On water waves with angled crests

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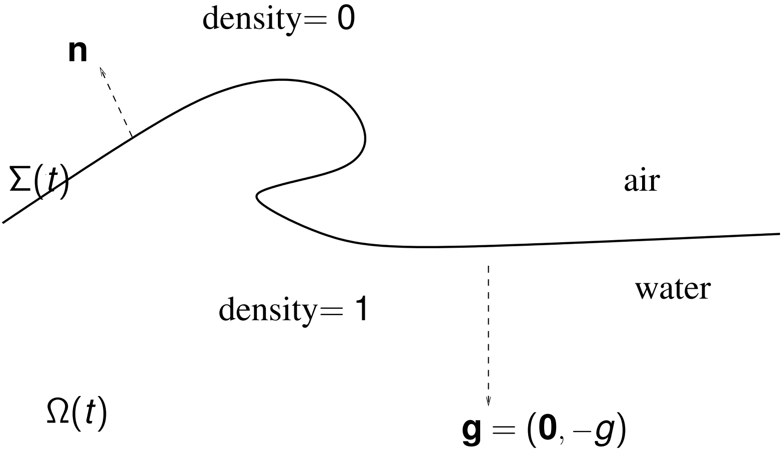
We consider the motion of the interface separating air from water.

We assume:

- air density= 0
- water density =1
- water region is below the air region. At time t, water region is $\Omega(t)$, the interface is $\Sigma(t)$.

We assume that the water is

- inviscid, incompressible, irrotational.
- The surface tension is zero.
- The water is subject to the influence of gravity $\mathbf{g} = (\mathbf{0}, -g)$.
- $g > 0$


density= 0
n
Σ(t)
air
water
density= 1
Ω(t)
g = (0, −g)

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = (\mathbf{0}, -g) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \Sigma(t) \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

When surface tension is zero, the motion can be subject to the Taylor instability.

- Taylor condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$$

  on the interface $\Sigma(t)$. $\mathbf{n}$ is the unit normal to $\Sigma(t)$ pointing out of the water region $\Omega(t)$.

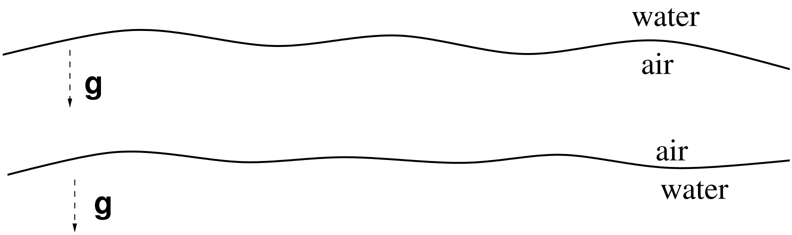
G. I. Taylor: linearize about the flat interface,

- air above water

$$-\frac{\partial P}{\partial \mathbf{n}} > 0 : \quad \text{stable.}$$

- water above air

$$-\frac{\partial P}{\partial \mathbf{n}} < 0 : \quad \text{unstable.}$$



Known results:

local wellposedness for arbitrary data

S. Wu: (1997) 2-D, (1999) 3-D.

- The Taylor condition always holds for the water wave motion, i.e.
$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$$
as long as the interface is non-selfintersecting.
- Local existence and uniqueness in Sobolev spaces: There exist a unique solution for a finite time period, for any initially non-selfintersecting interface, and any given initial velocity (incompressible & irrotational).

The proof of the fact $-\frac{\partial P}{\partial \mathbf{n}} > 0$:

Applying div to both sides of the Euler equation gives

$$\Delta P = -|\nabla \mathbf{v}|^2 \leq 0$$

Maximum principle implies $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$.

For C^1 interface, $-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$, by Green's identity.

Earlier results:

1. T. Beale, T. Hou & Lowengrub (1992).
   Linear wellposedness assuming the presumed solution satisfies the Taylor sign condition:

   $$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0.$$

2. Nalimov(1974): infinite depth, 2-D, small data, local wellposedness
3. Yosihara(1982): finite depth, 2-D, small data. local wellposedness

Recent works:

Local wellposedness with additional effects: nonzero surface tension, finite depth, nonzero vorticity, assuming the Taylor sign condition holds.

- Iguchi(2001), Ogawa & Tani (2002), Ambrose & Masmoudi(2005), D. Lannes (2005), Christodoulou & Lindblad (2000), Lindblad (2005), Coutand & Shkoller (2007), P. Zhang & Z. Zhang (2007), Shatah & Zeng (2008)

Global behavior for small and smooth data

More recently, there are results on almost global or global well-posedness for small and smooth initial data, for the infinite depth zero surface tension water wave problem, in two and three dimensions.

Global behavior for small and smooth data

- S. Wu (2009): almost global well-posedness for 2-D,
- S. Wu (2011): global well-posedness for 3-D
- Germain, Masmoudi & Shatah (2012): global well-posedness for 3-D
- Alazard & Delort (2013): 2-D water waves, global existence and modified scattering
- Ionescu & Pusateri (2013): similar result
- Hunter, Ifrim & Tataru (2014): 2-D water wave, almost global existence, –modified energy method.
- Ifrim & Tataru (2014): 2-D water wave, global existence.

Singularities:

- Alazard, Burq, Zuily (2012): Local wellposedness in low regularity Sobolev space–the interface is $C^{3/2+\epsilon}$.
- Alazard, Burq, Zuily (2014): Local wellposedness in low regularity Sobolev space–the interface is $C^{3/2-\epsilon}$.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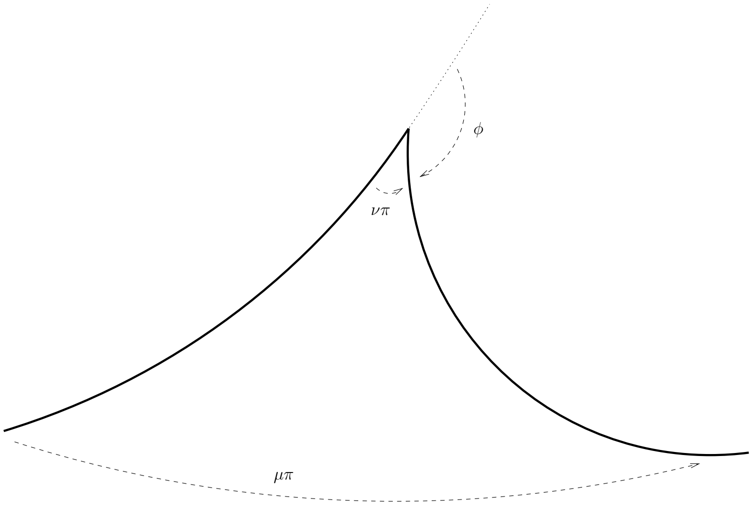
Singularities:

What are some typical singular behaviors? How does it form? What are some basic structures of the singularities?

S. Wu (2012): construction of self-similar solution for 2-D water waves in the regime where convection is in dominance:

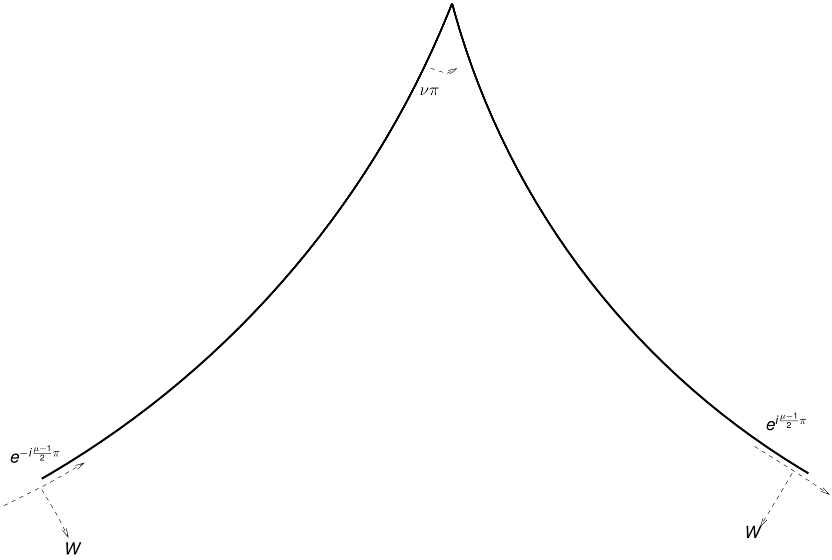
- $z \sim t$, or in hyperbolic scaling: $s = 1$.

— neglecting gravity and surface tension.
—- satisfies the Taylor sign condition $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$.



ϕ
$\nu\pi$
$\mu\pi$

- $\nu < \frac{1}{2}, \qquad \mu > \frac{1}{2}$



$\nu\pi$
$e^{-i\frac{\mu-1}{2}\pi}$
$e^{i\frac{\mu-1}{2}\pi}$
W
W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Question:

Q: How relevant are the self-similar solutions? Are they stable?

In all earlier work, either it is assumed there is no bottom, or there is a bottom Υ, of a positive distance away from the interface $\Sigma(t)$

$$dist(\Sigma(t), \Upsilon) \geq h_0 > 0$$

Question

We consider the following problem:
Q: the interaction of the free surface with a fixed rigid boundary?

In the presence of a fixed rigid boundary Υ, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = (\mathbf{0}, -g) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \Sigma(t) \\ \mathbf{v} \cdot \mathbf{n} = 0, & \text{on } \Upsilon \end{cases} \tag{2}$$

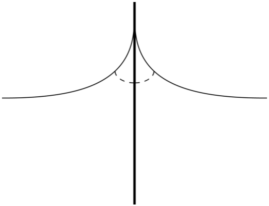
$\mathbf{v}$ is the fluid velocity, P is the fluid pressure. $\mathbf{n}$ is a normal vector to Υ.

$\partial\Omega(t) = \Sigma(t) \cup \Upsilon$.

While this problem maybe hard....

We look at a specific problem:

If the fixed rigid boundary Υ is a vertical wall $\{x = 0\}$, and the fluid domain $\Omega(t)$ is the domain to the right of $\{x = 0\}$. Then the velocity field $\mathbf{v} = (v_1, v_2)$ satisfies $v_1(0, y; t) = 0$.
By Schwarz reflection:
$\mathbf{v}(-x, y; t) = (-v_1(x, y; , t), v_2(x, y; t))$; $P(-x, y; t) = P(x, y, ; t)$
we can reduce the problem to the one on the symmetric domain without a fixed wall.



This is new only when the angle of the wave with the wall is other than 90°.

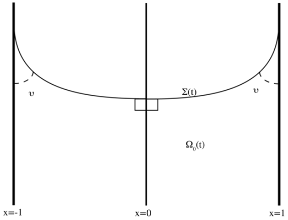
Q: Is it possible for the angle between the interface and the wall to be other than $\frac{\pi}{2}$?

We study the following problem:

Assume the rigid boundary Υ is consisting of two vertical walls:

$$\Upsilon = \{x = 0\} \cup \{x = 1\}$$

Assume the free interface $\Sigma(t)$ makes a 90° angle with the wall $\{x = 0\}$, but we allow a possible non-trivial angle at $\{x = 1\}$. We make a Schwarz reflection about $\{x = 0\}$:



- Q: Can the angle ν be other than 90°?
- Q: local existence in this framework?

Main Result:

- Yes, there is local existence in this framework. In fact, besides a non-trivial angle ν, the interface can have angled crests.
- The angle ν must be no more than $\frac{\pi}{2}$, and the interior angles of the angled crests cannot be more than π.

The water wave problem admits such solutions.

- A prior estimate: joint work with Rafe Kinsey.

Difficulty

In our framework, we have

- $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$, but
- $-\frac{\partial P}{\partial \mathbf{n}} = -\mathbf{n} \cdot \nabla P = 0$ at the wall when there is a non-trivial angle, and at the points on the interface where there are angled crests.
- $\mathbf{n}$ outward unit normal.

Let the free surface be

$$\Sigma(t) : z = z(\alpha, t), \quad \alpha \text{ Lagrangian coordinate.}$$

- $z = x(\alpha, t) + iy(\alpha, t)$, in complex form;
- $z_t = z_t(\alpha, t)$ is the velocity;
- z_{tt} is the acceleration;
- $-i$ is the gravity;
- $P = 0$ on $\Sigma(t)$ implies: $\nabla P \perp \Sigma(t)$
- $\nabla P = i\mathfrak{a} z_\alpha$, where $\mathfrak{a} = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}}$

- div $\mathbf{v}$ = curl $\mathbf{v} = 0$ implies $\bar{\mathbf{v}}$ is holomorphic in $\Omega(t)$.
- $\mathbf{v}(-1 + iy; t) = \mathbf{v}(1 + iy; t)$ is purely imaginary.
  $\mathbf{v} \cdot \mathbf{n} = 0$ on $x = \pm 1$.
- $\bar{z}_t(\alpha, t) = \bar{\mathbf{v}}(z(\alpha, t), t)$, the boundary value of periodic holomorphic function $\mathbf{v}$.
- $\bar{z}_t = \mathfrak{H}\bar{z}_t + c$

  Equation of the free surface:

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t + c \end{cases} \tag{3}$$

Quasilinear equation:

$$\bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha$$

Let

- $\mathfrak{u} = \bar{z}_t$
- $i\bar{z}_{t\alpha} = \nabla_{\mathbf{n}}\mathfrak{u}$

Free surface equation:

$$(\partial_t^2 + \mathfrak{a}\nabla_{\mathbf{n}})\mathfrak{u} = l.o.t$$

is degenerate hyperbolic, if $\mathfrak{a}$ can be zero.

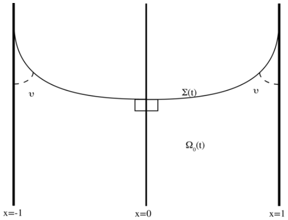
$\mathfrak{a} = 0$ at the singularities:

Free surface equation: $z_{tt} + i = i\mathfrak{a} z_\alpha := \nabla P$, $\mathfrak{a} \in \mathbb{R}$ implies:

$$-\frac{x_\alpha}{y_\alpha} = \frac{y_{tt}+1}{x_{tt}}. \tag{4}$$

$$\tan\nu = -\frac{x_\alpha}{y_\alpha} = \frac{y_{tt}+1}{x_{tt}}. \tag{5}$$

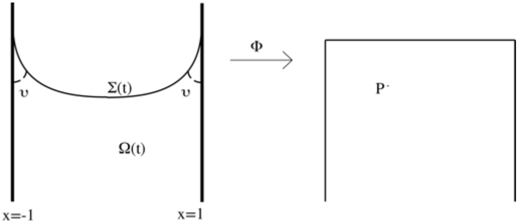
- $x_t(\pm 1; t) = 0$ implies $x_{tt}(\pm 1; t) = 0$.
- If $\nu \neq \frac{\pi}{2}$, then $y_{tt} + 1 = 0$ at $x = \pm 1$.
- Therefore $\nabla P = 0$ at the corner $x = \pm 1$.
- $\mathfrak{a} = 0$ at the corner $x = \pm 1$.



υ
Σ(t)
υ
Ω0(t)
x=-1
x=0
x=1

$\nu \leq \frac{\pi}{2}$.

Let $\Phi : \Omega(t) \to P^{-} := [-1, 1] \times (-\infty, 0]$ be the Riemann mapping, taking corners of $\Omega(t)$ to $(\pm 1, 0)$, $-\infty$ to $-\infty$:



Let $h(\alpha; t) := \Phi(z(\alpha; t); t)$.
$h(\cdot; t) : [-1, 1] \to [-1, 1]$ is strictly increasing.

Let

- h^{-1} be: $\quad h(h^{-1}(\alpha'; t); t) = \alpha'$
- $Z(\alpha'; t) = z(h^{-1}(\alpha'; t), t) := z \circ h^{-1}$; $Z_{,\alpha'} = \partial_{\alpha'} Z(\alpha', t)$
- $Z_t(\alpha'; t) := z_t \circ h^{-1}$; $Z_{tt}(\alpha'; t) := z_{tt} \circ h^{-1}$
- $A \circ h = \mathfrak{a} h_\alpha$
- $\mathbb{H} f(\alpha') = \frac{1}{2i} \int_{-1}^{1} \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta')\, d\beta'$
  be the Hilbert transform

Free surface equation in Riemann mapping variable α':

$$\begin{cases} Z_{tt} + i = iAZ_{,\alpha'} \\ \overline{Z}_t = \mathbb{H}\overline{Z}_t + c \end{cases} \tag{6}$$

$h(\alpha; t) := \Phi(z(\alpha; t); t)$ implies that

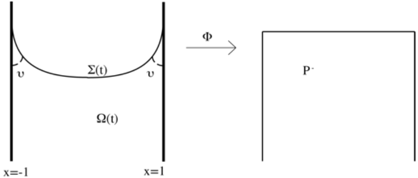
$$Z(\alpha', t) = \Phi^{-1}(\alpha'; t); \quad Z_{,\alpha} = \partial_{z'} \Phi^{-1}(\alpha'; t)$$

$$\overline{Z}_{,\alpha'}(Z_{tt} + i) = iA|Z_{,\alpha'}|^2 := iA_1$$

We can show that $A_1 \geq 1$ (this was shown in [S. Wu 1997] for the whole line case.)

$$\frac{1}{Z_{,\alpha'}} = i \frac{\overline{Z}_{tt} - i}{A_1}$$

- Fact 1: $\nu \leq \frac{\pi}{2}$
- at the corner:
  $\Phi^{-1}(z') \approx (z')^r$, where $\nu = \frac{\pi}{2} r$.
- $Z_{,\alpha'} = \partial_{z'} \Phi^{-1}(z') \approx (\alpha')^{r-1}$.
- if $\nu > \frac{\pi}{2}$, i.e. if $r > 1$, then $Z_{,\alpha} = 0$ at the corner, so $Z_{tt} = \infty$, so $y_{tt} = \infty$ at the corner, since $x_{tt} = 0$,



Recall

$$\tan\nu = -\frac{x_\alpha}{y_\alpha} = \frac{y_{tt}+1}{x_{tt}}. \tag{7}$$

this implies

$$\tan\nu = \infty$$

therefore

$$\nu = \frac{\pi}{2}$$

So ν cannot be greater than $\frac{\pi}{2}$.
Similarly,

- Interior angle of the angled crests cannot be more than π.

- Fact 2:
$$-\frac{\partial P}{\partial \mathbf{n}} = \mathfrak{a}|z_\alpha| = \frac{A_1}{|Z_{,\alpha'}|} \circ h \geq 0$$
Recall: $Z_{,\alpha'} = \partial_{z'} \Phi^{-1}(\alpha'; t)$
- (S.Wu, 1997) If the interface $\Sigma(t) \in C^1$, then $0 < c_0 \leq |\partial_{z'} \Phi^{-1}(\alpha'; t)| \leq C_0 < \infty$, then
$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_1 > 0$$
- If the angle $\nu < \frac{\pi}{2}$, or if the free surface has angled crests with interior angle $< \pi$, then $r < 1$, then $\frac{1}{Z_{,\alpha'}} \to 0$ at the corner or at the crests, this implies
$$-\frac{\partial P}{\partial \mathbf{n}} = 0$$
at the corner if $\nu < \frac{\pi}{2}$ and at the crests where the interior angle is $< \pi$.

Energy

We introduce a special derivative

$$D_\alpha f = \frac{1}{z_\alpha}\partial_\alpha f, \quad D_{\alpha'} g = \frac{1}{Z_{,\alpha'}}\partial_{\alpha'} g$$

If f is the boundary value of a periodic holomorphic function F on $\Omega(t)$, $f(\alpha, t) = F(z(\alpha, t), t)$, then

$$D_\alpha f = \partial_z F(z(\alpha, t); t) = -i\partial_y F(z(\alpha, t); t)$$

D_α preserves the holomorphicity and periodicity of periodic holomorphic functions.

Recall the quasilinear equation of the free surface:

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\bar{z}_t = -i\mathfrak{a}_t\bar{z}_\alpha, \tag{8}$$

higher order equation

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\theta = G_\theta. \tag{9}$$

where $\theta = D_\alpha^k \bar{z}_t$, $G_\theta = D_\alpha^k(-i\mathfrak{a}_t\bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^k]\bar{z}_t$.
Energy:

- $e = \int |\theta_t|^2 + \Re \int (i\mathfrak{a}\partial_\alpha\theta)\bar{\theta}$

We construct the energy: let $\alpha_0 \in [-1, 1]$ be fixed

$$E = E_{a,D_\alpha^2 \bar{z}_t} + E_{b,D_\alpha \bar{z}_t} + |\bar{z_{tt}}(\alpha_0; t) - i|$$

where

$$E_{a,\theta} = \int_{-1}^{1} \frac{h_\alpha}{A_1 \circ h} |\theta_t|^2 \, d\alpha + \Re \int_{-1}^{1} \frac{h_\alpha}{A_1 \circ h} (i\mathfrak{a}\partial_\alpha \theta)\bar{\theta} \, d\alpha + l.o.t.$$

$$E_{b,\theta} = \int_{-1}^{1} \frac{1}{\mathfrak{a}} |\theta_t|^2 \, d\alpha + \Re \int_{-1}^{1} (i\partial_\alpha \theta)\bar{\theta} \, d\alpha + l.o.t.$$

$$\mathfrak{a} \approx h_\alpha \approx -\frac{\partial P}{\partial \mathbf{n}}$$

E_a and E_b have roughly inverse singular weights h_α and $\frac{1}{\mathfrak{a}}$.

$$A_1 \circ h = \frac{\mathfrak{a} |z_\alpha|^2}{h_\alpha}$$

Main Result

Theorem (A priori estimate, joint work with Rafe Kinsey)

There exists a polynomial $p = p(x)$ with universal coefficients, such that, for any solution of water wave equations with $E(t) < \infty$ for all $t \in [0, T]$,

$$\frac{d}{dt}E(t) \leq p(E(t)) \tag{10}$$

for all $t \in [0, T]$.

Theorem (local existence)

For any initial data satisfying $E(0) < \infty$, there exists $T > 0$, depending only on $E(0)$, such that the water wave equation is solvable for time $t \in [0, T]$, with $E(t) < \infty$ for $t \in [0, T]$.

A characterization of the energy

$$E(t) \le C\Big(\|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D^2_{\alpha'}\bar{Z}_t\|_{L^2}, \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2},$$
$$\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}, \|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\bar{Z}_t\|_{\dot{H}^{1/2}}, \|D_{\alpha'}\bar{Z}_t\|_{\dot{H}^{1/2}}, \|\frac{1}{Z_{,\alpha'}}\|_{L^\infty}\Big), \quad (11)$$

$$\|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D^2_{\alpha'}\bar{Z}_t\|_{L^2}, \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2},$$
$$\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\|_{L^2}, \|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\bar{Z}_t\|_{\dot{H}^{1/2}}, \|D_{\alpha'}\bar{Z}_t\|_{\dot{H}^{1/2}}, \|\frac{1}{Z_{,\alpha'}}\|_{L^\infty} \le C(E) \quad (12)$$

C: universal polynomial.

Remarks:

- $$\frac{1}{Z_{,\alpha'}} = i\frac{\overline{Z}_{tt} - i}{A_1} \approx \overline{Z}_{tt} - i$$
- The self-similar solution (S. Wu 2012) has finite energy.
- In general, surfaces that have angled crests of interior angle $< \frac{\pi}{2}$, and the angle ν of the wave with the vertical wall $\nu < \frac{\pi}{4}$ have finite energy.
- Stokes wave of maximum height do not have finite energy.

Thank you very much for your attention!