Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Laurent Seppecher
DMA, Ecole Normale Supérieure de Paris

February 13, 2014

Joint work with H. Ammari, P. G. Grasland-Mongrain, and P. Millien
projet ERC: MULTIMOD
1. How to create currents with an acoustic beam and a constant magnetic field?
   - The ultrasonically induced Lorenz force tomography
   - Ionic description of the conductivity in aqueous tissues
   - Boundary measurements

2. From boundary measurements to meaningful internal data
   - Introduction of a virtual potential
   - Deconvolution
   - Geometric integral transform or asymptotic formula

3. Recovering the conductivity from an internal current
   - By optimization
   - By solving a transport equation
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

How to create currents with an acoustic beam and a constant magnetic field?

The ultrasonically induced Lorenz force tomography

**Assumptions**

Ω mechanically homogeneous and is a conductive medium. Γ₁ and Γ₂ are perfect conductors. Γ₀ is a perfect isolator. B is constant.
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

How to create currents with an acoustic beam and a constant magnetic field?

The ultrasonically induced Lorenz force tomography

**Velocity field**

For any \( x \in \Omega \), written \( x = y + z\xi + r \) with \( z > 0 \), \( r \in \xi^\perp \),

\[
v_{y,\xi}(y + z\xi + r, t) = A(z, |r|)w(z - ct)\xi
\]
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography.

How to create currents with an acoustic beam and a constant magnetic field?

The ultrasonically induced Lorenz force tomography

As $\Omega$ is electrically neutral, can we explain the origin of the current measured at the electrodes?
Assume that Ω is an electrolyte medium (saline gel, living tissues, . . .) the conductivity phenomenon is due to the presence of ions. Assume that we have \( N \) types of ions of charge \( q_i \) and volume density \( n_i(x) \), \( i \in \{1, \ldots, N\} \). We have, for any \( x \in \Omega \)

**Neutrality**

\[
\sum_i q_i n_i(x) = 0
\]

**Kolhrausch’s law**

\[
\sigma(x) = e^+ \sum_i \mu_i q_i n_i(x)
\]

with \( \mu_i \in \mathbb{R} \), satisfying \( \mu_i q_i > 0 \) is called the ionic mobility and \( e^+ \) is the elementary charge.
We can understand now the source of current as the deviation of the ions by the magnetic field $B$.

Consider an ion $i$ at position $x$ at time $t$. The acoustic beam imposes to it a velocity in the direction $\xi : v(x, t)\xi$. The Lorentz force applied to $i$ is

$$F_i = q_i v \xi \times Be_3$$

and the ion get almost immediately an additional drift speed

$$v_{d,i} = \frac{\mu_i}{q_i} F_i = B\mu_i v \tau$$

where $\tau = \xi \times e_3$. At first order in the displacement length, its total velocity is

$$v_i = v \xi + B\mu_i v \tau.$$  

Defining the current as the total amount of charges displacement,

$$j_s = \sum_i n_i q_i v_i = (\sum_i n_i q_i) v \xi + B (\sum_i n_i \mu_i q_i) v \tau = \frac{B}{e^+ \sigma} v \tau.$$
The interaction between the velocity field $v(x, t)\xi$ and the magnetic field $Be_3$ create a source of current

$$j_S(x, t) = \frac{B}{e^+} \sigma(x) v(x, t) \tau$$

Our measure is the indirect effect of $j_S$ on the boundary. Assume that the electromagnetic propagation is much faster than the acoustic propagation, we adopt the electrostatic approximation.

$$j = j_S + \sigma \nabla u$$

satisfying

$$\nabla \cdot j = 0$$

then the potential satisfies at a fixed time $t$,

$$-\nabla \cdot (\sigma \nabla u) = \nabla \cdot j_S \quad \text{in } \Omega$$
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography.

How to create currents with an acoustic beam and a constant magnetic field?

Boundary measurements

The intensity that we measure is

$$ I = \int_{\Gamma_2} \sigma \partial_\nu u $$
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

From boundary measurements to meaningful internal data

Introduction of a virtual potential

In order to understand the measurements, we multiply the potential equation by a well chosen test function $U$ called virtual potential defined by

\[
\begin{align*}
-\nabla \cdot (\sigma \nabla U) &= 0 \quad \text{in } \Omega \\
U &= 0 \quad \text{on } \Gamma_1 \\
U &= 1 \quad \text{on } \Gamma_2 \\
\partial_{\nu} U &= 0 \quad \text{on } \Gamma_0
\end{align*}
\]

and through integration by part it comes

\[
I = \int_{\Omega} js \cdot \nabla U = \frac{B}{e^+} \int_{\Omega} \nu(x, t) \sigma(x) \nabla U(x) dx \cdot \tau
\]

and we define the measurements function as

\[
M_{y, \xi}(z) = \int_{\Omega} \nu_{y, \xi} \left(x, \frac{Z}{c}\right) \sigma(x) \nabla U(x) dx \cdot \tau_{\xi}
\]
The inverse problem posed by this hybrid method is

**Inverse problem**

Find $\sigma : \Omega \to \mathbb{R}$ from the knowledge of

\[
M_{y,\xi} : z \to \int_{\Omega} v_{y,\xi} \left( x, \frac{z}{c} \right) \sigma(x) \nabla U(x) dx \cdot \tau_\xi
\]

known for any $y \in Y \subset \mathbb{R}^d$ and $\xi \in \Theta \subset S^{d-1}$

In general, $Y$ is supposed to be a bounded smooth surface of $\mathbb{R}^d$.

**Idea**

If $Y$ and $\Theta$ are well chosen, we show that the virtual current $J(x) = (\sigma \nabla U)(x)$ can be recovered.
Step 1: Deconvolution

As $v_{y,\xi}(y + z'\xi + r, \frac{\xi}{c}) = w(z' - z)A(z', |r|)$ we rewrite the measurements $M_{y,\xi}$ as

$$M_{y,\xi}(z) = (w \ast \Phi_{y,\xi})(z)$$

where

$$\Phi_{y,\xi}(z) = \int_{\xi_{\perp}} (\sigma \nabla U)(y + z\xi + r)A(z, |r|)dr \cdot \tau_{\xi}$$

To recover $\Phi_{y,\xi}$ with stability, we need short pulses and/or changes of the frequency. To recover the largest spectral band in the Fourier domain.
Step 2 : Getting the current

Once we know

$$\Phi_{y,\xi}(z) = \int_{\xi} (\sigma \nabla U)(y + z\xi + r)A(z, |r|)dr \cdot \tau_{\xi}$$

we can notice that it looks like a weighted Radon transform of the current density. If we assume that the support of $A$ is thin,

$$\Phi_{y,\xi}(z) = (\sigma \nabla U)(y + z\xi) \int_{\xi} A(z, |r|)dr \cdot \tau_{\xi} + O(R)$$

where $R$ is such that $\text{supp}(\rho \mapsto A(z, \rho)) \subset [0, R]$ and with a remainder depending on $|\sigma \nabla U|_{TV(\Omega)}$. Finally, choosing $x \in \Omega$ and consider $\Phi_{y,\xi}(z)$ for any $(y, \xi, z)$ such that $x = y + z\xi$ we reconstruct

$$J(x) = (\sigma \nabla U)(x)$$
Now the problem is the following,

Recover \( \sigma \) from \( J = \sigma \nabla U \) where \( U \) is defined as

\[
U = F[\sigma] = \begin{cases} 
-\nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega \\
U = 0 & \text{on } \Gamma_1 \\
U = 1 & \text{on } \Gamma_2 \\
\partial_{\nu} U = 0 & \text{on } \Gamma_0 
\end{cases}
\]

Classical approach is to minimize

\[
K_\varepsilon[\sigma] = \frac{1}{2} \int_{\Omega} |\sigma \nabla F[\sigma] - J|^2 + \varepsilon |\sigma|_{TV(\Omega)}
\]

This works but the convexity is not good (numerically).
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current
By optimization

Figure: Conductivity map $\sigma$ to be reconstructed and the reconstruction by optimisation.
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

**Orthogonal field transport equation**

If we know $\sigma \nabla U$, we know the direction of $\nabla U$. From this we can try to reconstruct the potential $U$. Let us construct a vectorial field $F$ such that

\[
\nabla U \cdot F = 0 \quad \text{in} \quad \Omega
\]

and $U|_{\Gamma_1} = 0$, $U|_{\Gamma_2} = 1$ and if the variations of $\sigma$ are supposed far from $\Gamma_0$, we can look for $U$ in $H^1(\Omega)$ as a solution of

\[
\begin{cases}
  F \cdot \nabla U = 0 & \quad \text{in} \quad \Omega \\
  U = x_2 & \quad \text{on} \quad \partial \Omega
\end{cases}
\]

This idea is good only if the previous problem admits a unique solution!
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current
By solving a transport equation

The transport problem

\[
\begin{align*}
F \cdot \nabla U &= 0 \quad \text{in } \Omega \\
U &= x_2 \quad \text{on } \partial \Omega
\end{align*}
\]

is highly related to the corresponding characteristic flow problem

\[
\begin{align*}
\partial_t X(x, t) &= F(X(x, t)) \quad \text{on } [0, T[ \\
x(x, 0) &= x \in \Omega
\end{align*}
\]

because \( t \mapsto U(X(x, t)) \) would be a constant function. We would need \( F \) to be local Lipschitz in \( \Omega \)...

Problem

\( F \) is not even continuous!
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

About Cauchy problem with non smooth field

**Theorem [DiPerna-Lions 89]**

Consider $u \in L^1(\Omega)$ satisfying

\[
\begin{align*}
F \cdot \nabla u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

with $F \in L^1(\Omega) \cap W^{1,1}_{loc}(\Omega)^d$, $\nabla \cdot F \in L^\infty(\Omega)$, then

\[u = 0.\]

Controlling the divergence is necessary to control the measure transport by the flow. We have

\[e^{-ct} \lambda \leq \lambda \circ X(t) \leq \lambda e^{ct}\]

where $c = \|\nabla \cdot F\|_{L^\infty(\Omega)}$ and $\lambda$ is the Lebesgue measure. Basically, this prevents two different characteristic lines from touching each other. Then Lions in 96 extended it to "piecewise" $W^{1,1}$ regularity.
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

And with $BV$ regularity?

**Theorem [Ambrosio 03]**

Assume that $F \in L^\infty(\Omega) \cap BV_{loc}(\Omega)$, $\nabla \cdot F \in L^\infty_{loc}(\Omega)$, then there exists a unique lagrangian flow $X$ satisfying

$$X(x, t) = x + \int_0^t F(X(x, u))du.$$ 

That would assure the uniqueness for our transport equation. But in our case if we compute formally $\nabla \cdot F = \nabla \cdot (\sigma \nabla U \times e_3) = \nabla \sigma \times \nabla U \cdot e_3 +$ something. No chance to fit in $L^\infty(\Omega)$ even locally. We shall try another approach.
We remarked that we need only existence of a flow and we do not really care about uniqueness. To fixe the ideas,

existence of outgoing flow $\Rightarrow$ uniqueness for the transport

**Theorem [Bressan-Shen 98]**

Assume that $F(x) = g(\tau(x), x)$ where

$\tau : \mathbb{R}^d \to \mathbb{R}$ is $C^1$, $t \mapsto g(t, x)$ is measurable

$x \mapsto g(t, x)$ is Lipschitz.

If there exist a compact set $K$ such that $f(x) \in K$ and

$\nabla \tau(x) \cdot z > 0$ for all $x \in \Omega, z \in K$

Then the Cauchy problem

$$
\begin{cases}
\partial_t X(x, t) = F(X(x, t)) \quad \text{on } [0, T[ \\
X(x, 0) = x \in \Omega
\end{cases}
$$

has at least solution.

Problem: $F$ cannot be tangent to its own discontinuities. This is called by Bressan the "transversality condition".
Dead end?

Our flow cannot be Lagranian so neither fits with the DiPerna-Lions theory nor the Ambrosio’s one. The flow can be tangent to the discontinuities so it does not fit with the Bressan-Shen Cauchy problem.

We can try our own (local) existence of a characteristic flow which may fit our problem.
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

For any surface \( S \in \Omega \) of class \( C^2 \) cutting \( \Omega \) in connected Lipschitz domains \( \Omega_i \), we say that \( f \in C_S^{k,\alpha}(\overline{\Omega}) \) if \( f|_{\Omega_i} \in C^{k,\alpha}(\overline{\Omega_i}) \)

**Theorem: Local existence for characteristic flow**

Consider a smooth surface \( S \subset \Omega \) and \( F \in C_S^{k,\alpha}(\overline{\Omega})^2 \). Assume that the jump of \( F \) on \( S \) can be written

\[
F^+ = f\tau + gh^+ \nu \\
F^- = f\tau + gh^- \nu
\]

where \( \nu \) is the normal to \( S \) and \( \tau \) the tangent vector and with \( f, g, h^+ \) and \( h^- \) are in \( C^{0,\alpha}(S) \), \( h^+ \) and \( h^- \) are positive and \( g \) locally signed. Then for any \( x \in \Omega \), there exists \( T > 0 \) and \( X \in C^1([0, T], \Omega) \) such that \( t \mapsto F(X(t)) \) is measurable and

\[
X(t) = x + \int_0^t F(X(s))ds \quad \forall t \in [0, T[.
\]
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

Enough difficulties! To assure that the characteristics reach the boundary, we add the hypothesis

\[ F \cdot e_1 \geq c > 0 \]

**Theorem: Existence of outgoing characteristics**

If \( F \) satisfies the previous conditions, for any \( x \in \Omega \) there exists \( T \in ]0, diam(\Omega)/c[ \) and \( X \in C^0([0, T[, \Omega) \) such that \( t \mapsto F(X(t)) \) is measurable and

\[
X(t) = x + \int_0^t F(X(s))ds \quad \forall t \in [0, T[ \]

and

\[
\lim_{t \to T} X(t) \in \partial \Omega.
\]
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

We have a uniqueness result,

**Corollary**

If $F$ satisfies the previous conditions, and $u \in C^0(\Omega) \cap C^{0,\alpha}_S(\Omega)$ satisfies

\[
\begin{aligned}
F \cdot \nabla u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

then $u = 0$ in $\Omega$. 
If the current is such that $F$ satisfies all the previous conditions, the virtual potential $U$ can be found solving

$$\begin{cases} 
F \cdot \nabla U = 0 & \text{in } \Omega \\
U = x_2 & \text{on } \partial \Omega,
\end{cases}$$

To solve this we introduce the regularized problem

$$\begin{cases} 
-\nabla \cdot (\varepsilon(I + FF^T)\nabla U_\varepsilon) = 0 & \text{in } \Omega \\
U_\varepsilon = x_2 & \text{on } \partial \Omega,
\end{cases}$$

and prove

**Proposition**

The sequence $(U_\varepsilon - U)_{\varepsilon>0}$ converges strongly to zero in $H^1_0(\Omega)$. 
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current
By solving a transport equation

Sketch of proof:

- \( \nabla (U_\varepsilon - U) \) is bounded in \( L^2(\Omega) \)
- up to an extraction \( (U_\varepsilon - U) \) converges in \( H^1_0(\Omega) \) for the weak \(-\star\) topology.
- The limit \( U^* \) satisfies
  \[
  \begin{cases}
    F \cdot \nabla U^* = 0 & \text{in } \Omega \\
    U^* = 0 & \text{on } \partial \Omega,
  \end{cases}
  \]
  so using the previous work, \( U^* = 0 \).
- We prove that the convergence is strong and we do not need extraction.

Corollary

The sequence \( \frac{1}{\sigma_\varepsilon} := \frac{J \cdot \nabla U_\varepsilon}{|J|^2} \) converges to \( \frac{1}{\sigma} \) strongly in \( L^2(\Omega) \).
Recovering discontinuous conductivity from internal current: case of the ultrasonically-induced Lorentz force electrical impedance tomography

Recovering the conductivity from an internal current

By solving a transport equation

Figure: Conductivity map $\sigma$ to be reconstructed and the reconstruction through transport equation.