Higher representation theory in algebra and geometry: Lecture VI

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References

For this lecture, useful references include:

- Chuang and Rouquier, *Derived equivalences for symmetric groups and \( sl_2\)-categorification* (introduces definition of categorical \( sl_n \) actions via Hecke algebras)

- Khovanov and Lauda, *A categorification of quantum \( sl(n) \)* (covers connection to partial flag varieties in great detail)

- Brundan and Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras* (introduced grading on symmetric groups and affine Hecke algebras)

- B. W., *A note on isomorphisms between Hecke algebras* (contains proof we’ll discuss today)

The slides for the talk are on my webpage at:


You can also find some proofs that I didn’t feel like going through in class at:

https://pages.shanti.virginia.edu/Higher_Rep_Theory/
Thus far...

We’ve talked a lot about categorical actions of $\mathfrak{sl}_2$. We’ve seen them show up in the representation theory of symmetric groups, in the geometry of Grassmannians and in the construction of knot invariants.

This leads us down a lot of interesting roads, but it’s not very satisfying. After all, $\mathfrak{sl}_2$ is just the most basic of a big class of Lie algebras: the simple Lie algebras (or if you like to be fancier, the Kac-Moody algebras).

So, this is a natural direction to hunt. As with categorical $\mathfrak{sl}_2$-actions, the thing to look for is natural examples.
Simple Lie algebras

Recall, every simple Lie algebra over $\mathbb{C}$ (or more generally, Kac-Moody algebra) has a presentation by generators $E_i, F_i, H_i$ for $i \in I$ with relations

\[
[E_i, F_i] = H_i \quad [H_i, E_i] = 2E_i \quad [H_i, F_i] = -2F_i
\]

\[
\text{ad}_{E_i}^{-a_{ij}+1} E_j = 0 \quad \text{ad}_{F_i}^{-a_{ij}+1} F_j = 0 \quad [E_i, F_j] = 0 \quad [H_i, H_j] = 0
\]

for some Cartan matrix $A = (a_{ij})$.

The most popular source of these is from graphs, where $I$ is the vertex set, and $-a_{ij}$ is the number of edges joining $i$ and $j$.

The algebra $\mathfrak{sl}_{n+1}$ corresponds to a linear graph with $n$ nodes.
Examples

Theorem (Beilinson-Lusztig-Macpherson)

There is a natural $U_{\sqrt{p}(\mathfrak{sl}_n)}$ action on the space of functions on the set of all $n - 1$-step flags in a fixed vector space $S$, such that:

- The maps $E_i$ and $F_i$ are induced by the pullback and pushforward are induced by the correspondence

$$
\begin{align*}
\text{Gr}(\cdots \subset k_i \subset \cdots , S) & \overset{\pi_2}{\longrightarrow} \text{Gr}(\cdots \subset k_i - 1 \subset k_i \subset \cdots , S) \\
\text{Gr}(\cdots \subset k_i - 1 \subset \cdots , S) & \overset{\pi_1}{\longleftarrow} \text{Gr}(\cdots \subset k_i - 1 \subset \cdots , S)
\end{align*}
$$

- The functions concentrated on $\text{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1, S)$ are of weight $(d - k_1, k_1 - k_2, \ldots, k_{n-1})$.

- The functions constant on each $\text{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1, S)$ form a copy of the simple representation $\text{Sym}^d(\mathbb{C}^n)$. 
Symmetric groups

Fix a field \( \mathbb{k} \), and let \( K = \bigoplus_m K(\mathbb{k}[S_m]\text{-proj}) \) be the Grothendieck group of projective \( \mathbb{k}[S_m] \) modules.

Earlier, we defined functors \( \mathcal{F}_a \) and \( \mathcal{E}_a \) for \( a \in \mathbb{Z} \) by

\[
\mathcal{E}_a(M) = \{ h \in \text{Res}_{S_{m-1}}^{S_m} M \mid (X_m - a)^N h = 0 \text{ for } N \gg 0 \}.
\]
\[
\mathcal{F}_a(M) = \{ s \otimes h \in \text{Ind}_{S_{m-1}}^{S_m} M \mid s(X_m - a)^N \otimes h = 0 \text{ for } N \gg 0 \}.
\]

The functors \( \mathcal{E}_i \) and \( \mathcal{F}_i \) induce operators \( E_i \) and \( F_i \) on \( K \).

**Proposition**

Each \( (E_i, F_i) \) generate a copy of \( \mathfrak{sl}_2 \), and

\[
[E_i, [E_i, E_i \pm 1]] = 0 \quad [E_i, E_j] = 0 \quad (i \neq j \pm 1)
\]
\[
[E_i, F_j] = 0 \quad (i \neq j) \quad [F_i, [F_i, F_i \pm 1]] = 0 \quad [F_i, F_j] = 0 \quad (i \neq j \pm 1)
\]
That is, we have an action of the Lie algebra whose Dynkin diagram has vertices given by the image of \( \mathbb{Z} \) in \( k \) where \( i \) and \( i + 1 \) are adjacent.

- When \( \text{char}(k) = 0 \), then this is \( \mathfrak{sl}_\infty \).
- When \( \text{char}(k) = p \), then this is the affine Lie algebra \( \hat{\mathfrak{sl}}_p \).

**Proposition**

*The Grothendieck group \( K \) is an irreducible representation of \( \mathfrak{sl}_\infty \) or \( \hat{\mathfrak{sl}}_p \), generated by a highest weight vector satisfying \( H_i v = \delta_{i,0} v \) (“the basic representation”).*
Categorifications of $\mathfrak{sl}_n$

These examples suggest there’s some underlying structure here, so let’s find it. The place that’s most conducive to doing the calculation is on flag varieties. Recall that:

**Proposition**

The cohomology $\text{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1)$ is isomorphic to $C_{d}^{k_{n-1}, \ldots, k_1}$, generated by $e_i(1, k_{n-1}), e_i(k_{n-1} + 1, k_{n-2}), \ldots, e_i(k_1 + 1, d)$ in the coinvariant algebra $C_d$.

Whatever a categorical action of $\mathfrak{sl}_n$ is, we have an example of one on the categories $\bigoplus_k C_{d}^{k_{n-1}, \ldots, k_1} \text{-mod}$ with functors $\mathcal{E}_i$ and $\mathcal{F}_i$ given by the tensor products

\[
\mathcal{E}_i = C_{d}^{\ldots, k_i-1, k_i, \ldots} \otimes C_{d}^{\ldots, k_i, \ldots} \quad \rightarrow \quad C_{d}^{\ldots, k_i-1, \ldots} \\
\mathcal{F}_i = C_{d}^{\ldots, k_i, k_i+1, \ldots} \otimes C_{d}^{\ldots, k_i, \ldots} \quad \rightarrow \quad C_{d}^{\ldots, k_i+1, \ldots}
\]
Categorifications of $\mathfrak{sl}_n$

First of all, the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ for each $i$ generate a categorical action of $\mathfrak{sl}_2$. That is $\mathcal{E}_i$ and $\mathcal{F}_i$ satisfy $\mathfrak{sl}_2$ relations up to isomorphism, are biadjoint, and $\mathcal{F}_i^m$ has an action of the nilHecke algebra.

Well, that’s a good start, but presumably it’s not enough; we need to have some kind of interaction between these different $\mathfrak{sl}_2$’s.

Thinking geometrically, $\mathcal{F}_i \mathcal{F}_j$ corresponds to tensor product with $H^*$ of the space of triples of flags

$$X_{i,j} = \{(V_\bullet, V'_\bullet, V''_\bullet) \mid V_k \supset V'_k, \dim(V_k/V'_k) = \delta_{k,i}, V'_m \supset V''_m, \dim(V'_m/V''_m) = \delta_{m,j}\}$$

If $i \neq j \pm 1$, then switching the order of $i$ and $j$ gives the same space (Exercise: write down the isomorphism), and so $\mathcal{F}_i \mathcal{F}_j \cong \mathcal{F}_j \mathcal{F}_i$. 

Categorifications of $\mathfrak{sl}_n$

However, if $j = i + 1$, it’s a different story. We may as well forget all the other spaces in the flag, and assume that $n = 3$, $i = 1$ and $j = 2$. Thus, we have that

$$X_{1,2} = \{ V_1 \supset V_2, V_1'' \supset V_2'' \}$$

$$X_{2,1} = \{ V_1 \supset V_1'' \supset V_2 \supset V_2'' \}$$

Thus, we have an obvious inclusion map $X_{2,1} \to X_{1,2}$ and thus pullback and pushforward maps in cohomology.

These maps aren’t isomorphisms. Instead, their composition in either direction is multiplication by the Euler class of the line bundle $\text{Hom}(V_2/V_2'', V_1/V_1'')$ since $X_{2,1}$ is the vanishing set of a section of this bundle (the induced canonical map).
In the case of $\mathfrak{sl}_2$, the structure of a categorical action is controlled by the nilHecke action on powers of the functor $\mathcal{F}$. What will this structure look like in the $\mathfrak{sl}_n$ case?

Consider the sum $\mathcal{F} := \bigoplus \mathcal{F}_i$. What acts on $\mathcal{F}^k$?

- there are idempotents $e_i$ which project to $\mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1}$.
- the action of dots $y_1, \ldots, y_k$ induced by the individual $\mathfrak{sl}_2$ actions.
- elements $\psi_j$ which acts on $\mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1}$ by
  - the Demazure operator if $i_j = i_{j+1}$
  - the pushforward by the map $X_{i_{j+1},i_j} \rightarrow X_{i_j,i_{j+1}}$ if $i_{j+1} = i_j + 1$
  - the pullback by the map $X_{i_j,i_{j-1}} \rightarrow X_{i_{j-1},i_j}$ if $i_{j+1} = i_j - 1$
  - the map induced the isomorphism $X_{i_{j+1},i_j} \cong X_{i_{j+1},i_j}$ if $i_{j+1} \neq i_j, i_j \pm 1$. 
Diagrams

We represent these with diagrams much like before.

\[
\begin{align*}
\text{deg} = 0 & \quad e_i \\
\text{deg} = 2 & \quad y_j \\
\text{deg} = -\langle \alpha_{ij}, \alpha_{ij+1} \rangle & \quad \psi_j
\end{align*}
\]

Definition

The Khovanov-Lauda-Rouquier (KLR) or quiver Hecke algebra \( R_m \) for \( \mathfrak{sl}_n \) is the algebra generated by these elements with \( m \) strands modulo the relations on the next slide.
Diagrams

\[
\begin{align*}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$j$};
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$j$};
\end{tikzpicture}
\text{ unless } i = j
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i$};
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i$};
\end{tikzpicture}
\quad + \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i$};
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\quad - \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\quad \quad \text{ unless } i = k = j \pm 1
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i$};
\end{tikzpicture}
\quad = \quad 0
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\quad = \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\quad + \quad
\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
  \node at (0.5,0.5) {$i$};
  \node at (1.5,0.5) {$i-1$};
\end{tikzpicture}
\end{align*}
\]
Diagrams

\[
\begin{align*}
\text{if } i & \neq j, \\
\begin{array}{c}
\text{unless } i = j \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{unless } i = k = j \pm 1
\end{align*}
\]
Categorical actions of $\mathfrak{sl}_n$

Definition

A categorical action of $\mathfrak{sl}_n$ is a collection of categorical $\mathfrak{sl}_2$ actions $(F_i, E_i)$ for $i = 1, \ldots, n - 1$, together with an extension of the nilHecke action to an action of $R_m$ on $(\bigoplus F_i)^m$.

We’ve deliberately set things up so we know at least one example of these: the modules over $C_{d^{k_n-1,\ldots,k_1}}$ for all different choices of $k$. 

Serre relations

Why do the Serre relations hold? Consider the relation

\[
\begin{array}{c}
\begin{array}{ccc}
i & i & i \\
i & i - 1 & i
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
i & i & i \\
i & i - 1 & i
\end{array}
\end{array}
\]

This is splitting of a sum as two orthogonal idempotents. In fact, this shows directly that:

\[
\mathcal{F}_i \mathcal{F}_{i-1} \mathcal{F}_i \cong \mathcal{F}_i^{(2)} \mathcal{F}_{i-1} \oplus \mathcal{F}_{i-1} \mathcal{F}_i^{(2)}
\]

which is just a rearranged Serre relation!

Proposition

Any categorical $\mathfrak{sl}_n$ action actually does induce an $\mathfrak{sl}_n$ action on the Grothendieck group. (Note, I haven’t checked that $[E_i, F_j] = 0$.)
Categorical actions for other Lie algebras

It’s relatively easy to guess how to extend this to the case of $\hat{\mathfrak{sl}}_n$.

- You can construct a KLR algebra for $\hat{\mathfrak{sl}}_n$ with the same sort of diagrams and relations, but with strands labeled by $\mathbb{Z}/n\mathbb{Z}$, rather that $\{1, \ldots, n - 1\}$.
- A categorical $\hat{\mathfrak{sl}}_n$-action is a collection of $\mathfrak{sl}_2$ actions which have an action of this KLR algebra on $(\oplus E_i)^m$.

Now, two things should bother you about this:

1. Many of you probably know that an $\hat{\mathfrak{sl}}_n$ or $\mathfrak{sl}_n$ was defined in a different way in Chuang-Rouquier, using an affine Hecke algebra instead.
2. It’s not clear that there are any examples of categorical $\hat{\mathfrak{sl}}_n$-actions “in the wild” though you may recall that symmetric groups looked promising.
There’s an alternate definition of categorical actions via affine Hecke algebras.

**Definition**

The degenerate affine Hecke algebra $h_m$ of rank $m$ is generated by $t_1, \ldots, t_{m-1}, x_1, \ldots, x_m$ with relations:

\[
\begin{align*}
t_i^2 &= 1 \\
t_it_{i\pm 1}t_i &= t_{i\pm 1}t_it_{i\pm 1} \\
t_it_j &= tjt_i \ (i \neq j \pm 1) \\
x_ix_j &= x_jx_i \\
t_ix_it_i &= x_{i+1} - t_i \\
x_it_j &= t_ix_j \ (i \neq j, j + 1)
\end{align*}
\]

**Definition**

The affine Hecke algebra $H_m(v)$ of rank $m$ is generated by $T_1, \ldots, T_{m-1}, X_1^{\pm 1}, \ldots, X_m^{\pm 1}$ with relations:

\[
\begin{align*}
(T_i + 1)(T_i - v) &= 0 \\
T iT_{i\pm 1}T_i &= T_{i\pm 1}T iT_{i\pm 1} \\
T iT_j &= TjT_i \ (i \neq j \pm 1) \\
X_iX_j &= X_jX_i \\
T iT_{i+1} &= vX_{i+1} \\
X iT_j &= T_iX_j \ (i \neq j, j + 1)
\end{align*}
\]
Consider a pair of adjoint functors \((\mathcal{F}, \mathcal{E})\) such that \(\mathcal{F}^m\) carries an action of the affine Hecke algebra over \(\mathbb{k}\) for some \(v \in \mathbb{k}\). Let \(U \subset \mathbb{k}\) be the union of the spectra of \(X\) acting on \(\mathcal{F}\) (and that \(\mathcal{F}(M)\) is the sum of its generalized eigenspaces). We give \(U\) a graph structure by adding an edge \(u_1 \to u_2\) if \(u_2 = vu_1\).

We can decompose \(\mathcal{F} = \bigoplus_{u \in U} \mathcal{F}_u\) into its generalized eigenspaces for \(X\), and similarly \(\mathcal{E} = \bigoplus_{u \in U} \mathcal{E}_u\).

**Alternate definition**

If the functors \(\mathcal{E}_i\) and \(\mathcal{F}_i\) define a weak \(\mathfrak{sl}_2\) action, then the functors \(\mathcal{E}_i\) and \(\mathcal{F}_i\) define an action of \(\mathfrak{g}_U\), the algebra with Dynkin diagram \(U\).

Since \(U\) is a union of linear and cyclic graphs, this just means a bunch of commuting \(\hat{\mathfrak{sl}}_r\) and \(\mathfrak{sl}_s\) actions.
Consider a pair of adjoint functors \((\mathcal{F}, \mathcal{E})\) such that \(\mathcal{F}^m\) carries an action of the degenerate affine Hecke algebra over \(k\). Let \(u \subset k\) be the union of the spectra of \(x\) acting on \(\mathcal{F}\) (and that \(\mathcal{F}(M)\) is the sum of its generalized eigenspaces). We give \(u\) a graph structure by adding an edge \(u_1 \to u_2\) if \(u_2 = u_1 + 1\).

We can decompose \(\mathcal{F} = \bigoplus_{u \in U} \mathcal{F}_u\) into its generalized eigenspaces for \(x\), and similarly \(\mathcal{E} = \bigoplus_{u \in U} \mathcal{E}_u\).

### Alternate definition

If the functors \(\mathcal{E}_i\) and \(\mathcal{F}_i\) define a weak \(\mathfrak{sl}_2\) action, then the functors \(\mathcal{E}_i\) and \(\mathcal{F}_i\) define an action of \(\mathfrak{g}_U\), the algebra with Dynkin diagram \(U\).

Since \(U\) is a union of linear and cyclic graphs, this just means a bunch of commuting \(\hat{\mathfrak{sl}}_r\) and \(\mathfrak{sl}_s\) actions.
For example, we have an action of $h_m$ on $\text{Ind}_S^{S_k+m}$ with

$$t_i \mapsto (i + k, i + k + 1) \quad x_i \mapsto \sum_{g < i} (g, i)$$

The spectrum $u$ is the image of $\mathbb{Z}$ in $\mathbb{k}$; thus, as we saw before, it’s a cycle if $\text{char}(\mathbb{k}) = p$, and an infinite line if $\text{char}(\mathbb{k}) = 0$.

**Proposition**

*This defines a categorical $\widehat{sl}_p$ or $sl_\infty$-action on $\bigoplus_m \mathbb{k}[S_m]$-mod (according to our alternate definition).*

Similarly, we’ll get an $\widehat{sl}_p$-action if we look at representations of the (finite) Hecke algebra with $\nu \in \mathbb{k}$ a primitive $p$th root of unity. This is just one aspect of the analogy of this case with the case $\text{char}(\mathbb{k}) = p$. 
So, why are these “the same thing”? Well, of course, it would be great if the (d)AHA were isomorphic to a KLR algebra, but a moment’s thought shows this can’t be so (for which graph?).

**Proposition**

Instead, there’s a natural completion of $H_m(v)$ at the collection of quotients where $X_i$ has spectrum $U$ which is isomorphic to the KLR algebra $R_m$ with graph $U$ completed according to its grading.

For example, if we let $U$ be the $p$th roots of unity in $\mathbb{k}$ of characteristic prime to $p$, we’ll get the KLR algebra for $\hat{sl}_p$. 
So, why are these “the same thing”? Well, of course, it would be great if the (d)AHA were isomorphic to a KLR algebra, but a moment’s thought shows this can’t be so (for which graph?).

**Proposition**

*Instead, there’s a natural completion of $h_m$ at the collection of quotients where $x_i$ has spectrum $u$ which is isomorphic to the KLR algebra $R_m$ with graph $u$ completed according to its grading.*

For example, if we let $u$ be $\mathbb{Z}/p\mathbb{Z}$ in $\mathbb{k}$ of characteristic $p$, we’ll get the KLR algebra for $\tilde{\mathfrak{sl}}_p$. 
Polynomial representations

In order to think about KLR algebras, one of the most useful tools is a representation on a sum of polynomial rings. Let $I$ be the vertex set of an oriented graph. Consider the sum of polynomial rings $\bigoplus_{i \in I} k[Y_1, \ldots, Y_m] \epsilon_i$. Consider the operators

$$e_i \cdot f(Y_1, \ldots, Y_m) \epsilon_j = f(Y_1, \ldots, Y_m) \delta_{i,j} \epsilon_j$$

$$y_k \cdot f(Y_1, \ldots, Y_m) \epsilon_j = Y_k f(Y_1, \ldots, Y_m) \epsilon_j$$

$$\psi_k \cdot f(Y_1, \ldots, Y_m) \epsilon_j = \begin{cases} \frac{f-f^{s_i}}{Y_{k+1}-Y_k} \epsilon_{s_k \cdot j} & j_k = j_{k+1} \\ (Y_{k+1} - Y_k) \# \{j_k \rightarrow j_{k+1}\} f^{s_k} \epsilon_{s_k \cdot j} & j_k \neq j_{k+1} \end{cases}$$

**Proposition**

*The algebra these generate is the KLR algebra of $I$.***
Polynomial representations

In order to think about KLR algebras, one of the most useful tools is a representation on a sum of polynomial rings. Let $I$ be the vertex set of an oriented graph. Consider the sum of polynomial rings $\bigoplus_{i \in I^m} \mathbb{k}[Y_1, \ldots, Y_m] \epsilon_i$. Consider the operators

\[
e_i \cdot f(Y_1, \ldots, Y_m) \epsilon_j = f(Y_1, \ldots, Y_m) \delta_{i,j} \epsilon_j
\]

\[
y_k \cdot f(Y_1, \ldots, Y_m) \epsilon_j = Y_k f(Y_1, \ldots, Y_m) \epsilon_j
\]

\[
\psi_k \cdot f(Y_1, \ldots, Y_m) \epsilon_j = \begin{cases} 
\frac{f-f^s_i}{Y_{k+1}-Y_k} \epsilon_{s_k \cdot j} & j_k = j_{k+1} \\
P_{i_ki_{k+1}}(Y_{k+1}, Y_k)f^{s_k} \epsilon_{s_k \cdot j} & j_k \neq j_{k+1}
\end{cases}
\]

**Proposition**

The algebra these generate is the KLR algebra of $I$.

Generally, we can choose a Cartan matrix $A$ and polynomials $P_{ij}(u, v)$ with

\[
P_{ij}(u, v) P_{ji}(v, u) = t_{ij} u^{-a_{ji}} + \cdots + t_{ji} v^{-a_{ij}}.
\]
The affine Hecke algebra also has a (signed) polynomial representation on \( \mathbb{k}[X_1^\pm 1, \ldots, X_m^\pm 1] \) given by

\[
T_i F(X_1, \ldots, X_m) = -F^{s_i} + (1 - q)X_{i+1}\frac{F^{s_i} - F}{X_{i+1} - X_i}
\]

We’ll also complete this representation at the quotients where the spectrum of \( X_i \) has spectrum \( U \). For each \( u = (u_1, \ldots, u_m) \), we have the \( \epsilon_u \) of 1 to the generalized \( u_i \)-eigenspace for \( X_i \).

The idea of the isomorphism is to match up these two representations after completion. Fix any power series \( B(z) = 1 + \sum_{i=1}^{\infty} B_i z^i \) with \( B_1 \neq 0 \).

\[
X_1^{a_1} \cdots X_m^{a_m} \epsilon_u \leftrightarrow (u_1 B(y_1))^{a_1} \cdots (u_m B(y_m))^{a_m} \epsilon_u
\]
Polynomial representations

The degenerate affine Hecke algebra also has a (signed) polynomial representation on $\mathbb{k}[x_1, \ldots, x_m]$ given by

$$t_i f(x_1, \ldots, x_m) = -f^{s_i} - \frac{f^{s_i} - f}{x_{i+1} - x_i}$$

We’ll also complete this representation at the quotients where the spectrum of $x_i$ has spectrum $u$. For each $u = (u_1, \ldots, u_m)$, we have the $\varepsilon_u$ of 1 to the generalized $u_i$-eigenspace for $x_i$.

The idea of the isomorphism is to match up these two representations after completion. Fix any power series $b(z) = b_1 z + \ldots$ with $b_1 \neq 0$.

$$x_1^{a_1} \cdots x_m^{a_m} \varepsilon_u \leftrightarrow (b(y_1) + u_1)^{a_1} \cdots (b(y_m) + u_m)^{a_m} \varepsilon_u$$
Polynomial representations

The trick is to consider the “intertwining elements”

\[ \Phi_r := T_r + \sum_{u \text{ s.t. } u_r \neq u_{r+1}} \frac{1 - qe^h}{1 - X_r X_{r+1}^{-1}} e_u + \sum_{u \text{ s.t. } u_r = u_{r+1}} e_u \]

If you look at how this element acts in the polynomial representation, you can match it up with the element \( \psi_r \):

\[
\Phi_r \cdot F = \begin{cases} 
  u_r B(y_r) - vu_{r+1} B(y_{r+1}) & u_r \neq u_{r+1} \\
  u_{r+1} B(y_{r+1}) - u_r B(y_r) & B(y_r) - vB(y_{r+1}) \left( F^{s_r} - F \right) e_u & u_r = u_{r+1}
\end{cases}
\]

We then use the fact that \( B(y_r) - aB(y_{r+1}) = (1 - a) + B_1(y_r - y_{r+1}) + \cdots \) is invertible if \( a \neq 1 \), whereas \( (B(y_r) - B(y_{r+1}))/ (y_r - y_{r+1}) = B_1 + \cdots \) is well-defined and invertible.
Polynomial representations

The trick is to consider the “intertwining elements”

$$
\phi_r := t_r + \sum_{u \text{ s.t. } u_r \neq u_{r+1}} \frac{1}{x_i - x_{i+1}} e_u + \sum_{u \text{ s.t. } u_r = u_{r+1}} e_u
$$

If you look at how this element acts in the polynomial representation, you can match it up with the element $\psi_r$:

$$
\Phi_r \cdot F = \begin{cases} 
  \frac{b(y_r) + u_r - b(y_{r+1}) - u_{r+1} - 1}{b(y_{r+1}) + u_{r+1} - b(y_r) - u_r} F_{sr}^s \epsilon_{u^s} & u_r \neq u_{r+1} \\
  \frac{b(y_r) - b(y_{r+1}) - 1}{b(y_{r+1}) - b(y_r)} (F_{sr}^s - F) \epsilon_u & u_r = u_{r+1}
\end{cases}
$$

We then use the fact that $b(y_r) - b(y_{r+1}) + a = a + b_1 (y_r - y_{r+1}) + \cdots$ is invertible if $a \neq 0$, whereas $(b(y_r) - b(y_{r+1}))/ (y_r - y_{r+1}) = b_1 + \cdots$ is well-defined and invertible.
Polynomial representations

That is, we send

\[ \Phi_{r \mathbf{e_u}} \mapsto \begin{cases} 
\frac{(B(y_r) - vB(y_{r+1}))(y_r - y_{r+1})}{B(y_{r+1}) - B(y_r)} \psi_r e_u & u_r = u_{r+1} \\
\frac{B(y_{r+1}) - B(y_r)}{B(y_r) - B(y_{r+1})} \psi_r e_u & u_r = vu_{r+1} \\
\frac{(v^{-1}B(y_{r+1}) - B(y_r))(y_{r+1} - y_r)}{u_r B(y_r) - vu_{r+1} B(y_{r+1})} \psi_r e_u & u_r \neq u_{r+1}, vu_{r+1} \\
\frac{u_{r+1} B(y_{r+1}) - u_r B(y_r)}{u_r B(y_r) - vu_{r+1} B(y_{r+1})} \psi_r e_u & \psi_r e_u 
\end{cases} \]

**Theorem**

This induces an isomorphism of completions between the KLR algebra \( R \) for \( U \) and \( H_m(v) \).

Note that this isomorphism is *highly* non-unique. I kind of like to use \( B(x) = e^x \), but Brundan and Kleshchev use \( B(x) = 1 + x \).
Polynomial representations

That is, we send

$$\phi_re_u \mapsto \begin{cases}
\frac{(b(y_r) - b(y_{r+1}) - 1)(y_r - y_{r+1})}{b(y_{r+1}) - b(y_r)} \psi_{r}e_u & u_r = u_{r+1} \\
\frac{b(y_{r+1}) - b(y_r)}{b(y_r) - b(y_{r+1})} \psi_{r}e_u & u_r = u_{r+1} + 1 \\
\frac{(b(y_{r+1}) - b(y_r) - 1)(y_{r+1} - y_r)}{b(y_r) + u_r - b(y_{r+1}) - u_{r+1} - 1} \psi_{r}e_u & u_r \neq u_{r+1}, u_{r+1} + 1
\end{cases}$$

Theorem

This induces an isomorphism of completions between the KLR algebra $R$ for $u$ and $h_m$.

Note that this isomorphism is highly non-unique. I kind of like to use $B(x) = e^x$, but Brundan and Kleshchev use $B(x) = 1 + x$. 
One really interesting consequence of this isomorphism is that \( \mathbb{k}[S_m] \) has an unexpected grading:

**Proposition**

*The group algebra \( \mathbb{k}[S_m] \) is the quotient of \( h_m \) by \( x_1 = 0 \). The other \( x_i \)'s go to the Jucys-Murphy elements.*

Thus, we can also describe \( \mathbb{k}[S_m] \) as a quotient of the KLR algebra by the two sided ideal generated by \( e_i \) with \( i_1 \neq 0 \), and \( y_1 \).

For example, if \( m = 2 \), then as long as \( p > 2 \), this quotient is spanned by \( e_{0,1} \) and \( e_{0,p-1} \), the idempotents projecting to the trivial and sign reps. If \( p = 2 \), then these are the same, and \( \mathbb{k}[S_2] = \mathbb{k}[y_2]/(y_2^2) \).
Gradings on $\mathbb{k}[S_m]$

If interpreted correctly, most natural modules over $\mathbb{k}[S_m]$ such as simples, Specht modules, permutation modules, and projectives can be graded.

Thus, interesting questions like:

(*) what are the multiplicities of simples modules in Specht modules?

have $q$-analogues where we consider graded multiplicities. These have, for example, interesting connections to Kazhdan-Lusztig polynomials.

To be precise, if one looks instead at the (finite) Hecke algebra over $\mathbb{C}$ at $v = e^{2\pi i/p}$, the graded multiplicities of simples in Specht modules will be affine parabolic Kazhdan-Lusztig polynomials. The multiplicities over $\mathbb{F}_p$ are not the same (though they are bounded below by the char 0 multiplicities) for all $p$, but for $p$ in some range $c(m) \leq p \leq m$.

A long-standing conjecture of James had been that $c(m) \leq \sqrt{p}$. This was recently disproven by Williamson.