Dynamics of a small rigid body in a perfect incompressible fluid

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Based on joint works with
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Presentation of the model: a rigid body immersed in an incompressible perfect fluid

- We consider the motion of a rigid body immersed in an incompressible perfect fluid in a regular domain $\Omega \subset \mathbb{R}^2$.

$\Omega = \mathbb{R}^2$ or is a bounded domain.

- The solid occupies at each instant $t \geq 0$ a closed subset $S(t) \subset \Omega$, and the fluid occupies $\mathcal{F}(t) := \Omega \setminus S(t)$. 
Fluid equation

▶ In $\mathcal{F}(t)$, the fluid satisfies the incompressible Euler equation:

$$
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, & t \in [0, T], \ x \in \mathcal{F}(t), \\
\text{div } u = 0 & t \in [0, T], \ x \in \mathcal{F}(t)
\end{cases}
$$

▶ At the boundaries, the fluid satisfies the no-penetration/slip condition:

$$
u \cdot n = 0 \text{ for } x \in \partial \Omega,$$

$$
u \cdot n = V_S \cdot n = [h'(t) + \vartheta'(t)(x - h(t))^\perp] \cdot n \text{ for } x \in \partial S(t).$$

Here:

▶ $u = u(t, x) : \mathcal{F}(t) \to \mathbb{R}^2$ is the fluid velocity, $p = p(t, x) : \mathcal{F}(t) \to \mathbb{R}$ the pressure,

▶ $n$ is the outer normal to the boundaries $\partial \Omega$ and $\partial S(t),

▶ $h(t)$ is the position of its center of mass (say $h(0) = 0$), $\vartheta$ is the angle with respect to the initial position (so $\vartheta(0) = 0$).
Dynamics of the solid

The dynamics of the solid is driven by the action of the pressure on its surface:

\[ m \ h''(t) = \int_{\partial S(t)} p \ n \ ds, \]
\[ \mathcal{J} \ \vartheta''(t) = \int_{\partial S(t)} p (x - h(t))\perp \cdot n \ ds, \]

where

- \( m > 0 \) is the mass of the body,
- \( \mathcal{J} > 0 \) denotes the moment of inertia.

Remark. D’Alembert’s paradox does not apply here, because it concerns fluids which are potential in \( \mathbb{R}^2 \), stationary and constant at infinity. In that case (only), D’Alembert’s paradox states that the fluid does not influence the dynamics of the solid.
Other formulations

▶ **Vorticity formulation.** In 2-D, the fluid part of the system can also be written

\[ \partial_t \omega + (u \cdot \nabla) \omega = 0 \quad \text{in} \quad F(t), \]

and

\[ \text{curl } u = \omega \quad \text{in} \quad F(t), \]
\[ \text{div } u = 0 \quad \text{in} \quad F(t), \]

\[ \oint_{\partial S(t)} u \cdot \tau \, ds = \oint_{\partial S_0} u_0 \cdot \tau \, ds = \gamma \quad \text{(Kelvin’s law)}, \]

+ boundary conditions on \( u \cdot n. \)

▶ As for the Euler equation alone, the complete system can be viewed as an **equation of geodesics** on an infinite dimensional Riemannian manifold, in the spirit of Arnold’s work, see also Ebin-Marsden. (G.-Sueur)
References for the Cauchy problem

- Classical solutions (say at least $C^1$) solutions with finite energy:
  - Ortega-Rosier-Takahashi in the full plane.
  - Rosier-Rosier in the full space.
  - Houot-San Martin-Tucsnak in a bounded domain in Sobolev spaces.

- Weaker solutions (Yudovich or DiPerna-Majda type solutions):
  - G.-Sueur
  - Yun Wang-Zhouping Xin
Cauchy problem (2D, Yudovich-type solutions)

**Theorem (G.-Sueur)**

Let $S_0$ be a smooth, bounded domain in $\Omega \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$. For any $u_0 \in C^0(\overline{F}_0; \mathbb{R}^2)$, $(h'_0, \varphi'_0) \in \mathbb{R}^3$ such that

$$\text{div } u_0 = 0, \text{ curl } u_0 = \omega_0 \in L_\infty^c(\overline{F}_0),$$

$$u_0 \cdot n = (h'_0 + \varphi'_0(x - h_0)^\perp) \cdot n \text{ on } \partial S_0, \quad \lim_{|x| \to +\infty} |u_0| = 0 \text{ or } u \cdot n = 0 \text{ on } \partial \Omega.$$

there exists a unique solution

$$(h, r, u) \in C^2([0, T^*)) \times C^1([0, T^*)) \times L^\infty([0, T^*); \mathcal{L}\mathcal{L}(\mathcal{F}(t)))),$$

where $T^* \in (0, +\infty]$ is the first meeting time between $S(t)$ and $\partial \Omega$.

Here $\mathcal{L}\mathcal{L}(U) := \{ f \in C^0(U) / \exists C > 0, \forall x, y \in U,$

$$|f(x) - f(y)| \leq C|x - y|(1 + \ln^{-}|x - y|)\}.$$

**Remark**

When $\Omega = \mathbb{R}^2$, in general $u(t, \cdot) \notin L^2(\mathcal{F}(t); \mathbb{R}^2)$. Finite energy solutions would be too particular in the sequel...
The problem of a small body

Let us be given $S_0$ smooth, etc. and $h'_0, \vartheta'_0, \gamma, \omega_0 \in L^\infty_c(\mathbb{R}^2)$ fixed as above.

**Question.** What can be said for a small solid, that is, how behaves the solution when the initial position of the solid is:

$$S_0^\varepsilon := \varepsilon S_0,$$

as $\varepsilon$ goes to 0?

We will be interested in the following two particular regimes:

- A massive point in the limit:
  $$m_{\varepsilon} = m_1 \quad \text{and} \quad \mathcal{J}_{\varepsilon} = \varepsilon^2 \mathcal{J}_1,$$

- A constant density:
  $$m_{\varepsilon} = \varepsilon^2 m_1 \quad \text{and} \quad \mathcal{J}_{\varepsilon} = \varepsilon^4 \mathcal{J}_1,$$

where $m_1$ and $\mathcal{J}_1$ are fixed constants.
We have obtained results in that direction in two situations:

- **Situation 1**: $\Omega = \mathbb{R}^2$, $\omega_0 \in L^\infty_c(\mathbb{R}^2 \setminus \{0\})$. Hence for small $\varepsilon > 0$, 
  \[
  \text{dist}(\text{Supp}(\omega_0), S_0^\varepsilon) > 0 \ldots
  \]

- **Situation 2**: $\Omega$ is a smooth bounded domain, $\omega_0 = 0$.

Two regimes, two situations, four possibilities.
### Four cases

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#### Massive particle:
- $m_\varepsilon = m_1$
- $\mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J}_1$

#### Light particle:
- $m_\varepsilon = \varepsilon^2 m_1$
- $\mathcal{J}_\varepsilon = \varepsilon^4 \mathcal{J}_1$
Motivations

- Obtain simplified fluid-solid models. This would be particularly interesting in the case of many bodies. . .

- Give another justification of classical vortex models.

- One would like to study control problems associated to a fluid-rigid body system.
First case: a massive small solid in $\mathbb{R}^2$

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First case: a massive small solid in $\mathbb{R}^2$

Notations.

- Recall the Biot-Savart formula in $\mathbb{R}^2$, $K[\omega]$ describes the velocity generated by the vorticity $\omega$

$$K[\omega] := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)\perp}{|x - y|^2} \omega(t, y) \, dy,$$

so that

$$\text{curl } u = \omega, \quad \text{div } u = 0 \text{ in } \mathbb{R}^2, \quad \lim_{|x| \to +\infty} u(x) = 0,$$

but in general $u \notin L^2(\mathbb{R}^2)$

- A point vortex located at $h$, with intensity $\gamma$ corresponds to:

$$K[\gamma \delta_h] := \frac{\gamma}{2\pi} \frac{(x - h)\perp}{|x - h|^2}.$$
A massive small solid in $\mathbb{R}^2$

Theorem (G.-Lacave-Sueur). Up to a subsequence, one has:

- $h^\varepsilon \rightharpoonup h$, $\varepsilon \vartheta^\varepsilon \rightharpoonup 0$ weakly-* in $W^{2,\infty}(0, T; \mathbb{R}^2)$,
- $\omega^\varepsilon \longrightarrow \omega$ in $C^0([0, T]; L^\infty(\mathbb{R}^2) - w^*)$,
- $u^\varepsilon \longrightarrow K[\omega + \gamma \delta_{h(t)}]$ in $C^0([0, T]; L^q_{loc}(\mathbb{R}^2))$, $q < 2$,
- one has

$$\frac{\partial \omega}{\partial t} + \text{div} \left( K[\omega + \gamma \delta_{h(t)}] \omega \right) = 0 \text{ in } [0, T] \times \mathbb{R}^2,$$

$$mh''(t) = \gamma \left( h'(t) - K[\omega(t, \cdot)](h(t)) \right)^\perp,$$

$$\omega|_{t=0} = \omega_0, \ h(0) = h_0, \ h'(0) = h'_0.$$
Kutta-Joukowski force

The force appearing in the equation of the point in the limit

$$mh''(t) = \gamma \left( h'(t) - K[\omega(t, \cdot)](h(t)) \right) \perp,$$

is similar to the Kutta-Joukowski lift force of the irrotational theory:

$$F = \gamma (v - u_\infty) \perp.$$

the force applied to the body at speed $v$, with fluid velocity $u_\infty$ at infinity and circulation $\gamma$ around the body is

\[ F = \gamma (v - u_\infty) \perp. \]
In the case of a fixed obstacle, a similar result was obtained by Iftimie, Lopes-Filho and Nussenzveig-Lopes. Of course in that case, the vortex in the limit is \( \delta_0 \).

An analogous system was introduced and studied in the irrotational case with several solids by Grotta-Ragazzo, Koiller and Oliva:

\[
m_j \ddot{q}_j = \gamma_j \left( \dot{q}_j - \sum_{i \neq j} K [\gamma_i \delta_{q_i}] \right) \perp.
\]

(See also Kanso-Marsden-Vankerschaver.) The result above gives a rigorous derivation of their system in the “single-body + vorticity” case.
A light small solid in $\Omega$, $\omega_0 = 0$

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A light small solid in $\Omega$, $\omega_0 = 0$

- We let $\psi(h, \cdot)$ be the solution of the following Dirichlet problem (no $S_0$!):

$$\Delta \psi = 0 \text{ in } \Omega, \quad \psi(h, \cdot) = G(\cdot - h) \text{ on } \partial \Omega,$$

where $G(r) := -\frac{1}{2\pi} \ln |r|$.

- The Kirchhoff-Routh stream function $\psi_\Omega$/velocity $u_\Omega$ is defined as

$$\psi_\Omega(x) := \frac{1}{2} \psi(x, x) \text{ and } u_\Omega := \nabla \perp \psi_\Omega.$$

- The limit system is as follows

$$h' = \gamma u_\Omega(h) \quad \text{for } t \in [0, \infty), \quad h(0) = 0.$$

- This system describes a single vortex point in a bounded domain. It has global in time solutions which can be obtained as limits of regular solutions where the vorticity concentrates to a point (Turkington).
Theorem (G.-Munnier-Sueur). Let $h$ the maximal solution on $[0, +\infty)$ of

$$h' = \gamma u_{\Omega}(h) \quad \text{for } t \in [0, \infty), \quad h(0) = 0.$$ 

Let $h^\varepsilon$ the maximal solution on $[0, T^\varepsilon)$ of the fluid-solid system with solid of size $\varepsilon$, $\omega_0 = 0$ and $h'_0$, $\vartheta'_0$ and $\gamma \neq 0$ fixed.

Then one has, as $\varepsilon \to 0$,

- $\lim T^\varepsilon = +\infty,$
- $h^\varepsilon \rightharpoonup h$ in $W^{1,\infty}([0, T]; \mathbb{R}^2)$ weak-$\star$ for all $T > 0$,
- $\varepsilon \vartheta^\varepsilon \rightharpoonup 0$ in $W^{1,\infty}([0, T]; \mathbb{R})$ weak-$\star$ for all $T > 0$. 


## Summary & other cases

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### Situation 1:

- $\Omega = \mathbb{R}^2$
- $\omega_0 \in L^\infty_c(\mathbb{R}^2 \setminus \{0\})$

#### Massive particle:

- $mh'' = \gamma(h' - \tilde{u})^\perp$
- $\tilde{u} = K[\omega(t, \cdot)](h(t))$

#### Light particle:

- $h' = \tilde{u}$
- $\tilde{u} = K[\omega(t, \cdot)](h(t))$

### Situation 2:

- $\Omega$ bounded domain
- $\omega_0 = 0$

#### Massive particle:

- $mh'' = \gamma(h' - \tilde{u})^\perp$
- $\tilde{u} = \gamma u_\Omega(h(t))$

#### Light particle:

- $h' = \tilde{u}$
- $\tilde{u} = \gamma u_\Omega(h(t))$

\[
\partial_t \omega + \text{div} (K[\omega + \gamma \delta_{h(t)}]\omega) = 0
\]
The light particle in $\mathbb{R}^2$: the wave/vortex system

- The limit system here:

$$
\frac{\partial \omega}{\partial t} + \text{div} \left( K[\omega + \gamma \delta h(t)]\omega \right) = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^2,
$$

$$
h'(t) = K[\omega(t, \cdot)](h(t)),
$$

is known as Marchioro and Pulvirenti’s wave/vortex system.

- It can be obtained as limits of regular solutions of the Euler equation (as some vorticity concentrates in $h_0$)

- If $h_0 \notin \text{Supp} \, \omega_0$, then one has global existence and uniqueness, and $h(t) \notin \text{Supp} \, \omega(t)$ for all $t$.

- References: Marchioro-Pulvirenti, Lacave-Miot, Bjorland
Difficulties

\[ m^\varepsilon \ h''(t) = \int_{\partial S^\varepsilon(t)} p \ n \ ds, \]
\[ \mathcal{J}^\varepsilon \ \vartheta''(t) = \int_{\partial S^\varepsilon(t)} p (x - h(t))^\perp \cdot n \ ds, \]

- We have to study the pressure in details.

- The problem is singular in space since \( S^\varepsilon \) shrinks to a point and the circulation remains constant.

- The problem is singular in time when \( m^\varepsilon = \varepsilon^2 m^1 \) and \( \mathcal{J}^\varepsilon = \varepsilon^4 \mathcal{J}^1 \).

- The energy is not bounded (even infinite in the case \( \Omega = \mathbb{R}^2 \)) and moreover does not give a strong control when \( m^\varepsilon = \varepsilon^2 m^1 \) and \( \mathcal{J}^\varepsilon = \varepsilon^4 \mathcal{J}^1 \ldots \)
Some ideas of the proof (light particles)

- We consider the case $m^\varepsilon = \varepsilon^2 m^1$ and $J^\varepsilon = \varepsilon^4 J^1$.

- One of the main difficulties consists in obtaining uniform estimates. Even if we assume the total energy to be finite, this merely gives

$$\| (h^\varepsilon)' \|_{L^\infty} = O(1/\varepsilon) \quad \text{and} \quad \| (\varrho^\varepsilon)' \|_{L^\infty} = O(1/\varepsilon^2) \ldots$$

General strategy:

- Find a modulated energy which gives a better a priori estimate on $(h^\varepsilon)'$. (Cf. Brenier)

- For that purpose, find a normal form for the equation of the solid, with “modulated unknowns”.

- This equation will look like an equation of geodesics but with a right hand side.

- This additional terms will not be all conservative, but will give a “reasonable” contribution to the energy.
1. The added mass effect

Suppose that we are in the case $\Omega = \mathbb{R}^2$. We consider equations in the body frame. Consider:

\[
\begin{cases}
    v = Q^T(t)u(t, Q(t)x + h(t)), \\
    q = p(t, Q(t)x + h(t)), \\
    \ell = Q^T(t)h'.
\end{cases}
\]

with $Q$ the rotation of angle $\vartheta(t)$. The equations of the fluid/body system become

\[
\partial_t v + [(v - \ell - \vartheta' x^\perp) \cdot \nabla] v + \vartheta' v^\perp + \nabla q = 0 \text{ for } x \in F_0,
\]

\[
div v = 0 \text{ for } x \in F_0,
\]

\[
m\ell'(t) = \int_{\partial S_0} qn \, ds - m\vartheta' \ell^\perp
\]

\[
\mathcal{J} \vartheta''(t) = \int_{\partial S_0} x^\perp \cdot qn \, ds.
\]
One introduces Kirchoff’s potentials $\Phi_1, \Phi_2, \Phi_3$:

$$
\Delta \Phi_i = 0 \text{ in } F_0, \quad \nabla \Phi_i \xrightarrow{\infty} 0,
$$

$$
\partial_n \Phi_i = \begin{cases} 
  n_i & (i = 1, 2), \\
  \mathbf{x}^\perp \cdot n & (i = 3), 
\end{cases} \text{ on } \partial S_0.
$$

The solid equations become

$$
\begin{bmatrix}
  m \mathrm{Id}_2 & 0 \\
  0 & J
\end{bmatrix}
\begin{bmatrix}
  \ell \\
  \vartheta'
\end{bmatrix}' = 
\begin{bmatrix}
  \int_{F_0} \nabla q \cdot \nabla \Phi_i \, dx
\end{bmatrix}_{i=1,2,3} - \begin{bmatrix}
  m \vartheta' \ell^\perp \\
  0
\end{bmatrix}
$$

Let $P$ the Leray projector. The pressure decomposes as follows:

$$
\nabla q = (I - P)(\partial_t \mathbf{v}) + (I - P)(-(\mathbf{v} - \ell - \vartheta' \mathbf{x}^\perp) \cdot \nabla \mathbf{v} - \vartheta' \mathbf{v}^\perp)
$$

$$
= : \nabla \varphi \quad = : \nabla \mu
$$

Using that $\partial_t \mathbf{v}$ is already divergence-free, one easily deduces that

$$
\nabla \varphi = - \left( \frac{\ell}{\vartheta'} \right)' \cdot (\nabla \Phi_i)_{i=1,2,3}.
$$
We end up with this new equation for the solid:

\[
\mathcal{M} \begin{bmatrix} \ell' \\ \varphi' \end{bmatrix}' = \begin{bmatrix} mr\ell_0 \end{bmatrix} + \left[ \int_{\mathcal{F}_0} \nabla \mu \cdot \nabla \Phi_i \, dx \right]_{i=1,2,3},
\]

where

\[
\mathcal{M} := \begin{bmatrix} m \text{Id}_2 & 0 \\ 0 & J \end{bmatrix} + \left[ \int_{\mathcal{F}(t)} \nabla \Phi_i \cdot \nabla \Phi_j \, dx \right]_{i,j=1,2,3} =: \mathcal{M}_a
\]

The matrix \( \mathcal{M}_a \) is a matrix of added inertia, expressing how the fluid opposes the movement of the solid. It is positive, and even positive definite when \( S_0 \) is not a disk, as a Gram matrix of independent functions.
2. Reformulation of the solid equation (bounded $\Omega$, $\omega = 0$)

For $\epsilon = 1$, using arguments of Lagrangian mechanics and shape derivative, we obtain that

$$ q := (\vartheta, h_1, h_2) $$

satisfies the following ODE

$$ M(q)q'' + \langle \Gamma(q), q', q' \rangle = F(q, q'), $$

where

- $M(q) = M_g + M_a(q)$ (genuine inertia + added inertia),
- $\Gamma(q)$ contains the Christoffel symbols associated to the metric $M(q)$,
- $F(q, q')$ is a Lorentz-type force, that is, of the form:

$$ F(q, q') := \gamma^2 E(q) + \gamma q' \times B(q), $$

with strong conditions on $E$. 
A important particular case: without outer boundary

- When $\Omega = \mathbb{R}^2$, the ODE becomes:

$$M_{\partial \Omega}(\vartheta) \, q'' + \langle \Gamma_{\partial \Omega}(\vartheta), q', q' \rangle = F_{\partial \Omega}(\vartheta, q'),$$

with

$$F_{\partial \Omega}(\vartheta, q') = \gamma \begin{pmatrix} R(\vartheta) \zeta \cdot h' \\ (h')^{\perp} - \vartheta' R(\vartheta) \zeta \end{pmatrix}$$

$$= \gamma q' \times B_{\partial \Omega}(\vartheta),$$

where

$$B_{\partial \Omega}(\vartheta) = \begin{pmatrix} -1 \\ R(\vartheta) \zeta^{\perp} \end{pmatrix},$$

and $\zeta$ is a geometrical constant depending on $S_0$ only.

- One has: $M_{\partial \Omega}(\vartheta) = Q(\vartheta) M_{\partial \Omega}(0) Q(\vartheta)^T$, with $Q(\vartheta)$ the rotation of angle $\vartheta$ on the $(h_1, h_2)$ variables.
3. A normal form ($\Omega$ bounded, $\omega_0 = 0$)

- Denote $p := q' = (\vartheta', h'_1, h'_2)$

- We consider the modulated velocity

$$\tilde{p} = \left(\varepsilon\vartheta', h' - \gamma u_\Omega(h) - \varepsilon\gamma u_c(\vartheta, h)\right),$$

where $u_c$ is explicit and depends merely on $\Omega$, $S_0$ and $q$.

- The ODE can be put in the following form.

$$\varepsilon^2 (M_1^g + M_{a,\Omega}(\vartheta)) \tilde{p}' + \varepsilon \langle \Gamma_{a,\Omega}(\vartheta), \tilde{p}, \tilde{p} \rangle = F_{\partial\Omega}(\vartheta, \tilde{p}) + \varepsilon\gamma^2 G(q) + O(\varepsilon^2),$$

where $G(q)$ is weakly gyroscopic in the sense that it satisfies:

$$\left| \int_0^t \tilde{p} \cdot G(q) \right| \leq \varepsilon K(1 + t + \int_0^t |\tilde{p}|_{\mathbb{R}^3}^2),$$

so that one wins a factor $\varepsilon$ in the energy estimate.
4. To get to the normal form

The recipe consists in developing in powers of $\varepsilon$:

- the Kirchoff potentials,
- the “circulation potential” $\psi$ satisfying

$$\Delta \psi = 0 \text{ in } \mathcal{F}(t),$$
$$\psi = 0 \text{ on } \partial \Omega, \quad \psi = C(q) \text{ on } \partial S(t),$$
$$\int_{\partial S(t)} \partial_n \psi \, ds = 1.$$

- inject the developments in the velocity decomposition and in the coefficients of the equation,
- make computations (using the so-called Lamb’s lemma) and use some cancellations...
Expansions of the potentials

Fluid state in $\Omega \setminus S^\varepsilon(t) = \text{Fluid state as if } \partial \Omega$

$+ \text{ Correction as if } S^\varepsilon(t)$

$+ \text{ Correction(Correction) as if } \partial \Omega$

$+ \ldots$

Method: Potential / Fredholm theory ...
5. Passage to the limit

- With the normal form, one can obtain a modulated energy estimate implying that, as long as \( S^\varepsilon(t) \) is at a minimal distance from the boundary,

\[
\| (h^\varepsilon)' \|_{L^\infty} = O(1) \text{ and } \| (\varphi^\varepsilon)' \|_{L^\infty} = O(1/\varepsilon) \ldots
\]

- Next it remains to pass to the limit in

\[
\varepsilon^2 \left( M^1_g + M^1_{a,\varrho}(\varphi) \right) \tilde{p}' + \varepsilon \langle \Gamma^1_{a,\varrho}(\varphi), \tilde{p}, \tilde{p} \rangle
\]

\[
= F_{\varrho}(\varphi, \tilde{p}) + \varepsilon \gamma^2 G(q) + O(\varepsilon^2),
\]

with

\[
F_{\varrho}(\varphi, \tilde{p}) = \gamma \left( \begin{array}{c}
R(\varphi) \zeta \cdot \left[ h' - \gamma u_\Omega(h) - \varepsilon \gamma u_c(\varphi, h) \right] \\
\left[ h' - \gamma u_\Omega(h) - \varepsilon \gamma u_c(\varphi, h) \right] \perp - \varepsilon \varphi' R(\varphi) \zeta
\end{array} \right)
\]
6. In the context $\Omega = \mathbb{R}^2$ and $\omega_0 \neq 0$

- Here we can change the frame in order to work in $\mathbb{R}^2 \setminus S_0^\varepsilon$.
- On the other hand, we know the vorticity with little precision.
- Using $\text{dist} [S^\varepsilon(t), \text{Supp} (\omega^\varepsilon(t))] > 0$, we obtain the normal form:

$$\varepsilon^2 [\mathcal{M}_g^1 + \mathcal{M}_a^1] (\tilde{p})' + \varepsilon \langle \Gamma_p + \Gamma_a, \tilde{p}, \tilde{p} \rangle$$

$$= \gamma \tilde{p} \times B_{\partial \Omega}(0) + \varepsilon \gamma G(\varepsilon, t) + O(\varepsilon^2),$$

where

- $\tilde{p} := \left( \varepsilon \vartheta', R(\vartheta)^T (h'(t) - K[\omega(t, \cdot)](h(t)) - \varepsilon \nabla K_{\mathbb{R}^2}[\omega^\varepsilon(t, \cdot)](0) \cdot \zeta, ) \right)$,
- $\Gamma_g$ and $\Gamma_a$ are gyroscopic (skew-symmetric) terms,
- $\gamma \tilde{p} \times B_{\partial \Omega}(0)$ is gyroscopic as well,
- $G(\varepsilon, t)$ is weakly gyroscopic (and depends on the vorticity).
Approximation of the velocity on the solid’s boundary

- To compute the pressure force on the solid’s boundary one approaches the fluid velocity by using
  - Kirchhoff’s potentials and the circulation potential (part generated by the solid),
  - the 1st order approximation of the fluid contribution to this velocity:
    \[ K_{\mathbb{R}^2}[^\omega]_{x=0} + (\nabla K_{\mathbb{R}^2}[^\omega]_{x=0})x. \]

- One approaches the latter by using new harmonic potentials in the same spirit and Kirchhoff.

- We are led to compute many integrals on \( \partial S_0^\varepsilon \). Using Blasius’ lemma, these are transformed into complex integrals that one can compute using basic complex anaylsis (Laurent series and so on)

- We rely on many cancellations to reach the above normal form.
Perspectives

- Asymptotic expansion (multi-scale in time) of the solution,
- Unified treatment with a boundary and vorticity at the same time,
- Many-bodies systems
- The case of a 3D, of a viscous fluid (cf. Dashti-Robinson, Silvestre-Takahashi), . . .
- Control of these systems!