Using multiple frequencies to satisfy local constraints in PDE and applications to hybrid problems

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Introduction

- Hybrid imaging techniques have been developed in recent years in medical imaging. Two different acquisition modalities are coupled simultaneously to obtain high contrast and high resolution images.

- The reconstruction of the parameters usually involves two steps:
  1. Internal functionals are constructed inside the domain of interest
  2. The parameters have to be found from the knowledge of these internal data

- In step 2. one often needs to construct solutions to the Helmholtz equation (or Maxwell's equations) satisfying certain non-zero local constraints.

- Complex geometric solutions have been used, but cannot be constructed a priori and the coefficients need to be very smooth.

Outline of the talk

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b. Multiple frequency approach for local constraints for the Helmholtz equation

c. Multiple frequency approach for local constraints for Maxwell's equations

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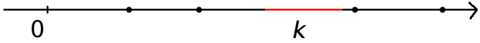
Microwave imaging by ultrasound deformation: Step 1

- $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$: C^2 bounded domain
- $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$: inverse of magnetic permeability
- $q \in L^\infty(\Omega)$: electric permittivity

$$\lambda \leq a, q \leq \Lambda \quad \text{in } \Omega$$

- $k \in \mathcal{A} = [K_{min}, K_{max}]$: admissible frequencies



The electric field $u_k^\varphi \in C^1(\overline{\Omega})$ satisfies

$$\begin{cases} -\text{div}(a\,\nabla u_k^\varphi) - k\,q\,u_k^\varphi = 0 & \text{in } \Omega, \\ u_k^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

By locally perturbing the medium with ultrasounds and measuring the difference of the boundary data we obtain the internal functionals (Ammari et al., SIAP 2011)

$$E_k^{\varphi\psi} = a\,\nabla u_k^\varphi \cdot \nabla u_k^\psi, \qquad e_k^{\varphi\psi} = q\,u_k^\varphi u_k^\psi \quad \text{in } \Omega'.$$

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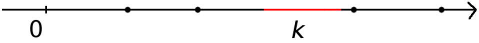

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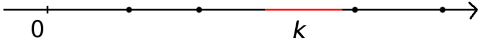

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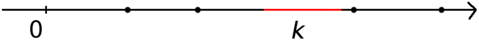

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$$E_k^{ij} = a\,\nabla u_k^i \cdot \nabla u_k^j, \qquad e_k^{ij} = q\,u_k^i u_k^j \qquad \xrightarrow{?} \qquad a, q$$

- Exact formula for a/q (Ammari et al., SIAP 2011)

$$|\nabla(e_k/\mathrm{tr}(e_k))|_2^2 \, \frac{a}{q} = 2\,\frac{\mathrm{tr}(e_k)\,\mathrm{tr}(E_k) - \mathrm{tr}(e_k E_k)}{\mathrm{tr}(e_k)^2}.$$

- Exact formula for q (GSA, IP 2013)

$$-\mathrm{div}\Big(\frac{a}{q}\,\mathrm{tr}(e)\,\nabla \log q\Big) = -\mathrm{div}\Big(\frac{a}{q}\,\nabla\,(\mathrm{tr}(e))\Big) + 2\sum_{k,i}\left(E_k^{ii} - k e_k^{ii}\right)$$

When are these formulae applicable?

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Complete Sets of Measurements

Given a set of measurements $K \times \{\varphi_1, \varphi_2, \varphi_3\}$, it turns out (Ammari et al., SIAP 2011 and GSA, IP 2013) that the reconstruction formulae are applicable if for every $x \in \Omega$ there exists $\bar{k}(x) \in K$ such that:

1. $\left|u^1_{\bar{k}}\right|(x) \geq C > 0$,
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This can be generalized to:

A set of measurements $K \times \{\varphi_i : i = 1, \ldots, d+1\}$ is *C-complete* if for every $x \in \Omega$ there exists $\bar{k}(x) \in K$ such that:

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The usual conditions are with a fixed frequency and possibly more boundary conditions, but this formulation better suits the following developments.

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Similar conditions arise in various contexts:

- Microwaves and ultrasounds:
  - stability: need 1. (Triki, IP 2010)
  - reconstruction formulae: need 1., 2. and 3. (Ammari et al., SIAP 2011)
- Quantitative thermo-acoustics:
  - stability: need 1. (Bal et al., IP 2011)
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- General elliptic equations (quantitative photo-acoustics, elastography):
  - need 1., 2., 3. and further conditions (Bal and Uhlmann, CPAM 2013)

How can we construct complete sets of measurements?

The construction of complete sets is non trivial since a and q are not constant.

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Classical approach: Complex Geometric Optics

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Complex geometric optics solutions (Sylvester and Uhlmann, AM 1987) are particular solutions to the Helmholtz equation in $\mathbb{R}^d$

$$u_{k_0}^t(x) = a^{-\frac{1}{2}} e^{tx_m} \left(\cos(tx_l) + i\sin(tx_l)\right)(1+\psi_t), \quad t \gg 0.$$

Theorem (Bal and Uhlmann, IP 2010)

If $t \gg 0$ then $u_{k_0}^t(x) \approx a^{-\frac{1}{2}} e^{tx_m} \left(\cos(tx_l) + i\sin(tx_l)\right)$ in $C^1(\overline{\Omega})$.

The traces on the boundary of these solutions give the required 1., 2. and 3..

Drawbacks:

- The result holds provided that a and q are smooth enough
- The construction of suitable illuminations depend on the parameters a and q
- Very oscillatory functions: numerically difficult to implement

Is there an alternative way to obtain these suitable illuminations?

Main idea: use different frequencies k.

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If $t \gg 0$ *then* $u_{k_0}^t(x) \approx a^{-\frac{1}{2}} e^{tx_m} \left(\cos(tx_l) + i\sin(tx_l)\right)$ *in* $C^1(\overline{\Omega})$.

The traces on the boundary of these solutions give the required 1., 2. and 3..

Drawbacks:

- The result holds provided that a and q are smooth enough
- The construction of suitable illuminations depend on the parameters a and q
- Very oscillatory functions: numerically difficult to implement

Is there an alternative way to obtain these suitable illuminations?

Main idea: use different frequencies k.

Classical approach: Complex Geometric Optics

1. $\left|u_{\bar{k}}^1\right|(x) \geq C$, 2. $\left|\det\left[\nabla u_{\bar{k}}^2 \quad \cdots \quad \nabla u_{\bar{k}}^{d+1}\right]\right|(x) \geq C$, 3. ...

Complex geometric optics solutions (Sylvester and Uhlmann, AM 1987) are particular solutions to the Helmholtz equation in $\mathbb{R}^d$

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Multi-Frequency Approach: basic idea I

As an example, let us consider the 1D case with $a = q = 1$.

1. $\left|u_k^1(x)\right| > 0$: the zero sets of u_k^φ move when k changes...

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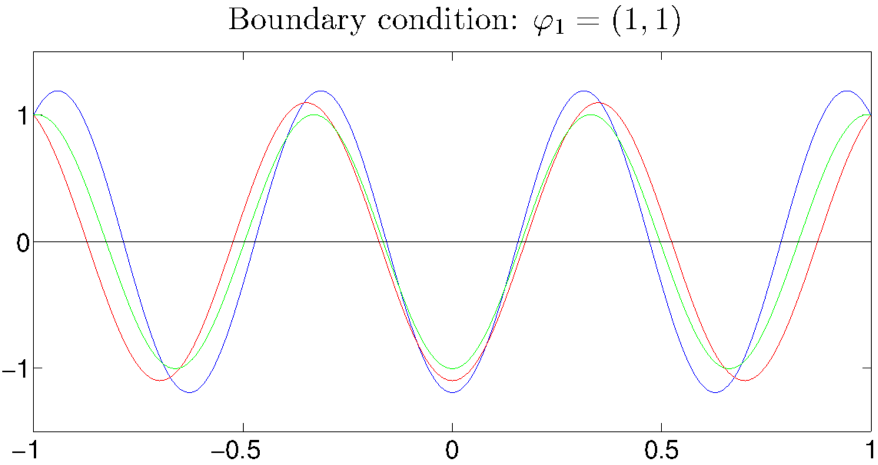

Multi-Frequency Approach: basic idea II

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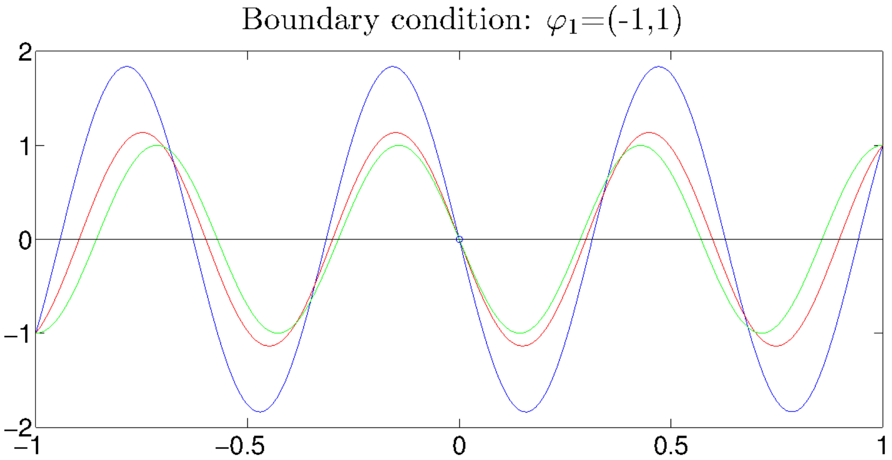
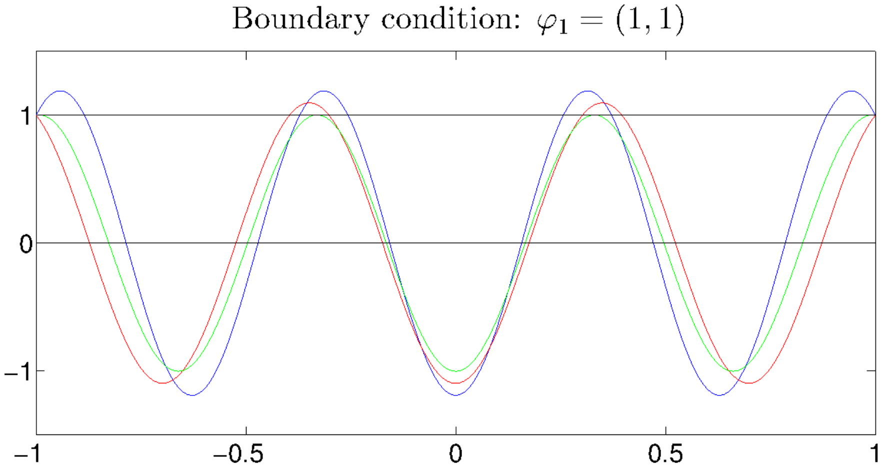


Multi-Frequency Approach: $k = 0$

1. $\left|u_k^1(x)\right| > 0$ everywhere for $k = 0$ $\implies$ the zeros "move"



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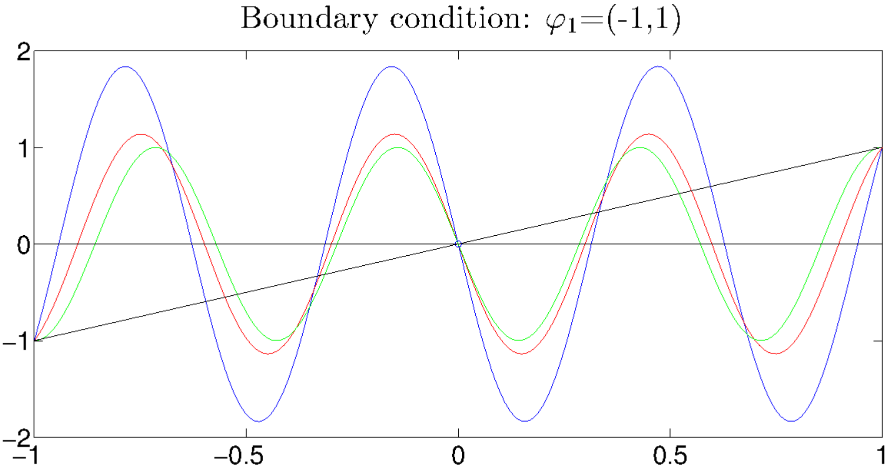

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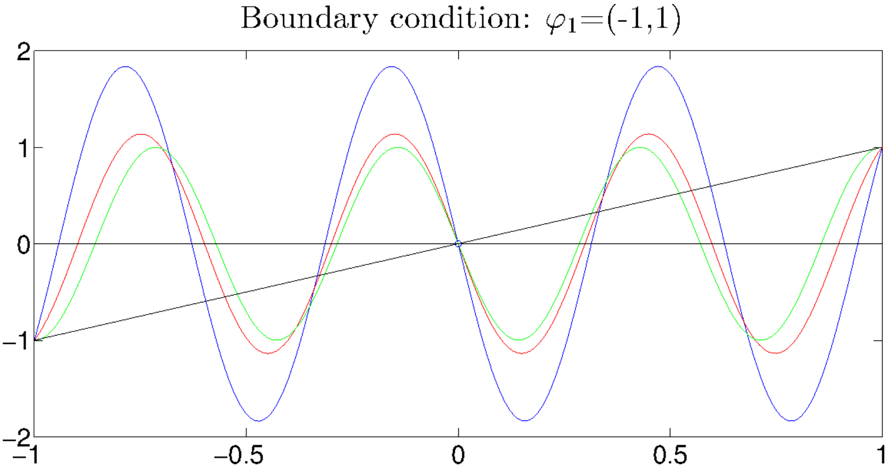

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What happens in $k = 0$?

$$\begin{cases} -\mathrm{div}(a\,\nabla u_k^i) - k\,q\,u_k^i = 0 & \text{in } \Omega, \\ u_k^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

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How to pass from 0 to k?

- Define $T\colon \mathbb{C}\setminus\Sigma \to C^1(\overline{\Omega})$, $k \mapsto u_k^\varphi$.

Lemma

The map $T\colon \mathbb{C}\setminus\Sigma \longrightarrow C^1(\overline{\Omega})$ is holomorphic.

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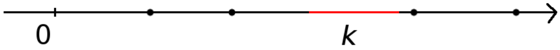

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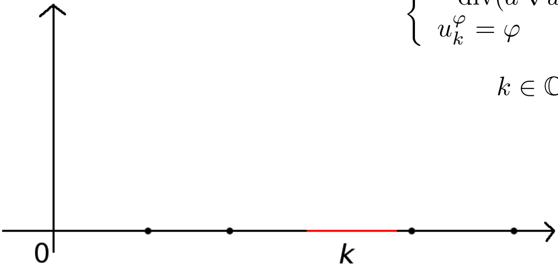

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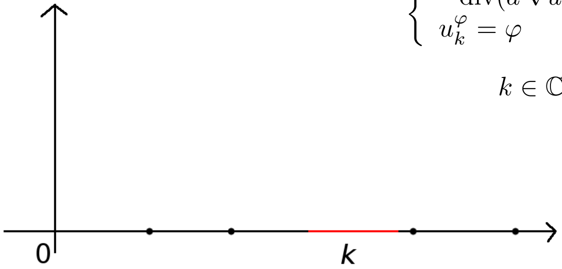

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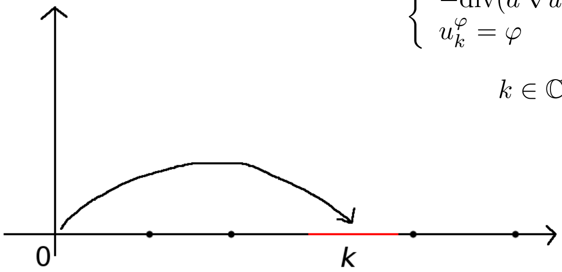

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Let $\Omega \subseteq \mathbb{R}^2$ be convex. We can find a finite $K \subseteq \mathcal{A} = [K_{min}, K_{max}]$ such that

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$K \times \{\varphi_i : i = 1, \dots, d+1\}$ is *C-complete* if for all $x \in \Omega$ there exists $\bar{k} \in K$ s.t.:

1. $\left|u_{\bar{k}}^1\right|(x) \geq C > 0,$
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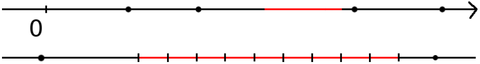

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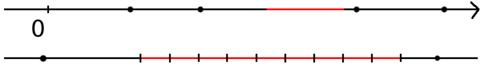

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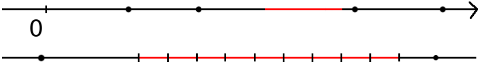

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In other words, (almost any) $d+1$ frequencies give a complete set.

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- Classical elliptic regularity theory implies that u_k^{φ} is real analytic
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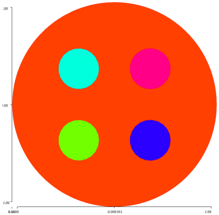
- A numerical test has been performed in 2D on 6561 different combinations of coefficients of the type



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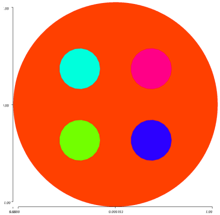

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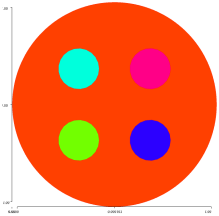

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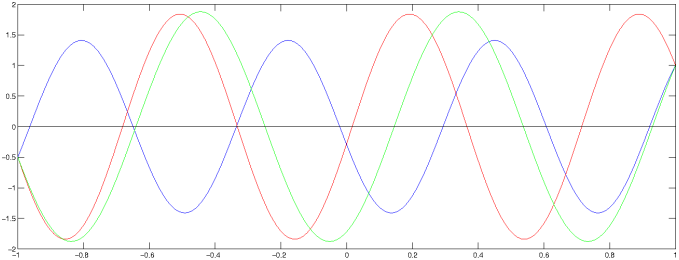

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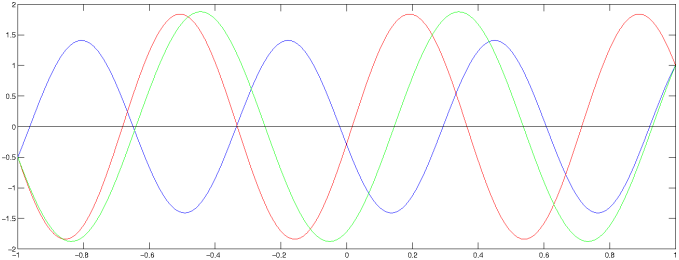

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What if $a \not\approx 1$ in 3D? (Proof)

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Suppose $a, q \in C^2(\mathbb{R}^3)$. *For a generic* C^2 *bounded domain* Ω *and a generic* $\varphi \in C^1(\overline{\Omega})$ *there exists a finite* $K \subseteq \mathcal{A}$ *such that*

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Some generalizations

- There is no need to consider these particular non-zero constraints. The only requirement is that the desired constraints are satisfied in $k = 0$.

Let $b, r \in \mathbb{N}^*$ be two positive integers, $C > 0$ and let

$$\zeta = (\zeta_1, \ldots, \zeta_r) \colon C^\nu(\overline{\Omega})^b \longrightarrow C(\overline{\Omega})^r \quad \text{be analytic.}$$

A set of measurements $K \times \{\varphi_1, \ldots, \varphi_b\}$ is *(ζ, C)-complete* if for every $x \in \overline{\Omega}$ there exists $\bar{k} \in K$ such that

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$$\nabla(k\varepsilon + \mathbf{i}\sigma) M_k(x) = F(k, (k\varepsilon + \mathbf{i}\sigma), H_k^i, \Delta H_k^i) \qquad \text{in } \Omega,$$

where $M_k(x)$ is a known 3×6 matrix and F is a given vector-valued function. If

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Let us state the main theorem in the simple case of the example:

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Assume $\mu, \varepsilon, \sigma \in W^{1,3+\delta}$ for some $\delta > 0$. Let $\hat{\sigma} \in \mathbb{R}^{3\times 3}$ be positive definite and assume $\|\sigma - \hat{\sigma}\|_{W^{1,3+\delta}} < \eta$ for some small $\eta > 0$. There exist C and n, depending only on a priori data and η, such that

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is a $(\det, C)$-complete set of measurements.

Idea of the proof.

- Regularity results give $E \in C^{0,\alpha}$ and $k \to E_k$ holomorphic
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- We propose an alternative to CGO by using a multi-frequency approach:
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  - A priori lower bounds and number of frequencies
- Same method for Maxwell, for which regularity can be easily derived

Future

- We need $d+1$ frequencies with real analytic coefficients. Can we drop this (very strong) assumption? (with Yves Capdeboscq)
- $a \in C^{0,\alpha}(\overline{\Omega}) \rightarrow a$ piecewise Hölder continuous (with L. Seppecher)
- In 3D, can we drop the assumption $a \approx 1$ completely?

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- $a \in C^{0,\alpha}(\overline{\Omega}) \rightarrow a$ piecewise Hölder continuous (with L. Seppecher)
- In 3D, can we drop the assumption $a \approx 1$ completely?

Conclusions

Past

- In order to use the reconstruction algorithms for several hybrid techniques, we need to find illuminations such that the solutions of the Helmholtz equation (or Maxwell's equations) satisfy some non-zero properties
- These are classically constructed with complex geometric optics solutions

Present

- We propose an alternative to CGO by using a multi-frequency approach:
  - A priori conditions on the illuminations which do not depend on the coefficients
  - The coefficients do not have to be smooth
  - A priori lower bounds and number of frequencies
- Same method for Maxwell, for which regularity can be easily derived

Future

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