$, and

$$e_+ = \frac{(\Gamma + 2)(\tau_* - \tau_{min})}{2\Gamma(\Gamma + 1)}, \quad e_* = \frac{\tau_*(\Gamma + 2 - \Gamma\tau_*)}{2\Gamma(\Gamma + 1)}. \tag{C.3}$$

We see from this analysis that the strong-Neumann-shock limit corresponds, for fixed Γ, to the limit $\tau_* \to \tau_{min}$, with $e_+ \to 0$ in the limit as $\tau_* \to \tau_{min}$. Thus, $\tau_* = \tau_{min}$ is the boundary for physical (positive e) shocks. Equivalently, this is the infinite Mach-number limit, or, in the language of the detonation literature, the infinite overdrive limit $f \to \infty$, where $1 \le f < \infty$ is defined as the square of the ratio of relative speed of the detonation (with respect to the ambient gas) and the minimum, Chapman–Jouget, detonation speed among all possible strong detonations [Er2, FW, LS, BMR].

Alternatively, computing by (C.3) the maximum value of e_+ (occurring for $\tau_* = \tau_+ = 1$), we can parametrize shock strength more conveniently by the specific internal energy

$$0 \le e_+ \le \frac{1}{\Gamma(\Gamma+1)} \tag{C.4}$$

of the unburned state, which convention we shall follow here.

Under the normalization $s = 1$, profile equation (1.9) becomes

$$-\bar\tau - \bar u = j_1, \quad -\bar u + \frac{\Gamma \bar e}{\bar\tau} = j_2, \quad -(\bar e + \bar u^2/2 + q(\bar z - 1)) + (\bar u \Gamma \bar e/\bar \tau) = j_3, \tag{C.5}$$

where $j_1 = -\tau_* - u_* = -\tau_+ - u_+ = -1$, $j_2 = -u_+ + \frac{\Gamma e_+}{\tau_+} = \Gamma e_+$, and $j_3 = -(e_+ + u_+^2/2) + (u_+ \Gamma e_+/\tau_+) = -e_+$. Solving successively for $\bar u = 1 - \bar\tau$ and $\bar e = \frac{\bar\tau(j_2 + 1 - \bar\tau)}{\Gamma}$ using equations one and two, then substituting into equation three and solving the resulting quadratic

$$\tau^2 - 2\beta\Big(\frac{\Gamma+1}{\Gamma}\Big)(j_2+1)\tau + \beta(1 + 2j_2 - 2q(z-1) - 2j_3) = 0, \quad 0 < \beta = \frac{\Gamma}{\Gamma+2} < 1,$$

in $\bar\tau$, we readily obtain

$$\bar u = 1 - \bar \tau, \quad \bar e = \frac{\bar\tau(\Gamma e_+ + 1 - \bar\tau)}{\Gamma}, \tag{C.6}$$

$$\bar\tau = \frac{(\Gamma+1)(\Gamma e_+ + 1) - \sqrt{(\Gamma+1)^2(\Gamma e_+ + 1)^2 - \Gamma(\Gamma+2)(1 + 2(\Gamma+1)e_+ - 2q(z-1))}}{\Gamma+2}, \tag{C.7}$$

where

$$0 \le q \le q_{cj} := \frac{(\Gamma+1)^2(\Gamma e_+ + 1)^2 - \Gamma(\Gamma+2)(1 + 2(\Gamma+1)e_+)}{2\Gamma(\Gamma+2)}. \tag{C.8}$$

The $\bar z$ component can then be solved via

$$\bar z' = k\phi(c\bar e(\bar z))z, \tag{C.9}$$

from which we may deduce uniform exponential decay as $x \to -\infty$, as required in (P1).

In the Chapman–Jouget limit $q = q_{cj}$,

$$\bar\tau = \frac{(\Gamma+1)(\Gamma e_+ + 1) - \sqrt{2(\Gamma(\Gamma+2)q_{cj} z}}{\Gamma+2}. \tag{C.10}$$

In the strong shock limit $e_+ = 0$, (C.6)–(C.8) greatly simplify, giving

$$\bar u = 1 - \bar\tau = \frac{1 + \sqrt{1 + 2\Gamma(\Gamma+2)q(z-1)}}{\Gamma+2}, \quad \bar e = \frac{\bar\tau(1-\bar\tau)}{\Gamma}, \tag{C.11}$$

and

$$\bar\tau = \frac{\Gamma + 1 - \sqrt{1 + 2\Gamma(\Gamma+2)q(z-1)}}{\Gamma+2}, \quad \text{with } 0 \le q \le q_{cj} := \frac{1}{2\Gamma(\Gamma+2)}. \tag{C.12}$$

The simultaneous strong shock and Chapman–Jouget limit yields still simpler formulae

$$q = q_{cj} = \frac{1}{2\Gamma(\Gamma+2)}, \quad \bar\tau = \frac{\Gamma+1-\sqrt{z}}{\Gamma+2}, \quad \bar u = \frac{1+\sqrt{z}}{\Gamma+2}, \quad \bar e = \frac{\bar\tau(1-\bar\tau)}{\Gamma}. \tag{C.13}$$

Appendix C.1.1. Dimensionless parametrization

We can parametrize all possible ZND profiles by $(e_+, q, \mathcal{E}, \Gamma)$, where

$$0 \le e_+ \le \frac{1}{\Gamma(\Gamma+1)}, \quad 0 \le q \le q_{cj}(e_+), \quad 0 \le \mathcal{E} < \infty, \quad \text{and } \; 0 < \Gamma < \infty, \tag{C.14}$$

leaving the characteristic length scale L as a free parameter to adjust for our convenience. A rescaling in space and time, L affects neither existence nor stability; however, as pointed out in [Er2], it is useful in renormalizing certain limits.

This parametrization is similar to that used by Erpenbeck in [Er2], but with wave speed s held fixed instead of e_+ as in [Er2]. The advantage of our choice, as pointed out in [HLyZ1], is that the linearized eigenvalue equations retain bounded coefficients in the strong Neumann shock limit, whereas in Erpenbeck's scaling, the Neumann temperature e_*, hence also coefficients, go to infinity, while the wave speed $s \to 0$.

Remark C.1. Converting from Erpenbeck's to our scaling amounts to rescaling the wave speed, so that $T \to T/s^2$ and $\mathcal{E} \to \mathcal{E}/s^2$, and $t \to ts^2$ (u is translation invariant, so irrelevant). Note, in the eigenvalue ODE, that this takes $\lambda \to \lambda/s^2$, dilating the frequency.

Appendix C.2. A classical benchmark problem

A classical test case dating back to [Er2, FW] is to take $\mathcal{E} = q = 50$, $\Gamma = .2$, and treat the overdrive f as a bifurcation parameter. Numerically, detonations are seen to be stable for overdrive above $f \approx 1.73$, and unstable below. This is equivalent in our scaling to taking $\mathcal{E} = \mathcal{E}_0 e_+$, $q = q_0 e_+$, and varying e_+ from $e_+ = e_{cj}(q_0)$ ($f = 1$) to 0 ($f = \infty$), where e_{cj} is determined implicitly by the relation $q_{cj}(e_{cj}) = q_0 e_{cj}$.

Appendix C.2.1. The high-overdrive limit

The high-overdrive limit $f \to \infty$ is thus seen to correspond with the limit $e_+ \to 0$, $\phi \to \phi_* \equiv 1$, and $q \to 0$, that is, the simultaneous *zero heat release* and *strong shock limits.* In this limit, the Evans–Lopatinski determinant converges to the Lopatinski determinant for a nonphysical shock with zero righthand internal energy e_+, which can be seen by direct computation to be stable.[8] See [HLyZ1] for related analysis in the more complicated viscous strong shock limit. Collecting information, we obtain the following analytical stability result, resolving an open question dating back to the work of Erpenbeck [Er2] in 1964.

Proposition C.1. *In the scaling of Erpenbeck [Er2], ZND detonations of* (C.1) *are strongly Evans–Lopatinski stable in the fixed-activation energy, fixed-heat release, high-overdrive limit $f \to \infty$.*

The high-overdrive limit, and its relation to the small-heat release limit, was discussed by Erpenbeck in [Er2]. However, he described this limit as "neutral" since eigenvalues of the associated Neumann shocks, though stable, approached the imaginary axis as $f \to \infty$, and concluded that determination of stability in this limit would therefore require a more refined analysis. Our analysis shows that this observed neutrality was an artifact of the unbounded scaling employed in [Er2]. Indeed, converting our scaling back to that of Erpenbeck takes frequency $\lambda \to \lambda e_+$, where $e_+ \to 0$

[8] Alternatively, we may use the numerical experiments for the associated viscous profile in [HLyZ1] together with the link to Lopatinski stablity in the zero-frequency limit [ZS, Z1] to make the same conclusion.

as $f \to \infty$; see Remark C.1. Thus, eigenvalues of the limiting shock are compressed by factor e_+ toward the origin, explaining the approach to the imaginary axis observed by Erpenbeck.

Proposition C.1 together with Theorem 1.1 implies a corresponding small-viscosity result for (rNS). Indeed, we have the following stronger result independent of viscosity $\varepsilon > 0$.

Proposition C.2. *In the scaling of Erpenbeck [Er2], for arbitrary fixed activation energy $\mathcal{E}$, heat release q, and viscosity parameter $\varepsilon > 0$, and overdrive f sufficiently large, (rNS) detonation of* (1.16) *are linearly and noninearly stable if the corresponding viscous Neumann shock is Evans stable.*[9]

Proof. This follows by the same rescaling/limiting argument as for Proposition C.1, noting as in [HLyZ1] that viscosity and heat conduction are invariant under our rescaling transformations, and that the limiting viscous shock with nonphysical value $e_+ = 0$ nonetheless has a well-defined Evans function to which the (rNS) Evans function converges as $q \to 0$.

Remark C.2. In the derivation of Maja's model from (C.1) by weakly nonlinear geometric optics, as described in [BMR], the activation energy is scaled as ε^{-1} and the heat release as ε^2 in Erpenbeck's scaling. A brief computation reveals that this limit, too, corresponds in our scaling to a small heat-release limit (though with ϕ merely bounded and not $\equiv 1$), hence stability. Thus, there is no particular reason to expect instability in Majda's model through the connection to (sometimes unstable) detonations of the full model; this perhaps explains somewhat the strong stability properties seen in Section B.

Appendix C.2.2. The Chapman-Jouget limit: Nonlinear bifurcation of viscous detonation profiles

As f is decreased toward the Chapman-Jouget limit $f = 1$, instabilities are seen numerically to occur [Er2]; indeed, this is a standard benchmark for numerical schemes [FW, BMR]. Specifically, there appears a spectral Hopf bifurcation in the form of a single complex conjugate pair of eigenvalues crossing the imaginary axis for the first time at about $f \approx 1.73$. Though we do not yet have an analytical proof that this occurs, we can deduce a great deal from these numerical observations (and, note that these could in principle be converted to numerical proof, since they consist of numerically well-conditioned approximations of the Evans–Lopatinski condition, as described in [HuZ2]). See also the asymptotic studies of [BN, CH, S1].

Proposition C.3. *Suppose that there exist values f_1 and f_2 such that, for $f \in [f_1, f_2]$, there is a single conjugate pair of roots $\lambda_\pm(f)$ of the Evans–Lopatinski determinant lying near the imaginary axis, with all others bounded away, and suppose that $\Re\lambda_\pm$ is negative for $f = f_2$ and positive for $f = f_1$.*[10] *Suppose further that the associated viscous Neumann shock is stable (as indicated by the numerical studies of [HLyZ1]). Then, for fixed viscosity ε sufficiently small, there occurs a nonlinear Hopf bifurcation from the associated (rNS) detonation to a family of time-periodic solutions of* (1.16) *at some value $f = f_* \in (f_1, f_2)$.*[11]

Proof. By Theorem 1.1, for ε sufficienty small, the roots of the (rNS) Evans function have the same properties at f_1 and f_2. By the Intermediate Value Theorem, they must cross the imaginary axis at some $f = f_*$. By the fact observed in [LyZ2] that they cannot cross through the origin, they must be separated, and thus, by analyticity of the reacting ideal gas equations with respect to its model parameters, hence, through the conjugation lemma, Lemma A.1, real analyticity of the Evans

[9] Recall that stability of arbitrary-amplitude viscous ideal gas was demonstrated numerically in [HLyZ1].
[10] Recall, [LyZ2], that roots cannot cross the origin, so this is the simplest way that instability can occur.
[11] See Theorem 1.14, [TZ4], for a precise description of the bifurcating time-periodic waves.

function, they are real analytic functions of the model parameters, in particular of f. It follows that $\Re\lambda_\pm(f)$ has a nonzero derivative at f_* with respect to the renormalized bifurcation parameter $g := (f - f_*)^n$, for some integer n, since derivatives of nonconstant real analytic functions cannot vanish to all orders. The result then follows by the general bifurcation result of Theorem 1.14, [TZ4].

Appendix C.3. Other limits, and problems for further investigation

Some further interesting limits relevant to the instability discussed above are the *high-activation energy limit*, which, in the scaling of Erpenbeck, corresponds to $\mathcal{E} \to \infty$, with $\tau_+ = 1$, $e_+ = 1$, q and f held fixed; the *high-heat release limit*, corresponding to $q \to \infty$, with $\mathcal{E}$, $\tau_+ = 1$, f held fixed, and, necessarily, $e_+ \to 0$; the *high-activation energy/high-heat release limit* corresponding to $\mathcal{E} = q \to \infty$, with $\tau_+ = 1$, f fixed, and $e_+ \to 0$; and, either separately or in conjunction with the previous limits, the *Newtonian limit* $\Gamma \to 0$. In each of these limits, asymptotics indicate that there is a simplification in behavior allowing predictions of stability boundaries; see the formal asymptotic analyses of [BN, S1, CH] and references therein shedding light on instability mechanisms. However, to our knowledge, in the one-dimensional setting there are up to now no rigorous results in this direction. (The celebrated work of [Er3] establishes high-frequency instabilities in the multi-dimensional case by quite different mechanisms.)

A very interesting open problem would be to investigate these limits, numerically if necessary, or by rigorous asymptotics if possible. A result, even numerical, of instability in a limiting case would be quite interesting, since it would then apply to all waves sufficiently near such a limit. For similar analyses in the viscous shock case, see, e.g., [HLZ, HLyZ1, BHZ]. Note that this proposal is qualitatively different from numerical studies already appearing in the detonation literature in that it involves at the first step the rigorous identification of a limit. Proposition C.1 gives a simple example of such argument structure, demonstrating the power of the approach.

We leave these investigations for the future, mentioning only how some of these translate in the scaling of this paper. For example, the high-heat release limit after rescaling the shock speed to unity translates to $\mathcal{E} \sim e_+ \to 0$, $q \to q_{lim}$, finite, so that the limiting problem corresponds to the strong shock/low-activation energy limit with $e_+ = 0$ and $\phi = \phi_* \equiv 1$, similarly as in the high-overdrive limit but with nonzero q. This would be a very simple and interesting limit to investigate, bearing on the question to what extent high activation energy/square-wave structure are essential mechanisms for instability.[12] Similarly, the high-activation energy/high-heat release limit translates in our scaling to $\mathcal{E} = q \to q_{lim}$, constant, and $e_+ \to 0$, a more complicated ignition structure but still different from the square wave type. Each of these limits are of regular perturbation type, and may be be investigated by the technique described above of simply carrying out numerics for the limiting case.

The high-activation energy and Newtonian limits on the other hand are true singular perturbation problems, for which asymptotics must be carried out analytically in order to establish a rigorous limit. For example, for $q = q_{cj}$ and $e_+ = 0$ held fixed (the strong shock/Chapman–Jouget limit) and $\mathcal{E} \to \infty$ (high-activation energy limit), we find that reaction occurs mainly at the maximum point $\bar\tau = 1/2$ for internal energy $\bar e = \frac{\bar\tau(1-\bar\tau)}{\Gamma}$, corresponding by (C.13) to $\bar z = \frac{\Gamma^2}{4}$, which, for typical physical values $\Gamma < 2$ lies strictly in the range $(0, 1)$ of $\bar z$. Moreover, for small values of Γ, in particular, for the value $\Gamma = 0.2$ of the benchmark problem above, this point lies very near to the square root singularity at $z = 0$ of $\tau(z)$ in (C.13), at which τ, u, and e vary rapidly with z. Rescaling L so as to keep a fixed half-reaction length, we find that these properties induces in the ZND profile the characteristic "square-wave" structure described in [BN, S1] of a nearly steady burnt

[12] The numerical results of [Er2] show instability without square wave structure, but still in the high-activation energy regime.

state followed by a rapid, nearly discontinuous rise in temperature, followed by an "induction zone" with arbitrarily slow reaction terminated finally by the Neumann shock. This structure should be amenable to rigorous singular perturbation techniques like those used in the present paper, and we view such treatment as an extremely interesting direction for future study.

Finally, we remark that such square-wave structure, and the associated nonmotonicity of $\bar{e}$, has no counterpart in the Majda model (B.1), where ZND profiles by (B.4) are always monotone. That is, the Majda model well models τ and u, but models e only when τ is constrained to the range for which e is monotonic, for example, the zero heat-release limit $q \to 0$ under which the Majda model may be derived by weakly nonlinear geometric optics [M, BMR]. This may bear on the question whether Majda's model admits the instabilities seen in the full flow.

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